Commun. math. Phys. 25, 1—61 (1972)

Representations of the CCR in the $(\phi^4)_3$ Model: Independence of Space Cutoff

JEAN-PIERRE ECKMANN

Département de Physique Théorique
Université de Genève

Received October 21, 1971

Abstract. The algebra of observables for the renormalized ϕ^4 interaction in 3-dimensional space-time is constructed. It is shown that the von Neumann algebras associated with observables in a bounded region B are independent of the space cutoff which is used in the construction of a Hamiltonian for this interaction. This result is shown to be useful in the construction of a translation invariant ϕ^4 theory in three dimensions. It also gives a physical criterion for the equivalence of non-Fock representations of the canonical commutation relations.

I. Introduction

Recently, there has been some interest in a certain class of non-Fock representations of the canonical commutation relations (CCR) which occurs in a natural way in the construction of a dense domain for a Hamiltonian for the $:\phi^4:$ interaction in $2+1$ dimensional space-time.

This construction was initiated by Glimm [10], who considered an interaction

$$H_\sigma(g) = H_0 + \int :\phi_\sigma^4:(x)\, g(x)\, d^2x + M_\sigma + E_\sigma \tag{1.1}$$

with a momentum cutoff σ (the momenta occurring in the interaction are bounded in absolute value by σ) and with a space cutoff g which is a smooth function with compact support. H_0 is the free Hamiltonian and $\phi_\sigma(x)$ is the cutoff free boson field at time zero:

$$\phi_\sigma(x) = \int_{|k| \leqq \sigma} e^{ikx}(k^2 + m_0^2)^{-1/4}(a^*(k) + a(-k))\, d^2k\,, \qquad m_0 > 0\,.$$

M_σ and E_σ are the mass and the additive counterterms respectively whose definitions are suggested by perturbation theory. In order to define a Hamiltonian $H_\infty(g)$ in the limit $\sigma \to \infty$, Glimm used a modified, truncated version of the formal wave operator. This operator $T_\sigma(g)$ is called a dressing transformation. His construction is summarized in the following

Theorem 1.1 (Glimm [10]). *Let* $\Lambda_\sigma(g) = \| H_0^{-1} \int :\phi_\sigma^4:(x) g(x) d^2x\, \Omega \|^2$, *where* Ω *is the Fock vacuum. There exists a family* $T_{\varrho\sigma}(g)$ *of dressing* *transformations satisfying for* $\psi, \psi' \in \mathscr{D}_0$ [1]:

I. $$\lim_{\sigma \to \infty} (T_{\varrho\sigma}(g)\psi, T_{\varrho'\sigma}(g)\psi') \exp - \Lambda_\sigma(g) \equiv (T_{\varrho\infty}(g)\psi, T_{\varrho'\infty}(g)\psi')_g \quad (1.2)$$

exists for all $\varrho, \varrho' \geqq 0$.

II. *The expression* (1.2) *defines a positive definite scalar product* $(.\,,.)_g$ *on the linear hull* $\mathscr{D}(g)$ *of* $\{T_{\varrho\infty}(g)\psi \mid \psi \in \mathscr{D}_0$, *all* $\varrho \geqq 0\}$. $\mathscr{D}(g)$ *together with* $(.\,,.)_g$ *is a prehilbert space whose completion* $\mathscr{F}(g)$ *is a separable Hilbert space.*

III. $\|H_\sigma(g)T_{\varrho\sigma}(g)\psi\|^2 \exp - \Lambda_\sigma(g)$ *is uniformly bounded in* $0 \leqq \sigma \leqq \infty$ *and* $\lim_{\sigma\to\infty}(T_{\varrho\sigma}(g)\psi, H_\sigma(g)T_{\varrho'\sigma}(g)\psi') \exp - \Lambda_\sigma(g)$ *exists and defines a symmetric operator* $H_\infty(g)$ *with domain* $\mathscr{D}(g)$.

It is easy to show that the multiplicative "renormalization" $\exp + \Lambda_\sigma(g)$ is infinite in the limit $\sigma \to \infty$; indeed $\Lambda_\sigma(g) = \mathcal{O}(\ln \sigma)$. This fact plays an important rôle in the following construction.

Consider the algebra $\mathfrak{A}_0(B)$ generated by $\{\exp i\phi(f), \exp i\,\pi(f), f \in \mathscr{D}_B\}$ where $\mathscr{D}_B$ is the set of all smooth functions with support contained in $B \subset \mathbb{R}^2$, and where $\pi(f)$ is the time derivative of the free field, at time zero. There exists a natural representation Π_g of $\mathfrak{A}_0(B)$ on $\mathscr{F}(g)$ obtained by defining $\Pi_g(C)$, $C \in \mathfrak{A}_0(B)$ by $(T_{\varrho\infty}(g)\psi, \Pi_g(C)\,T_{\varrho'\infty}(g)\psi')_g \equiv \lim_{\sigma\to\infty}(T_{\varrho\sigma}(g)\psi, C\,T_{\varrho'\sigma}(g)\psi') \exp - \Lambda_\sigma(g)$. The existence of this limit is proved in Theorem 4.3. Fabrey [8] and Hepp [16] have shown that such representations are inequivalent to the Fock representation. Let $\mathfrak{A}(B, g)$ be the weak closure in $\mathscr{F}(g)$ of $\Pi_g(\mathfrak{A}_0(B))$; $\mathfrak{A}(B, g)$ is a von Neumann algebra. It is natural to ask under which conditions are the two algebras $\mathfrak{A}(B, g)$ and $\mathfrak{A}(B, g')$ unitarily equivalent, which in turn would signify that the representation does not depend on the space cutoff g.

Similar problems have been treated in the literature. Chaiken [3] has developed necessary and sufficient criteria for representations of the CCR (in fact, Weyl systems) to be equivalent to the Fock representation. In addition, necessary and sufficient conditions for the equivalence of quasi-free representations of the CCR have been given by van Daele, Verbeure [20] and Araki [1]. No such general results have been found for our case of representations which appear not to be quasifree (and are thus not Fock representations). Fabrey [8] has shown that two representations are not disjoint if essentially $|\Lambda_\sigma - \Lambda'_\sigma|$ is bounded uniformly in σ. Osterwalder and the author [7] have shown that the representations obtained from different truncations in the definition of $T_{\varrho\sigma}(g)$ are all unitarily equivalent.

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[1] $\mathscr{D}_0$ is the set of all vectors in Fock space whose n-particle component is zero for n large and which have compact support in momentum space.

In this paper, we analyze the effects of a change of g to another function g'. If g is not equal to g' on a set whose volume is not zero, then $|A_\sigma(g) - A_\sigma(g')| = \mathcal{O}(\ln \sigma)$. The representations $\mathfrak{A}(B, g)$ and $\mathfrak{A}(B, g')$ are both non-Fock if $g \neq 0$, $g' \neq 0$ on B and so none of the above criteria will apply. One can, however, make use of the geometrical relations between g, g' and B, to establish unitary equivalence. This is expressed in our main result:

Theorem. *Let B be a bounded, open, convex*[2] *region in $\mathbb{R}^2$. Let $B_{d_0} = \{x \in \mathbb{R}^2, \text{dist}(x, B) \leq d_0\}$. If $g(x) = g'(x)$ on B_{d_0} for some $d_0 > 0$, then $\mathfrak{A}(B, g)$ and $\mathfrak{A}(B, g')$ are unitarily equivalent. The equivalence φ is natural in the sense that $\varphi(\Pi_g(C)) = \Pi_{g'}(C)$ for all $C \in \mathfrak{A}_0(B)$.*

Such a result does not come really unexpectedly. It signifies that the "local observables" derived from $\exp i\phi(f)$ and $\exp i\pi(f)$ can in fact not "see" a change of the space cutoff g if this change takes place outside of the support of f. We shall see below what this implies in terms of the dynamics of the $:\phi^4:$ model in 3 space-time dimensions.

The above theorem allows the construction of the time zero quasi-local (C^*-) algebra $\mathfrak{A}_\lambda$ for the coupling constant λ (see e.g. Haag [15] for a definition of quasi-local algebras). Indeed, let $\mathfrak{A}_\lambda(B)$ be the equivalence class of all algebras $\mathfrak{A}(B, g_{(B,\lambda)})$, $g_{(B,\lambda)}$ being smooth with compact support, $g_{(B,\lambda)} = \lambda$ on B_d for some $d > 0$. $\mathfrak{A}_\lambda(B)$ is an (abstract) C^*-algebra. For $B \subset B'$ there is a natural injection $\mathfrak{A}_\lambda(B) \to \mathfrak{A}_\lambda(B')$ defined on any representative $\mathfrak{A}(B, g_{(B',\lambda)}) \in \mathfrak{A}_\lambda(B)$. Therefore the inductive limit $\cup\{\mathfrak{A}_\lambda(B); B \text{ bounded}\}$ is defined, and is a normed $*$-algebra whose uniform closure we denote by $\mathfrak{A}_\lambda$, the quasi-local algebra for the coupling constant λ.

A more important application of the main theorem is its connection with the program of Glimm and Jaffe to construct a $:\phi^4:$ theory in 2 space-dimensions. The program can be visualized in the following diagram:[3] (see Fig. 1).

We give some explanations:

Point (1) is the construction of a domain for a symmetric operator $H_\infty(g)$, with a fixed space cutoff g. This construction, whose results we have summarized in Theorem 1.1, has been done by Glimm in [10]. In point (2) and point (3) important properties of the Hamiltonian $H_\infty(g)$ are derived. Point (2) is a proof of the semiboundedness of the Hamiltonian $H_\infty(g)$. This problem, and point (3) are under investigation by Glimm and Jaffe. Point (3) is the construction of a unique selfadjoint limit of the

[2] The convexity of B is inessential and we have restricted ourselves to this case for convenience.

[3] The program, conjectures and allusions to techniques being used in proofs which are still under work have been kindly communicated to me by Prof. Glimm in private discussions.

