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Black Holes in the Brans-Dicke
Theory of Gravitation

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Abstract. It is shown that a stationary space containing a black hole is a solution of the Brans-Dicke field equations if and only if it is a solution of the Einstein field equations. This implies that when the star collapses to form a black hole, it loses that fraction (about 7%) of its measured gravitational mass that arises from the scalar interaction. This mass loss is in addition to that caused by emission of scalar or tensor gravitational radiation. Another consequence is that there will not be any scalar gravitational radiation emitted when two black holes collide.

1. Introduction

In this paper I shall extend the arguments of the previous paper [1] to the Brans-Dicke theory of gravitation. Most of the results of the previous paper did not depend on the field equations in detail, but only on certain inequalities on the Ricci tensor such as

$$R_{ab} l^a l^b \geqq 0$$

for any null vector l^a. These inequalities are also satisfied in the Brans-Dicke theory if one expresses it in the conformal frame in which the gravitational constant is constant and the masses of particles vary with position. I shall call this the Einstein frame. In particular, the results that stationary black holes must be axi-symmetric and have spherical topology will hold in the Brans-Dicke theory also. It then follows that the scalar field which occurs in the Brans-Dicke theory must be constant everywhere in a stationary black hole solution. From this it follows that stationary black holes in Brans-Dicke theory are precisely the same as in general relativity and so presumably are represented by the Kerr family of solutions. What this seems to indicate is that if a massive body collapses behind an event horizon, its effect as a source of the scalar field decreases to zero. This has two important consequences. Firstly, the scalar monopole moment represents a fraction $1/(2\omega+4)$ of the measured active gravitational mass of a normal body (ω is the coupling constants which appear in the Brans-Dicke theory and which has a suggested value of about 6). As a body collapses it loses its scalar monopole moment so its measured active gravitational mass decreases by this fraction. This mass loss is in addition to any mass loss that may result from the emission of scalar or tensor radiation during the collapse. The resulting black hole will move on a geodesic in the Einstein conformal frame whereas small normal bodies move on geodesics in the conformal frame in which the masses of particles are constants (I shall call this the Brans-Dicke frame). This means that black holes do not obey the equivalence principle in the Brans-Dicke theory. The existence of such violations of the equivalence principle by bodies with significant gravitational binding energy has already been pointed out by Nordvedt [2]. A black hole may be regarded as extreme case of this effect where the binding energy is of the same order as the rest mass energy. One may express this violation by saying that the inertial mass is no longer equal to the passive and active gravitational mass.

The second important consequence is that no scalar gravitational radiation will be emitted when two black holes collide since the scalar field will be constant everywhere. This shows that the fact that the gravitational radiation pulses which Weber observes [3] do not appear to be scalar [4] is not inconsistent with the Brans-Dicke theory if the pulses arise from a black hole collisions. Indeed, this would seem to be the only possible process that could be producing them in view of the enormous energy [5] and the lack of associated electromagnetic [6] or neutrino [7] radiation.

2. The Brans-Dicke Theory

The Brans-Dicke theory was originally expressed in the conformal frame (the "Brans-Dicke frame") in which the masses of small particles remain constant while the locally measured gravitational constant G is $\phi^{-1}(4+2\omega)(3+2\omega)^{-1}$ where ϕ is a scalar field and ω is an adjustable coupling constant. The field equations have the form:

$$R_{ab}-\tfrac{1}{2}g_{ab}R = 8\pi\phi^{-1}c^{-4}T_{ab} + \omega\phi^{-2}(\phi_{;a}\phi_{;b}-\tfrac{1}{2}g_{ab}g^{cd}\phi_{;c}\phi_{;d}) + \phi^{-1}(\phi_{;ab}-g_{ab}g^{cd}\phi_{;cd}) \tag{1}$$

$$g^{cd}\phi_{;cd} = 8\pi(3+2\omega)^{-1}c^{-4}T \tag{2}$$

where semi-colon denotes the covariant derivative with respect to the metric g_{ab} and T_{ab} is the usual energy momentum tensor which obeys the conservation equation $T_{ab;c}g^{bc}=0$.

Dicke [9] has pointed out that the theory can also be expressed in another conformal frame in which the field equations resemble Einstein's. In this frame (the "Einstein frame") the value of G is defined to be a constant G_0 everywhere but the masses of particles vary as $\phi^{-\frac{1}{2}}$. The conformal transformation defining this frame is $\bar{g}_{ab}=\phi G_0 g_{ab}$, $\bar{T}_{ab}=\phi^{-1}G_0^{-1}T_{ab}$. The field equations then take the forms

$$\bar{R}_{ab}-\tfrac{1}{2}\bar{g}_{ab}\bar{R}=8\pi G_0 c^{-4}\bar{T}_{ab}+\frac{(3+2\omega)\,c^4}{16\pi G_0\phi^2}(\phi_{|a}\phi_{|b}-\tfrac{1}{2}\bar{g}_{ab}\bar{g}^{cd}\phi_{|c}\phi_{|d}) \tag{3}$$

$$\bar{g}^{cd}(\log\phi)_{|cd}=8\pi c^{-4}(3+2\omega)^{-1}\bar{T} \tag{4}$$

where stroke denotes differentiation with respect to $\bar{g}_{ab}$.