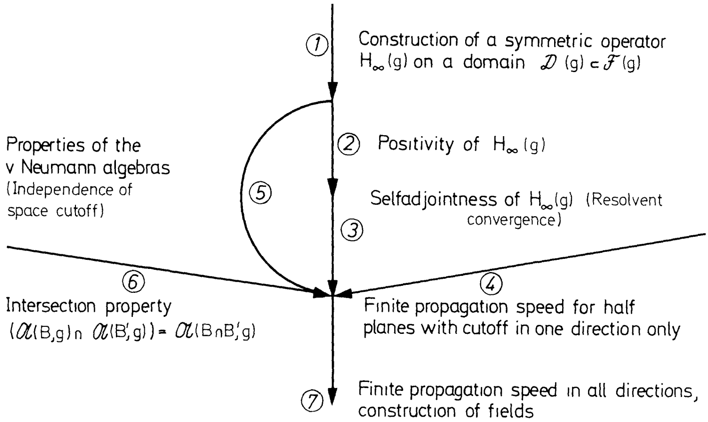


Fig. 1

operators $H_{\theta,\sigma}(g)$. The limit must be independent of the parameter θ of point (4). Point (4) is the following conjecture: Choose a cartesian coordinate system θ in $\mathbb{R}^2$ and let (k_1, k_2) be the two components of k in this frame. By $\mathring{\sigma}$ we denote the momentum cutoff which restricts only the k_1-component to values $|k_1| \leqq \sigma$, the other component being unrestricted. For fixed g and finite σ, one can expect $H_{\mathring{\sigma}}(g)$ to be "local" in the "2" direction, and $H_{\mathring{\sigma}}(g)$ should have "finite propagation speed" in the "2" direction; such results can be proved using ideas which go back to Guenin [14] and Segal [19], and which have been applied by Glimm and Jaffe in [11]. By "finite propagation speed" we understand the following: Let B be a bounded, convex region, and for $\varepsilon > 0$ and a given cartesian coordinate system θ, let $S^{(\theta)}_{|t|+\varepsilon}$ be the narrowest strip with boundaries parallel to the x_1-direction which contains $B_{|t|+\varepsilon}$. For every $\mathring{\sigma}$, one constructs an appropriate truncation $H_{0\mathring{\sigma}}$ of the free Hamiltonian H_0 and we define $H_{\theta.\sigma}(g) = H_{0\mathring{\sigma}} - H_0 + H_\sigma(g)$, where θ is the angle between the x_1-axis and some fixed coordinate system. Then "finite propagation speed" in the "2" direction means: For $A \in \mathfrak{A}(B, g)$, one has $\alpha_{t,\theta,\sigma,g}(A) \equiv \exp i(t H_{\theta,\sigma}(g))\, A \exp i(-t H_{\theta,\sigma}(g)) \in \mathfrak{A}(S^{(\theta)}_{|t|+\varepsilon}, g)$, and $\alpha_{t,\theta,\sigma,g}(A)$ is independent of g in the 2-direction if g is held fixed on $S_{|t|+\varepsilon}$ (cf. Fig. 2).

The "independence" of g is really nothing else than an application of our main theorem (5). Indeed, by the main theorem, $\mathfrak{A}(B, g)$ is unitarily equivalent to the algebra on Fock space, $\mathfrak{A}(B)$, whenever $\operatorname{supp} g \cap B_{d_0} = \emptyset$ for some $d_0 > 0$. Therefore $\mathfrak{A}(S^{\theta}_{|t|+\varepsilon}, g)$ is really defined and is, for example, equal to

$$\mathfrak{A}((\operatorname{supp} g \cap S^{(\theta)}_{|t|+\varepsilon})_\varepsilon, g) \cup \mathfrak{A}_{(0)}(S^{(\theta)}_{|t|+\varepsilon} \setminus (\operatorname{supp} g \cap S^{(\theta)}_{|t|+\varepsilon})_\varepsilon) .$$

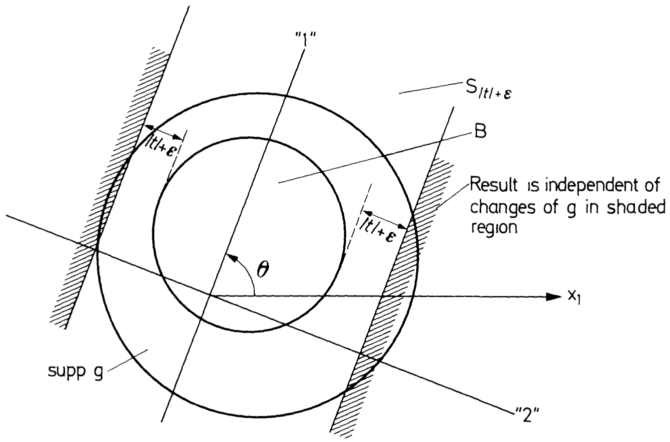


Fig. 2

But (supp $g \cap S^{(\theta)}_{|t|+\varepsilon})_\varepsilon$ is a convex bounded region if supp g is, and for such regions the main theorem gives the required equivalence criteria. With this result, one can easily go to the limit $\mathring{\sigma} \to \infty$. In point (6), one would like to prove that the intersection $\mathfrak{A}\,(B_1, g) \cap (B_2, g)$ is equal to $\mathfrak{A}\,(B_1 \cap B_2, g)$. We sketch the ideas of the proof of the "finite propagation speed" asserted in step (7) as a consequence of steps (3) – (6), which should show their respective rôles in proving this result. The statements should be understood as conjectures.

Let B be a bounded open convex region in $\mathbb{R}^2$. Let $g\,(x) = \lambda$ on $B_{|t|+\varepsilon}$, let g have compact support. For any cartesian coordinate system $\theta = (x_1, x_2)$, let $S^{(\theta)}$ be the narrowest strip with boundaries parallel to x_1 which contains B. It should follow at once from the selfadjointness of $H_\infty(g)$ and from the "finite propagation speed in half-planes" (steps (3) and (4)) that $\exp i\,(t\,H_\infty(g))\,\mathfrak{A}\,(B, g)\exp i\,(-tH_\infty(g)) \subset \mathfrak{A}\,(S^{(\theta)}_{|t|}, g)$. One now rotates the cartesian frame and thus finds

$$\exp i\,(t\,H_\infty(g))\,\mathfrak{A}\,(B, g)\,\exp i\,(-t\,H_\infty(g)) \subset \bigcap_\theta \mathfrak{A}(S^{(\theta)}_{|t|}, g)$$

where the intersection runs over all frames. By (6) one will find $\bigcap_\theta \mathfrak{A}\,(S^{(\theta)}_{|t|}, g) = \mathfrak{A}\left(\bigcap_\theta S^{(\theta)}_{|t|}, g\right) = \mathfrak{A}\,(B_{|t|}, g)$. So finite propagation speed is proved for each fixed g, and by the above remarks, the automorphism

$$\alpha_t : A \to \exp i\,(t\,H_\infty(g))\,A\,\exp i\,(-t\,H_\infty(g))\,, \qquad A \in \mathfrak{A}\,(B, g)$$

is independent of g if $g = \lambda$ on $B_{|t|+\varepsilon}$ for some $\varepsilon > 0$.

This paper is divided into two parts. In Part 1 we set up the definitions and state the results; Part 2 is devoted to the technical aspects of the proofs. Sect. II contains the definitions and Sect. III contains the statement of the main result and its proof as a consequence of the fact that a certain functional is normal on both algebras $\mathfrak{A}(B, g)$ and $\mathfrak{A}(B, g')$. This normality follows from the fact that a sequence of functionals approximating the above functional is norm convergent. These functionals are given in Sect. IV, where we also state the corresponding theorem. The sections of Part 2 deal with an approximation of operators in $\mathfrak{A}_0(B)$ in terms of Wick monomials with creation-annihilation operators in the complement of a region containing B, (Sects. V and VI). It is one of the main problems to show that this approximation converges weakly, whereas such a result cannot be expected to hold for the usual Wick expansion due to the non-Fock character of the representation of the canonical commutation relations induced on $\mathscr{F}(g)$. In Sect. VII we prove the norm convergence of the sequence of functionals defined in Sect. IV. Sect. VIII contains some purely technical estimates.

Part 1. Definitions and Results

II. Notations and Definitions

In this section, we introduce some notation. For the definition of Fock space $\mathscr{F}$, the reader is referred to the literature (e.g. in [16]).

The expression $\int :\phi_\sigma^4:(x)h(x)d^2x$ has an expansion

$$\sum_{i=0}^{4} W_{i\sigma} = \sum_{i=0}^{4} \int a^*(k_1)\dots a^*(k_i)\, a(k_{i+1})\dots a(k_4)\, w_{i\sigma h}(k_1, \dots, k_4)\, dk_1 \dots dk_4 . \tag{2.1}$$

Here,

$$w_{i\sigma h}(k_1, \dots, k_4) = \begin{cases} \binom{4}{i} \prod_{j=1}^{4} \mu(k_j)^{-1/2} \tilde{h}(k_1 + \dots + k_i - k_{i+1} - \dots - k_4) \\ \qquad\qquad \text{if } |k_j| \leqq \sigma, j = 1, \dots, 4, \\ 0 \text{ , otherwise ;} \end{cases} \tag{2.2}$$

$\sim$ denotes Fourier transform, $\mu(k) = (k^2 + m_0^2)^{1/2}$, $m_0 > 0$. In modifying Friedrich's perturbation theory, Glimm has defined "dressing transformations" [10, 16] which are defined on $\mathscr{D}_0 \subset \mathscr{F}$ (the set of all vectors whose n-particle component equals 0 if $n > N$, some N, and which have compact support in momentum space). These dressing transformations map into the domain of $H_0 + \int :\phi_\sigma^4:(x)h(x)dx +$ counter terms, and their limits $\sigma \to \infty$ define a domain for the renormalized $:\phi^4:$ interaction with space cutoff in 3 dimensional space-time. We define a simplified version of this dressing transformation, which gives rise to the same representation of the CCR as the transformation given by Glimm (see [7]). It is defined by

$$T_{r\sigma}(h) = \prod_{j=r}^{\infty} \exp_j V_{j\sigma}\,, \tag{2.3}$$

where

$$V_{j\sigma} = V_{j\sigma}(h) = -\int \chi_j(k_1, \ldots, k_4) \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-1} w_{4\sigma h}(k_1, \ldots, k_4) \cdot \prod_{i=1}^{4} a^*(k_i)\, dk_i\,; \qquad V_j \equiv V_{j\infty}\,. \tag{2.4}$$

Furthermore $\chi_j(k_1, \ldots, k_4)$ is the characteristic function of

$$I_j \equiv \left\{ (k_1, \ldots, k_4) \,|\, \max_i |k_i| \in [2^j, 2^{j+1}) \right\}, \quad \text{if} \quad j = 1, 2, \ldots, \tag{2.5}$$

$$I_0 \equiv \left\{ (k_1, \ldots, k_4) \,|\, \max_i |k_i| \leqq 2 \right\}, \tag{2.6}$$

and $\exp_j A \equiv \sum_{n=0}^{j} A^n/n!$.

Whenever the truncation is not specified, we shall write V for an operator with four creators.

We finally define

$$\Lambda_{j\sigma}(h) \equiv \| V_{j\sigma}(h)\, \Omega \|^2 = 4!\, \| v_{j\sigma}(h) \|_2^2\,, \quad \text{and} \quad \Lambda_\sigma(h) = \sum_{j=0}^{\infty} \Lambda_{j\sigma}(h) \tag{2.7}$$

where Ω is the Fock vacuum. We also set, for $\sigma \geqq 2^{j+1}$,

$$v_{j\sigma} = v_{j\sigma}(h) = -\chi_j \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-1} w_{4\sigma h} = v_j(h)$$

and

$$v(h) = -\left(\sum_{i=1}^{4} \mu(k_i) \right)^{-1} w_{4\infty h}\,. \tag{2.8}$$

Let $\mathscr{D}_B$ be the set of all functions in $\mathscr{S} = \mathscr{S}(\mathbb{R}^2)$ whose support is contained in B. Then $\mathfrak{A}_0(B)$ is defined as the *-algebra generated by

$$\{\exp i\, \phi(f)\,, \quad \exp i\, \pi(f)\,, \quad f \in \mathscr{D}_B\}\,. \tag{2.9}$$

Here ϕ and π are the time-zero free field and its time derivative respectively:

$$\phi(x) = 2^{-1/2} \int e^{ikx} \mu(k)^{-1/2} (a^*(k) + a(-k))\, d^2k\,, \tag{2.10}$$

$$\pi(x) = 2^{-1/2}\, i \int e^{ikx} \mu(k)^{+1/2} (a^*(k) - a(-k))\, d^2k\,. \tag{2.11}$$

The following result is standard [7]:

Lemma 2.1. *Let* $\psi_1, \psi_2 \in \mathcal{D}_0, r_1, r_2 \in \mathbb{N}, h \in \mathcal{S}, C \in \mathfrak{A}_0(B)$. *Then*

$$\begin{aligned}\omega(C) &= \omega(C|\psi_1, \psi_2, r_1, r_2, h) \\ &\equiv \lim_{\sigma\to\infty} (\psi_1, T^*_{r_1\sigma}(h)\; C\; T_{r_2\sigma}(h)\,\psi_2) \exp - \Lambda_\sigma(h)\end{aligned} \tag{2.12}$$

exists and defines a linear functional on $\mathfrak{A}_0(B)$. *The expression* $\omega(\mathbf{1}|\psi_1, \psi_2, r_1, r_2, h)$ *defines a positive definite scalar product* $(.,.)_h$ *on*

$$< T_{r\infty}(h)\,\psi|\psi \in \mathcal{D}_0, r \in \mathbb{N} > = \mathcal{D}(h), \tag{2.13}$$

where $< \;>$ *denotes the linear hull.*

The scalar product is given by

$$(T_{r_1\infty}(h)\,\psi_1, T_{r_2\infty}(h)\,\psi_2)_h \equiv \omega(\mathbf{1}|\psi_1, \psi_2, r_1, r_2, h). \tag{2.14}$$

$\mathcal{D}(h)$, together with $(.\,,\,.)_h$ is a pre-Hilbert space whose completion, a separable Hilbert space, will be called $\mathcal{F}(h)$.

We defer the proof of Lemma 2.1 to the end of this section. The remainder of this section is not needed in order to follow the statements which shall be made in Sects. III and IV. An expression of the form

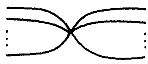
$$W_{mn} = \int a^*(k_1) \ldots a^*(k_m)\, a(k_{m+1}) \ldots a(k_{m+n})\, w_{mn}(k_1 \ldots k_{m+n})\, dk_1 \ldots dk_{m+n}$$

is called Wick monomial, w_{mn} is called its (numerical) kernel. We define $|W_{mn}|$ as the Wick monomial with kernel $|w_{mn}|$. We shall frequently use creation and annihilation operators in position space and we shall denote them by $A^*(x)$ and $A(x)$. We define the Wick expansions. Let W_{mn} and $W'_{m'n'}$ be two Wick monomials as above (with kernels w_{mn} and $w'_{m'n'}$). The product $W_{mn} W'_{m'n'}$ can be expanded as follows:

$$\begin{aligned} W_{mn} W'_{m'n'} = &\sum_{r=0}^{\min(n,m')} \sum_{P_r,\pi} \int \prod_{i=1}^{m} a^*(k_i)\, dk_i \prod_{i=1}^{m'-r} a^*(k'_{I'_i})\, dk'_{I'_i} \\ &\cdot \prod_{i=1}^{n-r} a(l_{I_i})\, dl_{I_i} \prod_{i=1}^{n'} a(l'_i)\, dl'_i \\ &\cdot \Big\{\int w_{mn}(k_1 \ldots k_m, l_1 \ldots l_n)\, w'_{m'n'}(k'_1 \ldots k'_{m'}, l'_1 \ldots l'_{n'}) \\ &\cdot \prod_{q=1}^{r} (\delta(l_{\pi_{i_q}} - k'_{i'_q})\, dk'_{i'_q})\Big\}. \end{aligned} \tag{2.15}$$

Here, $\sum_{P_r,\pi}$ extends over all partitions of $\{1, \ldots, n\}$ into two ordered sets $\{i_1, \ldots, i_r\}$, $\{I_1, \ldots, I_{n-r}\}$, and of $\{1 \ldots m'\}$ into two ordered sets $\{i'_1, \ldots, i'_r\}$, $\{I'_1, \ldots, I'_{m'-r}\}$, and over all permutations π of $\{i_1, \ldots, i_r\}$. This expansion is the Wick expansion and each term in the sum $\sum_r \sum_{P_r,\pi}$ is called a *Wick term* with numerical kernel { }. We extend this definition in the natural fashion to Wick polynomials.

It is customary [16] to represent Wick monomials and Wick terms as *graphs*. Each Wick monomial W_{mn} is drawn as

m lines  n lines

and each Wick term is represented by connecting the lines whose variables have been identified by $\prod_{q=1}^{r} \delta(\ldots)$ in (2.15). These lines are called internal lines, the others external lines. A *skeleton graph* is a graph which does not contain the graph

 , corresponding to Λ_σ, as a subgraph.

We introduce the notation $W_{mn} \underset{r}{\smile} W'_{m'n'}$ to denote the r-th term in the sum (2.15), i.e. the sum of all Wick terms of $W_{mn} W'_{m'n'}$ with r δ-functions. We also define $W_{mn} \smile W'_{m'n'} = \sum_{r>0} W_{mn} \underset{r}{\smile} W'_{m'n'}$.

A factor in a Wick term is sometimes called a *vertex*.

This ends our definitions.

Proof of Lemma 2.1. (We only sketch the proof for the case $C = \mathbf{1}$. The general case can be found in [8].) This proof is identical to proofs in [7], and [8], except for a minor modification we wish to include in order to make the proof more flexible with regard to changes of the space cutoff h or of the definition of T. Such changes will be needed in later sections. Glimm's analysis shows that, since

$$\exp \Lambda_{j\sigma} \exp(-\Lambda_{j\sigma}) \leqq 1 \quad \text{and} \quad \Lambda_{j\sigma} \leqq \lambda_0 , \qquad \text{for all } j, \tag{2.16}$$

$$j$$

$$|(T_{r_1\sigma}\psi_1, T_{r_2\sigma}\psi_2)\, e^{-\Lambda_\sigma}|$$

$$\leqq \sum_{p,q=0}^{\infty} \sum_{S_{pq}} (|\psi_1|, |S_{pq}|\, |\psi_2|) . \tag{2.17}$$

Here $\sum_{S_{pq}}$ runs over all Wick terms S_{pq} in the expansion of $(V^*)^p V^q$ whose graph is a skeleton graph. Let $s_{pq}(p_{\text{int}}, p_{\text{ext}})$ be the numerical kernel of S_{pq}; p_{ext} (p_{int}) stands for all the variables belonging to the external (internal) lines of the graph of S_{pq}. The following lemma is the basic estimate towards the proof of Lemma 2.1.

Lemma 2.2 (Glimm [10, Theorem 2.2.1]; [5]; [7]; [8]). *Let* $x_j^{(i)}$, $i = 1, \ldots, n$, $j \in \mathbb{N}$ *be a family of symmetric functions* $(\mathbb{R}^2)^4 \to \mathbb{C}$ *and let*

$X_j^{(i)} = \int dk_1 \dots dk_4 \; a^*(k_1) \dots a^*(k_4)\, x_j^{(i)}(k_1, \dots k_4)$. Suppose that there exist constants a, $\lambda_0 < \infty$ and $\gamma > 1$ such that the following inequalities hold for all $i, i' \in \{1, \dots, n\}$; $j, j' \in \mathbb{N}$:

$$\|\mu^{-a}(k_1)\, x_j^{(i)}\|_2 < \lambda_0 \gamma^{-j} \tag{2.18}$$

and

$$\begin{aligned} &\|(x_{j_1}^{(i_1)} \dots x_{j_\alpha}^{(i_\alpha)})\, x_j^{(i)} \underset{r}{\smile} x_{j'}^{(i')}\, (x_{j'_1}^{(i'_1)} \dots x_{j'_{\alpha'}}^{(i'_{\alpha'})})\|_2 \\ &\leqq \lambda_0 \gamma^{-j-j'-j_1-\dots-j_\alpha-j'_1-\dots-j'_{\alpha'}} \end{aligned} \tag{2.19}$$

where $r = 1, 2, 3$; $\alpha, \alpha' \leqq 4 - r$ and every vertex $x_{j_k^{(\prime)}}^{(i_k^{(\prime)})}$ is contracted to x_j or $x_{j'}$. Then there exists a constant $K = K(\lambda_0)$ such that for all Wick terms X_{pq} of

$$(X_{j_1}^{(i_1)} \dots X_{j_p}^{(i_p)})^* \, (X_{j'_1}^{(i'_1)} \dots X_{j'_q}^{(i'_q)}) \tag{2.20}$$

whose graph is a skeleton graph one has

$$\begin{aligned} &\left\| \prod_{p_i \in p_{\text{ext}}} \mu(p_i)^{-a} \int dp_{\text{int}} |x_{pq}(p_{\text{int}}, p_{\text{ext}})| \right\|_{2,\text{ext}} \\ &\leqq K^{p+q} \gamma^{-\sum_{n=1}^{p} j_n} \gamma^{-\sum_{n=1}^{q} j'_n}. \end{aligned} \tag{2.21}$$

Proof. [8]. We may suppose that the graph G of the Wick term is connected, since both sides of (2.21) are products of similar expressions involving connected components only. We proceed by induction on $p+q$. For $p+q=1$ the assertion follows from (2.18), the case $p+q=2$ follows from (2.19). Let $p+q \geqq 3$. The graph may be written as a disjoint union of subgraphs of one of four types: A central vertex connected to $1 \leqq r \leqq 4$ other vertices. The decomposition of G into these subgraphs is defined recursively as follows. For $p+q=3$, there will be one subgraph. Suppose the decomposition for $3 \leqq p+q \leqq N$ is given. Let $p+q = N+1$. Since G is connected, we may choose a subgraph $H \subset G$ of type $r=1$. Then $G-H$ is a disjoint union of connected components H'_j. Let H' be the union of H with all those H'_j which consist of a single vertex. If $H' \subsetneqq G$ we apply the decomposition prescription to the components of $G-H'$. We now apply (2.19) to each of the components to get (2.21).

We now return to the proof of Lemma 2.1, and we apply Lemma 2.2 with

$$n = 1, x_j^{(1)} = v_{j\infty} \equiv \chi_j(k_1, \dots, k_4) \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-1} w_{4\infty h}(k_1, \dots, k_4).$$

Inequalities (2.18) and (2.19) are known in this case, they follow from Weinberg's theorem [5, 22].

By the Schwarz inequality, we get as a bound on (2.17)

$$C_{\psi_1\psi_2} \sum_{p,q=0}^{\infty} 2^{4(p+q)}((4p)!\,(4q)!)^{1/2} \sum_{j_1\dots j_p, j'_1\dots j'_q \in J} \max_{x_{pq}} \left\| \prod_{p_i \in p_{\text{ext}}} \mu^{-a}(p_i) \int |x_{pq}(p_{\text{int}}, p_{\text{ext}})| dp_{\text{int}} \right\|_{2,\text{ext}} \tag{2.22}$$

where J is the set of allowed sequences $j_1 \dots j_p, j'_1 \dots j'_q$ if one takes into account the truncation $\prod_j \exp V_j$ (2.3)[4]. The maximum is over all Wick terms X_{pq} with kernel x_{pq} of (2.20) whose graph is a skeleton graph. It is known [7] that $j_1, \dots, j_p, j'_1, \dots, j'_q$ in J satisfies $j_i > i^{1/2}, j'_i > i^{1/2}$ and therefore we get as bound on (2.17)

$$C_{\psi_1\psi_2} \sum_{p,q=0}^{\infty} 2^{4(p+q)}((4p)!\,(4q)!)^{1/2} K^{p+q}(\gamma-1)^{-(p+q)} \cdot \gamma^{-\left(\sum_{n=1}^{p} n^{1/2}\right)-\left(\sum_{n=1}^{q} n^{1/2}\right)} < \infty \,. \tag{2.23}$$

This proves a uniform bound in σ for (2.12).