The Brans-Dicke frame is the most convenient for physical interpretations because in this frame small particles move on geodesics. However, from a mathematical point of view, the Einstein frame is better because in it there are no second derivatives of ϕ on the right hand side of the field equations. In fact one can regard the field equations as being simply the Einstein equations with a scalar field which interacts with all other matter fields through the trace of their energy-momentum tensor. Provided that the matter fields are reasonable, the Ricci tensor $\bar{R}_{ab}$ will satisfy the inequalities used in the previous paper and so the results derived there about the axial symmetry and the spherical topology of stationary black holes will hold in the Brans-Dicke theory also.

3. Stationary Black Holes

As stated in the previous section, a stationary black hole must be axially symmetric or static. In the former case there will be two Killing vector fields K^a, $\hat{K}^a$, the first of which is timelike and the second spacelike at infinity. The bivector $K^{[a}\hat{K}^{b]}$ will be timelike at infinity and its magnitude $h=K^{[a}\hat{K}^{b]}K_{[a}\hat{K}_{b]}$ will be negative. Carter [10] has shown thta the event horizon will occur where $h=0$. Outside the horizon h will be negative and so at each point there will be some linear combination of K^a and $\hat{K}^a$ which is timelike. The scalar field ϕ must be constant along the directions of $\hat{K}^a$, K^a since they are Killing vectors. It follows therefore that the gradient of ϕ must be spacelike or zero everywhere in the exterior region. The same will be true if the solution is static since in this case there will be one Killing vector K^a which is timelike everywhere in the exterior region. Now let $\mathscr{S}$ be a partial Cauchy surface for $\bar{J}^+(\mathscr{I}^-)\cap\bar{J}^-(\mathscr{I}^+)$ and $\mathscr{S}'$ the partial Cauchy surface obtained by moving each point of $\mathscr{S}$ a unit parameter distance along the integral curves of K^a. Let $\mathscr{V}$ be the region bounded by $\mathscr{S}$, $\mathscr{S}'$, a portion of the event horizon and a timelike 3-surface at infinity. If the exterior region is empty apart from an electromagnetic field, the scalar field will obey an equation

$$\bar{g}^{cd}(\log\phi)_{|cd}=0\,.$$

Multiply this equation by $\log(\phi/\phi_0)$ and integrate over $\mathscr{V}$ (ϕ_0 is the constant value which ϕ approaches at infinity; it can be normalized to unity). One can then integrate by parts to obtain a volume integral of minus the square of the gradient of $\log\phi$ and various surface integrals. Because of the isometry group the surface integral over $\mathscr{S}'$ cancels out that over $\mathscr{S}$. The surface integral at infinity is zero because $\log(\phi/\phi_0)$ is zero there. The surface integral over the horizon is zero because the gradient of ϕ is orthogonal to the null vector tangent to the horizon which is a linear combination of K^a and $\hat{K}^a$. This shows the volume integral of the square of the gradient of ϕ must be zero. Since the gradient can only be spacelike or zero, it must be zero everywhere and so ϕ must be constant. In this case the Brans-Dicke equations are the same as the Einstein equations. Thus, stationary black hole solutions in the Brans-Dicke theory are the same as stationary black hole solutions in the Einstein theory. It appears that the latter are completely represented by the Kerr family of solutions.

4. Mass Loss

Let K^a be the timelike Killing vector in a stationary asymptotically flat solution containing an uncollapsed body. The gravitational mass of the body measured from infinity by the orbits of small particles is given by the r^{-1} term in $K^aK^bg_{ab}$. More precisely

$$M=-\tfrac{1}{4}\pi\int x^{-2}x_{;a}g^{ab}K^c d\Sigma_{bc}$$

where $d\Sigma_{bc}$ is the surface element of a spacelike 2-surface near infinity and $x^2=K^aK^bg_{ab}$. One also defines the quantity M_t by a similar expression but with g_{ab} replaced by $\bar{g}_{ab}=\phi g_{ab}$. M_t represents the gravitational mass that one would calculate from the geodesics in the Einstein frame. One can then decompose the total mass M into the sum of a "tensor" component M_t and a scalar component M_s which causes small particles not to move on geodesics in the Einstein frame and which is given by the r^{-1} term in ϕ. M_t and M_s can be expressed in terms of integrals over the matter distribution. If the body is static and does not have high gravitational binding energy, they will be in the ratio of 1 to $(2+3\omega)^{-1}$. If the body now collapses to form a black hole, the field ϕ will become constant and so M_s will become zero. On the other hand, one can generalize the asymptotic conservation law of Penrose [10] and apply it to the Einstein frame to show that M_t will decrease by the amount of tensor gravitational waves and scalar field energy radiated to infinity.

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