The existence of the limit follows then by the fact that the kernels converge pointwise and by the bounded convergence theorem. The proof of the positive definiteness can be found in [10], [7]. The essential ingredient is the fact that

$$\lim_{r\to\infty} \lim_{\sigma\to\infty} \prod_{j \geq r} \exp A_{j\sigma}(h) \exp - A_{j\sigma}(h) = 1 \,. \tag{2.24}$$

III. Main Theorem

In this section, we formulate in a precise manner the main result which we mentioned already in the introduction.

Let B be a bounded open region in $\mathbb{R}^2$. Let $\mathscr{D}_B$ be the space of all functions in $\mathscr{S}(\mathbb{R}^2)$ whose support is contained in B and define $\mathfrak{A}_0(B)$ to be the $*$-algebra which is (algebraically) generated by

$$\{\exp i\,\phi(f), \quad \exp i\,\pi(f); \quad f \in \mathscr{D}_B\}\,.$$

Let $h \in \mathscr{D}$ be a (space cutoff) function. The construction of Lemma 2.1 defines a representation Π_h of $\mathfrak{A}_0(B)$ on the space $\mathscr{F}(h)$ in the natural way:

Let $C \in \mathfrak{A}_0(B)$, then $\Pi_h(C)$ is defined by

$$(T_{r_1\infty}(h)\,\psi_1, \Pi_h(C)\,T_{r_2\infty}(h)\,\psi_2)_h = \lim_{\sigma\to\infty} (\psi_1, T^*_{r_1\sigma}(h)\;C\;T_{r_2\sigma}(h)\,\psi_2)\,e^{-A_\sigma(h)}\,, \tag{3.1}$$

with $\psi_1, \psi_2 \in \mathscr{D}_0$, $r_1, r_2 \in \mathbb{N}$. We define $\mathfrak{A}(B, h)$ as the weak closure in $\mathscr{F}(h)$ of $\Pi_h(\mathfrak{A}_0(B))$; the algebra $\mathfrak{A}(B, h)$ is a v. Neumann algebra.

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[4] Explicitly, in $\pi_j \exp V_j$, the term V_j appears at most j times.

We note the following result which was proved in [7] and which justifies the particular choice of $T_{r\infty}(h)$ in (3.1).

Theorem [7, Theorem 7.1, Theorem 4.1]. *The space $\mathscr{F}(h)$ contains a dense domain for the renormalized Hamiltonian $H_\infty(h)$ (with space cutoff) of the $:\phi^4:$ boson interaction in 3 dimensional space-time.*

This means that although $T_{r_1\infty}(h)$ is not known to map into the domain of $H_\infty(h)$, it defines the correct Hilbert space and [7, Theorem 6.1] a representation of the commutation relations (in the Weyl form) which is the same as the one defined by a limit as in (1.2), where T was some dressing transformation which maps into the domain of $H_\infty(h)$. Since $T_{r\infty}(h)$ is a much simpler expression than T, these facts will make the ensuing definitions (in Sect. IV) simpler.

We shall distinguish between the following notations. If

$$\psi'_\alpha = T_{r_\alpha\infty}(h)\,\psi_\alpha,\ \psi_\alpha \in \mathscr{D}_0,\ r_\alpha \in \mathbb{N},\ \alpha = 1, 2,$$

we write

$$(\psi'_1, \Pi_h(C)\,\psi'_2)_h = (T_{r_1\infty}(h)\,\psi_1, \Pi_h(C)\,T_{r_2\infty}(h)\,\psi_2)_h, \tag{3.2}$$

which we consider in general as a functional over $\mathfrak{A}(B, h)$. But we can view it also as a functional over $\mathfrak{A}_0(B)$ only, in which case we write

$$\omega_h(C) = \omega_h(C|\psi_1, \psi_2, r_1, r_2) \equiv (\psi'_1, \Pi_h(C)\,\psi'_2)_h. \tag{3.3}$$

$\omega_h(C)$ in turn, may be viewed as a functional over any other representation of $\mathfrak{A}_0(B)$, and if continuity allows, this functional will extend to closures of such representations. For any set $S \subset \mathbb{R}^2$ we define S_d to be the set of points in $\mathbb{R}^2$ within distance d of S, $d \geqq 0$, and we define $\sim S$ to be the complement of S in $\mathbb{R}^2$.

We now formulate our main result.

Theorem 3.1. *Let $g, g' \in \mathscr{S}(\mathbb{R}^2)$ and let $B \subset \mathbb{R}^2$ be a bounded, open convex region. Suppose that for some $d_0 > 0$, $g(x) = g'(x)$ for all $x \in B_{d_0}$. Then the von Neumann algebras $\mathfrak{A}(B, g)$ and $\mathfrak{A}(B, g')$ are unitarily equivalent.*

The major step towards the proof of Theorem 3.1 is the following continuity statement about functionals. Let Π_g and $\Pi_{g'}$ denote the representations on $\mathscr{F}(g)$ and $\mathscr{F}(g')$, respectively.

Theorem 3.2. *Let g, g' and B satisfy the assumptions of Theorem 3.1. For any $\psi_1, \psi_2 \in \mathscr{D}_0$, $r_1, r_2 \in \mathbb{N}$, the functional $\omega_{g'}(\cdot|\psi_1, \psi_2, r_1, r_2)$ is continuous on $\mathfrak{A}_0(B)$ in the ultraweak topology from the representation Π_g.*

We postpone the proof of Theorem 3.2. and prove now Theorem 3.1. This proof is purely algebraical. We suppose a certain familiarity with von Neumann algebra terminology. Since $\Pi_g(\mathbf{1}) = \mathbf{1}_{\mathscr{F}(g)}$, $\Pi_{g'}(\mathbf{1}) = \mathbf{1}_{\mathscr{F}(g')}$, by construction of the representations, it follows that the representations are not identically zero.

We next show that the kernel of the two representations Π_g and $\Pi_{g'}$ is zero. Let $\mathfrak{M}$ be any finite dimensional subspace of $\mathscr{D}_B$, the space of smooth functions with support contained in B. We define $\mathfrak{A}(\mathfrak{M})$ to be the von Neumann algebra of fields over $\mathfrak{M}$, acting in Fock space $\mathscr{F}$:

$$\mathfrak{A}(\mathfrak{M}) = \{\exp i\,\phi(f), \exp i\,\pi(f), f \in \mathfrak{M}\}''$$

where $\{\ \}''$ denotes the bicommutant.

By v. Neumann's theorem [21], since $\mathfrak{M}$ is finite dimensional and since $\mathscr{F}$ is separable, we know that $\Pi_g(\mathfrak{A}(\mathfrak{M}))$ is unitarily equivalent to a direct sum of copies of $\mathfrak{A}(\mathfrak{M})$. It follows from this and from the proof of v. Neumann's theorem that $\mathfrak{A}(\mathfrak{M})$ is a factor of type I_∞. One can extend the representation Π_g to a C^*-algebra $\mathfrak{A}_1(B)$ defined by

$$\mathfrak{A}_1(B) = \Big(\bigcup_{\mathfrak{M} \subset \mathscr{D}_B, \dim(\mathfrak{M}) < \infty} \mathfrak{A}(\mathfrak{M})\Big)^-,$$

where $(\ )^-$ indicates uniform closure.

This algebra has been discussed by Segal [18], it is sometimes called the Weyl algebra (over $\mathscr{D}_B$) [3]. Evidently, one has $\mathfrak{A}_1(B) \supset \mathfrak{A}_0(B)$. In [9], Glimm has shown that $\mathfrak{A}_1(B)$ is simple; i.e. that every nontrivial representation of $\mathfrak{A}_1(B)$ is faithful. Therefore $\Pi_g\colon \mathfrak{A}_0(B) \to \Pi_g(\mathfrak{A}_0(B))$ is a $*$-isomorphism.

In order to show the unitary equivalence of the weak closures $\mathfrak{A}(B, g)$ (of $\Pi_g(\mathfrak{A}_0(B))$ or, what is the same, of $\Pi_g(\mathfrak{A}_1(B))$) and $\mathfrak{A}(B, g')$, we need the continuity properties established in Theorem 3.2 and general properties of $\mathfrak{A}_1(B)$. We first prove that the natural $*$-homomorphism

$$U\colon \Pi_g(\mathfrak{A}_1(B)) \to \Pi_{g'}(\mathfrak{A}_1(B)), \text{ defined by } U(\Pi_g(A)) = \Pi_{g'}(A)$$

for all $A \in \mathfrak{A}_1(B)$, extends to a normal homomorphism $\bar{U}$ from $\mathfrak{A}(B, g)$ onto $\mathfrak{A}(B, g')$.

The assertion of Theorem 3.2. holds for every vector of the form $T_{r\infty}(g')\,\psi$, $r \in \mathbb{N}$, $\psi \in \mathscr{D}_0$. Note that every element in the linear hull of the $T_{r\infty}(g')\,\psi$ is again of the form $T_{r'\infty}(g')\,\psi'$ for some $r' \in \mathbb{N}$, $\psi' \in \mathscr{D}_0$, and hence Theorem 3.2. holds for a dense set of vectors in $\mathscr{F}(g')$. Since the norm limit of normal functionals is normal, Theorem 3.2. holds on all of $\mathscr{F}(g')$.

The normality of U follows: Let $\{A_n\}$ be a sequence of operators in $\mathfrak{A}_0(B)$. If $\Pi_g(A_n) \to 0$, ultraweakly on $\mathscr{F}(g)$, then $(\psi, \Pi_{g'}(A_n)\,\psi')_{g'} \to 0$ since $(\psi, \cdot\,\psi')_{g'}$ is ultraweakly continuous on $\mathfrak{A}_0(B)$ with the topology of Π_g by Theorem 3.2. Since ψ, ψ' run over all of $\mathscr{F}(g')$ by the above remarks, it follows that $\Pi_{g'}(A_n) \to 0$ weakly on $\mathscr{F}(g')$. By going to infinite linear combinations of such functionals, it follows that $U\colon \Pi_g(\mathfrak{A}_0(B)) \to \Pi_{g'}(\mathfrak{A}_0(B))$ is ultraweakly-ultraweakly continuous, in the topologies defined by Π_g and $\Pi_{g'}$ respectively. Define now $\bar{U}(A)$ for $A \in \mathfrak{A}(B,g)$ by continuous extension of U: $\bar{U}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} U(A_n)$, where $A_n \in \Pi_g(\mathfrak{A}_0(B))$,

$A_n \to A$ weakly in the topology of Π_g. $\bar{U}$ is onto: If $A' \in \mathfrak{A}(B, g')$ then let $A'_n \in \Pi_{g'}(\mathfrak{A}_0(B))$; $A'_n \to A'$. Since U^{-1} is ultraweakly continuous, $A_n = U^{-1} A'_n$ is weakly convergent, and has a limit A. By definition $\bar{U}(A) = \lim_{n\to\infty} U(A_n) = \lim_{n\to\infty} A'_n = A'$, so $\bar{U}$ is onto. Thus $\bar{U}$ is a normal homomorphism $\bar{U}: \mathfrak{A}(B, g) \to \mathfrak{A}(B, g')$.

We now show that $\bar{U}$ is an isomorphism. Since $\bar{U}$ and $\overline{U^{-1}}$ are normal $\bar{U}\,\overline{U^{-1}}$ and $\overline{U^{-1}}\,\bar{U}$ are normal homomorphisms which equal identity on the dense subalgebras $\Pi_{g'}(\mathfrak{A}_0(B))$ and $\Pi_g(\mathfrak{A}_0(B))$; hence $\bar{U}\,\overline{U^{-1}} = \mathbf{1}$ everywhere and $\overline{U^{-1}} = (\bar{U})^{-1}$: $\bar{U}$ is an isomorphism.

We show that $\mathfrak{A}(B, g)$ has a cyclic vector and a separating vector. We note that by construction, $\mathscr{F}(g)$ is a separable Hilbert space. By construction $\mathfrak{A}(B, g) \supset \Pi_g(\mathfrak{A}_1(B))$ and since the $\mathfrak{A}(\mathfrak{M})$ are of infinite type, so is $\Pi_g(\mathfrak{A}_1(B))$ and hence $\mathfrak{A}(B, g)$ is of infinite type. It is known [4, III.8, Corollaire 11], that the (separable) commutant $\mathfrak{A}(B, g)'$ of the algebra $\mathfrak{A}(B, g)$ which is of infinite type has a separating vector, and this implies that $\mathfrak{A}(B, g)$ has a cyclic vector. The argument which proves the existence of a separating vector for $\mathfrak{A}(B, g)$ is similar: Let B' be a bounded open region which is contained in the complement of B. Now

$$\mathfrak{A}(B, g)' \supset \Pi_g(\mathfrak{A}_1(B')),$$

by the locality of the free field and by weak limits. So by the argument above $\mathfrak{A}(B, g)'$ is of infinite type and so $\mathfrak{A}(B, g) = (\mathfrak{A}(B, g)')'$ has a separating vector. By [4, III.1, Théorème 3] it is known that every normal isomorphism between two von Neumann algebras with cyclic and separating vector is unitarily implemented. Hence Theorem 3.1 is proved.

We now return to Theorem 3.2, and reduce it to the following technical theorem, whose proof will take up the remainder of this paper.

Theorem 3.3. *Let g, g', and B satisfy the assumption of Theorem 3.1. To every $\psi_1, \psi_2 \in \mathscr{D}_0$; $r_1, r_2 \in \mathbb{N}$, there exists a sequence $\omega^{(n)}$ of ultraweakly continuous functionals on $\mathfrak{A}(B, g)$ such that for all $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ one has*

$$|\omega^{(n)}(\Pi_g(C)) - \omega_{g'}(C|\psi_1, \psi_2, r_1, r_2)| < \varepsilon \|C\|, \tag{3.4}$$

for all $C \in \mathfrak{A}_0(B)$.

One says that $\omega_{g'}(\cdot|\psi_1, \psi_2, r_1, r_2)$ is the norm limit of the $\omega^{(n)}$. We shall choose functionals of the form

$$\omega^{(n)} = \omega_g(\cdot|\theta(\psi_1, r_1, n, g'), \theta(\psi_2, r_2, n, g'), r_1(n), r_2(n))$$

in the proof of (3.4).

Proof of Theorem 3.2 as a consequence of Theorem 3.3: By Theorem 3.3, $\|\omega^{(n)} - \omega^{(n')}|_{\Pi_g(\mathfrak{A}_0(B))}\| < \varepsilon$ for n, n' large. By Kaplansky's density Theorem, since the $\omega^{(n)}$ are ultraweakly continuous on $\mathfrak{A}(B, g)$

and since $\Pi_g(\mathfrak{A}_0(B))$ is ultraweakly dense in $\mathfrak{A}(B,g)$, we have that $\|\omega^{(n)}-\omega^{(n')}\|<\varepsilon$. The norm limit of normal states is normal, and therefore the limit ω of the $\omega^{(n)}$ is ultraweakly continuous. Since, by Theorem 3.3, $\omega|_{\Pi_g(\mathfrak{A}_0(B))}=\omega_{g'}$, we find that $\omega_{g'}$ is ultraweakly continuous on $\mathfrak{A}_0(B)$ in the topology of the representation Π_g; Theorem 3.2 is proved.

IV. The Approximating Sequence of Functionals

We motivate first our choice of the functionals $\omega^{(n)}$ which approximate $\omega_{g'}(\,.\,|\,\psi_1,\psi_2,r_1,r_2)$ of Theorem 3.3. We shall choose a first sequence $\omega^{(n)}$ of the form

$$\omega^{(n)}(C)=(\theta_{1n},\Pi_g(C)\theta_{2n})_g\,, \tag{4.1}$$

$\theta_{1n},\theta_{2n}\in\mathscr{D}(g)$. $(\,,\,)_g$ is the scalar product on $\mathscr{F}(g)$ as defined by Lemma 2.1. As a second approximation, we shall choose functionals $\omega_{mn}(C)=(\theta'_{1mn},\Pi_g(C)\theta'_{2mn})_g$, $\theta'_{1mn},\theta'_{2mn}\in\mathscr{D}(g)$. Both families of functionals are ultraweakly continuous on $\mathfrak{A}(B,g)$, by construction. Note that, although the functionals $\omega^{(n)}$ and ω_{mn} will be seen to converge in norm as $n\to\infty$, we cannot expect convergence of $\theta_{\alpha n}$ or $\theta'_{\alpha mn}$, as $n\to\infty$, $\alpha=1,2$.

We recall that

$$\omega_{g'}(C\,|\,\psi_1,\psi_2,r_1,r_2)=(T_{r_1\infty}(g')\psi_1,\Pi_{g'}(C)T_{r_2\infty}(g')\psi_2)_{g'}\,.$$

Our first approximation of $\omega_{g'}$ is by construction of vectors whose low-momentum part coincides with the low-momentum part of $T_{r\infty}(g')\psi_1$. More explicitly, we define for $n>r_\alpha$

$$\begin{aligned}\theta_{\alpha n\sigma}&=\prod_{j=n}^{\infty}\exp_j V_{j\sigma}(g)\prod_{j=r_\alpha}^{n-1}\exp_j V_{j\sigma}(g')\psi_\alpha\\ &\equiv T_{r_\alpha n\sigma}\psi_\alpha,\qquad \alpha=1,2\,.\end{aligned} \tag{4.2}$$

We let
$$\omega_\sigma^{(n)}(C)=(\theta_{1n\sigma},C\theta_{2n\sigma})\exp-\Lambda_{n\sigma} \tag{4.3}$$

where
$$\Lambda_{n\sigma}=4!\left\{\sum_{j=0}^{n-1}\|v_{j\sigma}(g')\|_2^2+\sum_{j=n}^{\infty}\|v_{j\sigma}(g)\|_2^2\right\}.$$

Finally we denote by $\omega^{(n)}(C)$ the limit $\lim_{\sigma\to\infty}\omega_\sigma^{(n)}(C)$, if it exists.

For technical reasons, we are forced to define below a more sophisticated approximation. Namely this approximation must be made in a way to allow the application of two major facts. The first fact is that any $C\in\mathfrak{A}_0(B)$ creates or annihilates smooth, exponentially decaying functions in the region localized outside of B_d, $d>0$. This follows essentially from the support properties of the test functions in $\mathscr{D}_B$ and from the fact that the $\mu^{\pm 1/2}$-factors which occur in $\phi(x)$ or $\pi(x)$ destroy localization only by exponential tails. The second fact is a materialization of the hypothesis that $g = g'$ on B_{d_0}. This implies that

$$\left\| P(B_{d_0/2}) \sum_{j=0}^{\infty} (V_{j\sigma}(g) - V_{j\sigma}(g'))\Omega \right\| < \infty ,$$

uniformly in σ, where $P(B_{d_0/2})$ is the projection of at least one of the four created variables onto $B_{d_0/2}$. It has been shown by Fabrey [8] that for such finite changes in norm between $v_{j\sigma}(g)$ and $v_{j\sigma}(g')$, say,

$$\left\| (v_{j\sigma}(g) - v_{j\sigma}(g')) \prod_{i=1}^{4} \mu_i^{t_i} \right\|_2 < \infty ,$$

there is a natural unitary map between $\mathscr{F}(g)$ and $\mathscr{F}(g')$ which, under simple additional conditions, intertwines the natural exponential Weyl systems on these spaces.

We shall not be able to separate completely the two arguments related to those two facts. This problem is due to the non-Fock character of the total representation. But we define now a second sequence of approximations to $\omega_{g'}(.)$ in which the two effects are better separated than in $\omega^{(n)}$. The approximating functional $\omega_{mn\sigma}$ will be constructed by replacing each $V_{j\sigma}(g)$ by a sum $V_{R_j} + V_{Q_j}$ which is almost equal to $V_{j\sigma}(g)$ for large j but in which V_{R_j} and V_{Q_j} have special supports in position space. Furthermore in the definition of $\omega_{mn\sigma}$, each $\exp_j(V_{R_j} + V_{Q_j})$ will be replaced by $\exp_j V_{R_j} \exp_j V_{Q_j}$.

We now start the explicit definitions. By the assumptions of Theorem 3.1, $g(x) = g'(x)$ on B_{d_0} and since B is convex, there exists a polygonally bounded region $Q \subset \mathbb{R}^2$ such that $g(x) = g'(x)$ on Q_d and $Q \supset B_d$ for some $d > 0$. Throughout the remainder of this paper, d will denote this fixed number and Q will denote this fixed region, $\sim Q$ its complement.

Let $1 < v < 2$; we shall fix v in Sect. VIII. We define space cutoff functions g_{+j} and g_{-j}, derived from g. Let j be large so that $4v^{-j} < d$. We define

$$g_{+j}(x) = \begin{cases} g(x) & \text{for} \quad \operatorname{dist}(x, Q) \geqq 4v^{-j} \\ 0 & \text{for} \quad \operatorname{dist}(x, Q) \leqq 3v^{-j} \\ \text{smooth interpolation by scaling of a smooth} \\ \text{function for } 3v^{-j} < \operatorname{dist}(x, Q) < 4v^{-j}; \end{cases} \tag{4.4}$$

$$g_{-j}(x) = \begin{cases} g(x) & \text{for} \quad \operatorname{dist}(x, \sim Q) \geqq 4v^{-j}, \\ 0 & \text{for} \quad \operatorname{dist}(x, \sim Q) \leqq 3v^{-j}, \\ \text{smooth interpolation by scaling of a smooth} \\ \text{function for } 3v^{-j} < \operatorname{dist}(x, \sim Q) < 4v^{-j}. \end{cases} \tag{4.5}$$

Let $g_{\Delta j}(x)=g(x)-g_{+j}(x)-g_{-j}(x)$. Similar functions are derived from g', we denote them by $g'_{\pm j}$ and $g'_{\Delta j}$. We define furthermore a smooth version of χ_j. Let φ be a function in $\mathscr{S}(\mathbb{R}^2)$ satisfying

$$\begin{aligned}&\tilde{\varphi}(x)=0 \quad \text{if} \quad |x|\geqq 1/2, \qquad \varphi(k)\geqq 0,\\ &\int \varphi(k)\,dk=1\,.\end{aligned} \tag{4.6}$$

We let $\varphi_j(k)=v^{-2j}\varphi(v^{-j}k)$, and we define

$$\zeta_j(k_1,\ldots,k_4)=\int\prod_{n=1}^{4}(dl_n\varphi_j(k_n-l_n))\,\chi_j(l_1,\ldots,l_4)\,, \tag{4.7}$$

χ_j was defined in (2.5).

Let finally $\chi_{\sim Q}$ be the characteristic function of

$$\{(x_1,\ldots,x_4)\,|\,x_i\notin Q,\ \text{for all } i,\ 1\leqq i\leqq 4\} \quad \text{and let} \quad \chi_Q=1-\chi_{\sim Q}\,.$$

With $v(g)$:

$$v(g\,|\,k_1,\ldots,k_4)\equiv-\prod_{n=1}^{4}\mu(k_n)^{-1/2}\left(\sum_{n=1}^{4}\mu(k_n)\right)^{-1}\tilde{g}(k_1+\cdots+k_4)\,,$$

we set

$$\begin{aligned}v_{Q_j}&=\chi_Q^{(x)}\zeta_j^{(p)}v(g_{-j})\\ v_{R_j}&=\chi_{\sim Q}^{(x)}\zeta_j^{(p)}v(g_{+j})\\ v'_{R_j}&=\chi_{\sim Q}^{(x)}\zeta_j^{(p)}v(g'_{+j})\,.\end{aligned} \tag{4.8}$$

Note that by the assumption of Theorem 3.1,

$$v'_{Q_j}\equiv\chi_Q^{(x)}\zeta_j^{(p)}v(g'_{-j})=v_{Q_j}\,.$$

In these definitions and later, $f^{(x)}$ will denote the operator "multiplication by f in position space" and $f^{(p)}$ will denote the operator "multiplication by f in momentum space". We let $V_{R_j}\equiv\int dk_1\ldots dk_4\,a^*(k_1)\ldots a^*(k_4)\cdot v_{R_j}(k_1,\ldots,k_4)$ and we define in an analogous way the operators V'_{R_j} and V_{Q_j}. Let furthermore

$$V_j^{(n)}=\begin{cases}V_{R_j} & \text{if} \quad j\geqq n\\ V'_{R_j} & \text{if} \quad j<n\,,\end{cases}$$

and we set

$$\hat{T}_{mns}=\prod_{j=m}^{s-1}\exp_j V_j^{(n)}\exp_j V_{Q_j}\,. \tag{4.9}$$

We always set $\prod_{j=a}^{b}A_j=1$ if $a>b$.

Finally, our second approximation to $\omega_{g'}(\,.\,|\,\psi_1, \psi_2, r_1, r_2)$ is defined for $n > m > \max(r_1, r_2)$ and is given by

$$\omega_{mn\sigma}(C) = (\hat{T}_{mns} T_{r_1 n\tau}(g')\,\psi_1, C\hat{T}_{mns} T_{r_2 n\tau}(g')\psi_2)\exp - \Lambda_{n\sigma}, \quad (4.10)$$

where $\tau = 2^m$, $\sigma = 2^s$; $m, n, s \in \mathbb{N}$, and we recall that

$$T_{r_1 n\tau}(g') = \prod_{j=r_1}^{m-1} \exp_j V_{j\tau}(g'),$$

and does not depend on n since $n > m$. Explicitly,

$$\omega_{mn\sigma}(C) = (\theta(r_1, m, n, s, \psi_1), C\theta(r_2, m, n, s, \psi_2)) \exp - \Lambda_{n\sigma},$$

and for $s > n$, $\theta(r, m, n, s, \psi)$ is given by

$$\theta(r, m, n, s, \psi) = \prod_{j=n}^{s-1} \left(\exp_j V_{R_j} \exp_j V_{Q_j}\right) \prod_{j=m}^{n-1} \left(\exp_j V'_{R_j} \exp_j V'_{Q_j}\right) \cdot \prod_{j=r}^{m-1} \exp_j V_{j\sigma}(g')\psi.$$

The ideas behind the construction of $\theta(r, m, n, s, \psi)$ are the following. This vector approximates the "low momentum" part of the vectors $\theta_r \psi = \prod_{j=r}^{s-1} \exp_j V_{j\sigma}(g')\psi$ used in the definition of $\omega_{g'}(\cdot\,|\,\psi_1, \psi_2, r_1, r_2)$. Indeed, the factors $\exp_j V_{j\sigma}(g')$ coincide exactly with those of $\theta_r\psi$ up to $j = m-1$ and approximately with those of $\theta_r\psi$ up to $j = n-1$. The product $\exp_j V'_{R_j} \exp_j V'_{Q_j}$ approximates $\exp_j V_{j\sigma}(g')$ up to the "strips" at the boundary of Q and up to the fact that $\exp_j(V'_{R_j} + V'_{Q_j})$ is replaced by $\exp_j V'_{R_j} \exp_j V'_{Q_j}$. For $j \geqq n$, the factors in $\theta(r, m, n, s, \psi)$ are approximations to $\exp_j V_j(g)$, making thus $\lim_{s\to\infty} \theta(r, m, n, s, \psi)\exp(-\Lambda_{n\sigma}/2)$ a vector in $\mathscr{F}(g)$.

Before going into boundedness and existence proofs for the limit $\sigma \to \infty$, let us give the *proof of Theorem 3.3.*, assuming these results. By construction of $\omega^{(n)}$, it is evident that $\omega^{(\infty)} = \omega_{g'}$. It follows from Theorem 4.4, which is our main technical estimate, that for $C \in \mathfrak{A}_0(B)$,

$$|\omega_{mn\infty}(C) - \omega_{mn'\infty}(C)| < \varepsilon \|C\|, \quad (4.11)$$

provided that $n, n' > N(\varepsilon, m)$, since $T_{rn\tau}\psi \in \mathscr{D}_0$ if $\psi \in \mathscr{D}_0$, $\tau < \infty$. On the other hand, we show in Lemma 4.5 that

$$\|\hat{T}_{mns} T_{rn\tau}\psi - T_{rn\sigma}\psi\|^2 \exp - \Lambda_{n\sigma} < \varepsilon \quad (4.12)$$

for $m > M(\varepsilon)$, uniformly in $n > m$ and $s > m$. Eq. (4.12) has nothing to do with the algebra $\mathfrak{A}_0(B)$, but simply expresses the fact that $\theta(r, m, n, s, \psi)$ approximates $T_{rn\sigma}\psi$ if m is large enough. It is immediate that (4.11) and

(4.12) prove
$$|\omega^{(n)}(C)-\omega^{(n')}(C)|<\varepsilon\|C\|,$$
provided $n, n'>N(\varepsilon)$, since
$$\begin{aligned}|\omega^{(n)}(C)-\omega^{(n')}(C)| &\leqq |\omega^{(n)}(C)-\omega_{mn\infty}(C)|+|\omega_{mn\infty}(C)-\omega_{mn'\infty}(C)|\\ &\quad+|\omega_{mn'\infty}(C)-\omega^{(n')}(C)|\,.\end{aligned} \tag{4.13}$$

One first chooses m so large that the L.H.S. of (4.12) is very small, such that the first and the third term in (4.13) are bounded by $\varepsilon/3\|C\|$ each, by the Schwarz inequality. The second term in (4.13) is then bounded using (4.11), and so this proves Theorem 3.3.

We now state the facts we have used in the derivation of (4.13).

Lemma 4.1. *For every $\psi\in\mathscr{D}_0$, there exists a constant $C(\psi)$ such that uniformly in $m, n, s\in\mathbb{N}$*
$$\|T_{rn\sigma}\psi\|^2\exp-A_{n\sigma}\leqq C(\psi)\,, \tag{4.14}$$
$$\|\hat{T}_{mn\sigma}T_{rn\tau}\psi\|^2\exp-A_{n\sigma}\leqq C(\psi)\,, \tag{4.15}$$
where $\sigma=2^s$, $\tau=2^m$.

We defer the proof of this lemma to the end of this section.

Lemma 4.2. *The limit $\sigma\to\infty$ of $\omega_\sigma^{(n)}$ exists and defines for each $n\in\mathbb{N}$ an ultraweakly continuous functional on $\mathfrak{A}(B, g)$.*

The proof of this lemma can be found essentially in [8], or [7]. The technical changes due to the fact that $T_{rn\sigma}$ depends on two space cutoffs g and g' are trivial, see also Lemma 2.2.

Theorem 4.3. *The limit $\sigma\to\infty$ of $\omega_{mn\sigma}$ exists and defines a functional on $\mathfrak{A}_0(B)$.*

At the end of this section, we shall only prove the existence of $\lim_{\sigma\to\infty}\omega_{mn\sigma}(1)$. The general assertion will then follow from the proof of Theorem 6.3.

Theorem 4.4. *For every $\theta_\alpha\in\mathscr{D}_0$, $m_\alpha\in\mathbb{N}$, $\alpha=1,2$, and for every $\varepsilon>0$ there is an $N=N(\varepsilon,\theta_1,\theta_2,m_1,m_2)$ such that for all $n,n'>N$ and for every $C\in\mathfrak{A}_0(B)$, uniformly in $s>\max(m_1,m_2)$, $\sigma=2^s$, one has*
$$|(\hat{T}_{m_1ns}\theta_1, C\hat{T}_{m_2ns}\theta_2)\exp-A_{n\sigma}-(\hat{T}_{m_1n's}\theta_1, C\hat{T}_{m_2n's}\theta_2)\exp-A_{n'\sigma}|\leqq\varepsilon\|C\|.$$

In the proof of Theorem 3.3. we used this theorem with $m_1=m_2=m$, $\theta_\alpha=T_{r_\alpha n2^m}\psi\in\mathscr{D}_0$, $\alpha=1,2$.

The second part of this paper will be devoted to the proof of Theorem 4.4.

The vector approximation is described in

Lemma 4.5. *For every $\psi\in\mathscr{D}_0$, $r\in\mathbb{N}$ and every $\varepsilon>0$ there exists an $M=M(\varepsilon,\psi,r)<\infty$ such that for all $m>M(\varepsilon)$, $s, n>m$, $\tau=2^m$, $\sigma=2^s$,* *one has*

$$\| T_{rn\sigma}\psi - \hat{T}_{mns} T_{rn\tau}\psi \|^2 \exp - \Lambda_{n\sigma} < \varepsilon .$$

This lemma will be proved at the end of this section.

Proof of Lemma 4.1. Inequality (4.14) follows from inequality (4.15) for $m = s$. The proof of (4.15) follows, at least in spirit, the proof of Lemma 2.1. We call Λ-component any Wick term whose graph has the form . Let $(\ )_{\text{skel}}$ denote the sum over all Wick terms whose graph is a skeleton graph, i.e. which contains no Λ-components as subgraphs. A simple combinatorial argument shows that, as a formal power series, (4.15) can be written as

$$\sum_{f \in F(s)} \left[\left(\psi, \prod_{j=r}^{m-1} \frac{V_j^{*r_j}}{r_j!} \prod_{j=m}^{s-1} \left(\frac{V_j^{(n)p_j}}{p_j!} \frac{V_{Q_j}^{q_j}}{q_j!} \right)^* \right.\right.$$
$$\left. \cdot \prod_{j=m}^{s-1} \left(\frac{V_j^{(n)p'_j}}{p'_j!} \frac{V_{Q_j}^{q'_j}}{q'_j!} \right) \prod_{j=r}^{m-1} \frac{V_j^{r'_j}}{r'_j!} \psi \right)_{\text{skel}} \tag{4.16}$$
$$\left. \cdot \sum_{G(s,f)} \left\{ \prod_{i,j=1}^{s-1} \prod_{\alpha=1}^{9} \frac{\Lambda_{ij,\alpha}^{k_{ij,\alpha}}}{k_{ij,\alpha}!} \exp - \Lambda_{n\sigma} \right\} \right] .$$

We have used the following notation:

$$\begin{aligned}
V_j &= V_{j\sigma}(g'), \\
\Lambda_{ij,1} &= 4!\,(v_i^{(n)}, v_j^{(n)}), && i,j \geqq m \\
\Lambda_{ij,2} &= 4!\,(v_{Q_i}, v_{Q_j}), && i,j \geqq m \\
\Lambda_{ij,3} &= 4!\,(v_i, v_j), && i,j < m \\
\Lambda_{ij,4} &= 4!\,(v_i^{(n)}, v_{Q_j}), && i,j \geqq m \\
\Lambda_{ij,5} &= 4!\,(v_{Q_i}, v_j^{(n)}), && i,j \geqq m \\
\Lambda_{ij,6} &= 4!\,(v_i^{(n)}, v_j), && i \geqq m, j < m \\
\Lambda_{ij,7} &= 4!\,(v_i, v_j^{(n)}), && i < m, j \geqq m \\
\Lambda_{ij,8} &= 4!\,(v_{Q_i}, v_j), && i \geqq m, j < m \\
\Lambda_{ij,9} &= 4!\,(v_i, v_{Q_j}), && i < m, j \geqq m .
\end{aligned} \tag{4.17}$$

We let $\Lambda_{ij,\alpha} = 0$, if i, j are not in the ranges indicated above. (Small letters indicate kernels of operators with same capital letters.) Finally, the sums extend over

$$\begin{aligned}
F(s) \equiv \{ f = (p_i, q_i, r_i, p'_i, q'_i, r'_i \mid r \leqq i \leqq s-1) \mid 0 \leqq p_i \leqq i, 0 \leqq q_i \leqq i, \\
0 \leqq r_i \leqq i; 0 \leqq p'_i \leqq i, 0 \leqq q'_i \leqq i, 0 \leqq r'_i \leqq i \}
\end{aligned} \tag{4.18}$$

and for $f \in F(s)$ over

$$G(s,f) \equiv \Bigg\{ k_{ij,\alpha} \geqq 0, \alpha = 1, \ldots, 9, r \leqq i < s, r \leqq j < s,$$

$$\sum_{j=m}^{s-1} k_{ij,1} + k_{ij,4} + \sum_{j=r}^{m-1} k_{ij,6} \leqq i - p_i ;$$

$$\sum_{j=m}^{s-1} k_{ij,2} + k_{ij,5} + \sum_{j=r}^{m-1} k_{ij,8} \leqq i - q_i ;$$

$$\sum_{j=r}^{m-1} k_{ij,3} + \sum_{j=m}^{s-1} k_{ij,7} + k_{ij,9} \leqq i - r_i ; \tag{4.19}$$

$$\sum_{i=m}^{s-1} k_{ij,1} + k_{ij,5} + \sum_{i=r}^{m-1} k_{ij,7} \leqq j - p'_j ;$$

$$\sum_{i=m}^{s-1} k_{ij,2} + k_{ij,4} + \sum_{i=r}^{m-1} k_{ij,9} \leqq j - q'_j ;$$

$$\sum_{i=r}^{m-1} k_{ij,3} + \sum_{i=m}^{s-1} k_{ij,6} + k_{ij,8} \leqq j - r'_j \Bigg\} .$$

By construction, $\Lambda_{ij,4} = \Lambda_{ij,5} = 0$, for all $i,j \geqq m$ and $\Lambda_{ij,3} = 0$ if $i \neq j$, $i,j < m$.

In order to prove (4.15) we have to prove the analogues of (2.23) and of the fact that $\prod_j \exp \Lambda_j \exp - \Lambda_j \leqq 1$. Note that the n^{th}-order contribution to $\hat{T}_{rts}$ is not a truncation of $V^n/n!$ in the usual sense that its kernel equals the kernel of $V^n/n!$, multiplied by a characteristic function. But $\hat{T}_{rts}$ is constructed in such a way that "up to" a square integrable "error" it is indeed such a truncation.

We start our estimates by bounding the $\Lambda_{ij,\alpha}$.

To this aim we relate v_{R_j} and v_{Q_j} to $v_j(g)$, as defined in Sect. II, Eq. (2.8).

Lemma 4.6. *There exist constants $\lambda_0 < \infty$ and $\gamma > 1$ such that the following inequalities hold.*

$$\|v_j(g_{\pm j})\|_2 < \lambda_0 , \tag{4.20}$$

$$\|v_{R_j} - v_j(g_{+j})\|_2 + \|v_{Q_j} - v_j(g_{-j})\|_2 < \lambda_0 \gamma^{-j} , \tag{4.21}$$

$$\|v_j(g) - v_j(g_{+j}) - v_j(g_{-j})\|_2 < \lambda_0 \gamma^{-j} . \tag{4.22}$$

If $i \neq j$ then

$$\sum_{\alpha = 1,2,6,7,8,9} |\Lambda_{ij,\alpha}| < \lambda_0 \gamma^{-i-j} . \tag{4.23}$$

We shall prove this Lemma in Sect. VIII.

We now use the inequalities (4.20)–(4.23) to prove (4.15). We first bound $A_{ii,1}$ and $A_{ii,2}$ as follows, using (4.21) and (4.22): Namely one finds

$$\begin{aligned} &|A_{ii,1} - 4!\,\|v_i^{(n)}(g_{+i})\|_2^2| < \lambda_2 \gamma_2^{-i} \quad \text{for some} \quad \gamma_2 > 1\,, \\ \text{and} \quad &|A_{ii,2} - 4!\,\|v_i(g_{-i})\|_2^2| < \lambda_2 \gamma_2^{-i}\,. \end{aligned} \tag{4.24}$$

Also,

$$|A_{ii,1} + A_{ii,2} - 4!\,\|v_i(g)\|_2^2| < \lambda_3 \gamma_3^{-i}, \quad \text{for some} \quad \gamma_3 > 1\,, \tag{4.25}$$

if $i \leqq n$, and the analogous relation holds with g' if $i > n$.

We now prove that $\sum_G \{\ \}$ in (4.16) is bounded by a constant, uniformly in $m < n$, s and $f \in F$. Note that for $A \geqq 0$, $\exp_j A \exp(-A) \leqq 1$, and that for

$$B < 0,\ \exp_j B \exp(-B) \leqq \exp_j |B| \exp|B| \leqq \exp 2|B|\,. \tag{4.26}$$

Using these facts, and replacing all $A_{ij,\alpha}$, $i \neq j$, $\alpha = 1, \ldots, 9$ by $|A_{ij,\alpha}|$, we get

$$\left|\sum_G \{\ \}\right| \leqq C_{\gamma,\lambda_0} \cdot \prod_{i \neq j} \exp\left\{2 \sum_{\alpha=1}^{9} |A_{ij,\alpha}|\right\} \leqq \text{const}\,, \tag{4.27}$$

by (4.26) and (4.23), uniformly in

$$m, n, s, p_i, q_i, r_i, p_i', q_i', r_i'\,.$$

Our next step is to prove the uniform boundedness of $\sum_F (\psi, \ldots \psi)_{\text{skel}}$ in (4.16), and we repeat the considerations of Sect. II. It follows at once from Lemma 4.6 that the set of functions $x_j^{(1)} = v_j^{(n)}$, $x_j^{(2)} = v_{Q_j}$ satisfies (2.18) and (2.19) since $v_j(g_{+j})$, $v_j(g_{-j})$, $v_j(g)$ and the corresponding primed quantities do. Let S_{pq} be a Wick term of $(V^*)^p(V)^q$ whose graph is a skeleton graph. We apply Lemma 2.2 with the above choice of $x_j^{(1)}$ and $x_j^{(2)}$ and get

$$\begin{aligned} &\left|\sum_F (\psi_1, \ldots \psi_2)_{\text{skel}}\right| \\ &\leqq \sum_{p,q=0}^{\infty} \sum_{S_{pq}} (|\psi_1|, |S_{pq}|\,|\psi_2|) \\ &\leqq \sum_{p,q=0}^{\infty} \max_{S_{pq}} (|\psi_1|, |S_{pq}|\,|\psi_2|)\,((4p)!\,(4q)!)^{1/2}\, 4^{4(p+q)} \\ &\leqq \sum_{p,q=0}^{\infty} C_{\psi_1,\psi_2} ((4p)!\,(4q)!)^{1/2}\, 4^{4(p+q)} (\gamma-1)^{-(p+q)} \gamma^{-\left(\sum_{n=1}^{p/3} n^{1/2}\right)} \gamma^{-\left(\sum_{n=1}^{q/3} n^{1/2}\right)} < \infty\,. \end{aligned} \tag{4.28}$$

The factor $\left(\gamma^{-\left(\sum_{n=1}^{p/3} n^{1/2}\right)} \gamma^{-\left(\sum_{n=1}^{q/3} n^{1/2}\right)}\right)$ comes from the fact that the set corresponding to J in Sect. II (after Eq. (2.22)) is now a subset of $\{j_i > (i/3)^{1/2}, j'_i > (i/3)^{1/2}\}$ since $\hat{T}$ is the product of three truncated exponentials. This completes the proof of Lemma 4.1.

Proof of Theorem 4.3. We prove here only the existence of $\lim_{\sigma\to\infty} \omega_{mn\sigma}(1)$. The existence of $\lim_{\sigma\to\infty} \omega_{mn\sigma}(C)$, $C \in \mathfrak{A}_0(B)$ will then follow in a similar way from Theorem 6.3. Usually, the existence of limits of this kind is proven by using the pointwise convergence of the kernels of $T_{r\sigma}$ and the bounded convergence theorem. We are forced, due to the more complicated structure of $\omega_{mn\sigma}$, to prove the convergence directly. This is easy if we use the explicit formula (4.16). We write

$$A(s) = \sum_{f \in F(s)} \left[S(s,f) \sum_{G(s,f)} L(s,f)\right]$$

in short for (4.16). We now assume that $s_1 > s_2$, and we want to show that $|A(s_1) - A(s_2)| < \varepsilon$ if s_2 is large enough. To this aim, we exhibit $A(s_1)$ as a sum of two expressions $A_1(s_1, s_2)$ and $A_2(s_1, s_2)$, where $A_1(s_1, s_2)$ "almost" cancels $A(s_2)$ term by term and where $|A_2(s_1, s_2)|$ is small.

Let $F_1(s_1, s_2)$ be that subset of $F(s_1)$ for which $p_j = q_j = r_j = p'_j = q'_j = r'_j = 0$ for $j \geqq s_2$. We have

$$F_1(s_1, s_2) = F(s_2) \times \{p_j = q_j = \dots = 0 \mid s_1 > j \geqq s_2\}.$$

If $f \in F(s_2)$, we let $\bar{f}$ be the corresponding element of $F_1(s_1, s_2)$. We set

$$A_1(s_1, s_2) = \sum_{f \in F_1(s_1,s_2)} \left[S(s_1,f) \sum_{G(s_1,f)} L(s_1,f)\right],$$

and $A_2(s_1, s_2) = A(s_1) - A_1(s_1, s_2)$. If $\bar{f} \in F_1(s_1, s_2)$, it follows from the definition of $S(s_1, \bar{f})$ that $S(s_1, \bar{f}) = S(s_2, f)$, and therefore

$$A_1(s_1, s_2) = \sum_{f \in F(s_2)} \left[S(s_2,f) \sum_{G(s_1,\bar{f})} L(s_1,\bar{f})\right].$$

We have thus prepared the cancellation in

$$\begin{aligned} &|A_1(s_1, s_2) - A(s_2)| \\ &= \left|\sum_{f \in F(s_2)} \left[S(s_2,f)\left\{\sum_{G(s_1,\bar{f})} L(s_1,\bar{f}) - \sum_{G(s_2,f)} L(s_2,f)\right\}\right]\right| \\ &\leqq \sum_{f \in F(s_2)} |S(s_2,f)| \sup_{f \in F(s_2)} \left|\sum_{G(s_1,\bar{f})} L(s_1,\bar{f}) - \sum_{G(s_2,f)} L(s_2,f)\right|. \end{aligned} \tag{4.29}$$

By the proof of Lemma 4.1,

$$\sum_{f \in F(s_2)} |S(s_2,f)| \leqq C(\psi), \quad \text{uniformly in } s_2.$$

We now argue that the sup in (4.29) goes to zero as $s_2 \to \infty$. Let

$$G'(s_1, s_2, f) = G(s_2, f) \times \{k_{ij,\alpha} = 0 \text{ if } i \neq j \text{ and } (i \geqq s_2 \text{ or } j \geqq s_2)\}.$$

Obviously, $G_1'(s_1, s_2, f) \subset G(s_1, \bar{f})$ and we set

$$G''(s_1, s_2, \bar{f}) = G(s_1, \bar{f}) \setminus G'(s_1, s_2, f).$$

In G', the $\Lambda_{ii,\alpha}$ $\alpha = 1, 2, 3$, will occur in all powers in the range $0, 1, \ldots, i$, if $i \geqq s_2$. Thus, for any $\varepsilon > 0$, we find

$$\begin{aligned}\sum_{G(s_1,\bar{f})} L(s_1, \bar{f}) &= \sum_{G(s_2,f)} L(s_2, f) \left(\prod_{i=s_2}^{s_1-1} \exp_i \left(\sum_{\alpha=1}^{3} \Lambda_{ii,\alpha} \right) \right) \exp(-\Lambda_{n\sigma_1} + \Lambda_{n\sigma_2}), \\ &= \sum_{G(s_2,f)} L(s_2, f) \cdot (1 \pm \varepsilon), \quad \text{for } s_2 \text{ large}.\end{aligned}$$

We have used $\sigma_i = 2^{s_i}$, and we used the fact that for $A > 0$,

$$0 \leqq 1 - \exp_j A \exp(-A) \leqq A^{j+1}/(j+1)!,$$

and the estimates of Lemma 4.6.

Therefore, for large s_2,

$$\begin{aligned}&|A_1(s_1, s_2) - A(s_2)| \\ &\leqq C(\psi) \sup_{f \in F(s_2)} \left| \sum_{G(s_2)} L(s_2, f) \right| \cdot \varepsilon + C(\psi) \sup_{f \in F(s_2)} \left| \sum_{G''(s_1, s_2, \bar{f})} L(s_1, \bar{f}) \right|. \end{aligned} \tag{4.30}$$

The first term in (4.30) is bounded uniformly in s_2 by the argument given in the proof of Lemma 4.1. The second term goes to zero as $s_2 \to \infty$ because every term in $\sum_{G''(s_1, s_2, \bar{f})}$ contains at least one small factor $\Lambda_{ij,\alpha}$ $i \neq j$, i or $j \geqq s_2$, and these factors go uniformly to zero as $s_2 \to \infty$, by Lemma 4.6. So we have shown that $|A_1(s_1, s_2) - A(s_2)| < \varepsilon$ for s_2 large.

In each term of $A_2(s_1, s_2)$ there occurs at least one $V_j^{(n)(*)}$, $V_j^{(*)}$ or $V_{Q_j}^{(*)}$ with $j \geqq s_2$. We use Eq. (2.22) and we apply the argument which led to (4.28) to get a bound $|A_2(s_1, s_2)| < \text{const.} \gamma^{-s_2}$, for some $\gamma > 1$. Thus $|A_2(s_1, s_2)| \to 0$ as $s_2 \to \infty$: the assertion "$\lim_{\sigma \to \infty} \omega_{mn\sigma}(\mathbf{1})$ exists" is proved.

Proof of Lemma 4.5. We want to bound the expression

$$\| T_{rn\sigma} \psi - \hat{T}_{mns} T_{rn\tau} \psi \|^2 \exp - \Lambda_{n\sigma}, \quad (\tau = 2^m) \tag{4.31}$$

for $n > m$, m large, and we want to show that this bound goes to zero as $m \to \infty$. The proof goes along lines described in [7, Lemma 4.1] with some new estimates.

We break (4.31) into two parts, writing

$$\begin{aligned} &\|T_{rn\sigma}\psi - \hat{T}_{mns}T_{rn\tau}\psi\| e^{-A_{n\sigma}/2} \\ &\quad \leqq \|T_{rn\sigma}\psi - \hat{T}_{mns}T_{rn\tau}T^{\Delta}_{mns}\psi\| e^{-A_{n\sigma}/2} \qquad (4.32.1) \\ &\qquad + \|\hat{T}_{mns}T_{rn\tau}T^{\Delta}_{mns}\psi - \hat{T}_{mns}T_{rn\tau}\psi\| e^{-A_{n\sigma}/2}. \qquad (4.32.2) \end{aligned}$$

Here

$$\begin{aligned} T^{\Delta}_{mns} \equiv \prod_{j=m}^{\min(s,n)-1} \underset{j}{\exp}(V_j(g') - V'_{R_j} - V_{Q_j}) \\ \cdot \prod_{j=n}^{s-1} \underset{j}{\exp}(V_j(g) - V_{R_j} - V_{Q_j}). \end{aligned} \qquad (4.33)$$

We show that the assertion of Lemma 4.5 holds for each of (4.32.1), (4.32.2) separately.

To bound (4.32.1), we write

$$T_{rn\sigma} - \hat{T}_{mns}T_{rn\tau}T^{\Delta}_{mns} = T_{rn\tau}\cdot(T_{mn\sigma} - \hat{T}_{mns}T^{\Delta}_{mns}) \equiv L_{mns}\cdot T_{rn\tau}.$$

Suppose for simplicity of notation that $s > n$; the proof of the case $s \leqq n$ is similar. We let

$$V_j(g^{(n)}) = \begin{cases} V_j(g) & \text{if} \quad j \geqq n \\ V_j(g') & \text{if} \quad j < n \end{cases},$$

$$\begin{aligned} L_{mns} &= \prod_{j=m}^{s-1} \underset{j}{\exp}(V_j^{(n)} + V_{Q_j} + (V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j})) \\ &\quad - \prod_{j=m}^{s-1} \underset{j}{\exp}(V_j^{(n)}) \underset{j}{\exp}(V_{Q_j}) \underset{j}{\exp}(V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j}) \\ &= \sum_{\gamma_m, \ldots, \gamma_{s-1} \geqq 0} \left\{ \prod_{j=m}^{s-1} \left[\underset{j}{\exp}(V_j^{(n)} + V_{Q_j} + (V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j}))\right]_{\gamma_j} \right. \\ &\quad \left. - \prod_{j=m}^{s-1} \left[\underset{j}{\exp}(V_j^{(n)}) \underset{j}{\exp}(V_{Q_j}) \underset{j}{\exp}(V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j})\right]_{\gamma_j} \right\}, \end{aligned} \qquad (4.34)$$

where $[\ ]_\gamma$ indicates "γ-th order term". By definition of $\underset{j}{\exp} A$,

$$\begin{aligned} &\left[\underset{j}{\exp}(V_j^{(n)} + V_{Q_j} + (V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j})\right]_\gamma \\ &\quad = \left[\underset{j}{\exp}(V_j^{(n)}) \underset{j}{\exp}(V_{Q_j}) \underset{j}{\exp}(V_j(g^{(n)}) - V_j^{(n)} - V_{Q_j})\right]_\gamma \end{aligned} \qquad (4.35)$$

if $\gamma \in \{0, 1, 2, \ldots j\}$. Therefore a term in the sum (4.34) is zero unless for at least one $j \geqq m$ one has $\gamma_j > j$.

This implies that all contributions from

$$\left|\underset{j}{\exp}(V_{R_j} + V_{Q_j} + (V_j(g) - V_{R_j} - V_{Q_j}))\right|$$

are cancelled in (4.34). We rewrite (4.32.1):

$$|(4.32.1)|^2 = \left\| L_{mns} \prod_{j=r}^{m-1} \exp_j V_j(g')\psi \right\|^2 \exp - A_{n\sigma}$$
$$= (\psi, T^*_{rn\tau} L^*_{mns} L_{mns} T_{rn\tau}\psi) \exp - A_{n\sigma} = I_1 + I_2 \,. \tag{4.36}$$

We describe the decomposition $I_1 + I_2$, which is obtained by distinguishing different Wick terms, cf. also the decomposition of (4.16) in the proof of Lemma 4.3. I_1 is the sum of all those terms in which V_{R_j} or V_{Q_j}, $j \geqq m$ and their adjoints occur only as $V^*_{R_j \smallsetminus 4} V_{R_j}$ or $V^*_{Q_j \smallsetminus 4} V_{Q_j}$ components resp., I_2 is the sum over the remaining terms.

$|I_2|$ is bounded by a standard argument, which we used already in the bound of $|A_2(s_1, s_2)|$ in the proof of Theorem 4.3. Each Wick term in I_2 has either a $V^{(*)}_{Q_j}$, $V^{(*)}_{R_j}$ with $j \geqq m$ in its skeleton, or a $V^{(*)}_{A_j}$, $j \geqq m$ in a skeleton or a A component of the form $A_{ij,1}$, $A_{ij,2}$, $i \neq j$, i or $j \geqq m$. Thus it follows from (4.28) and Lemma 4.6 that $|I_2| \to 0$ as $m \to \infty$; the uniformity in n follows at once from the fact that the bounds in Lemma 4.6 can all be given with the same constants for g and g'.

To estimate I_1, we use the representation (4.34) and the cancellation due to (4.35). We write

$$I_1 = \| T_{rn\tau}\psi \|^2 \exp(-A_{n\tau}) \cdot C_{mns} \,,$$

where
$$C_{mns} = \exp(-A_{n\sigma} + A_{n\tau}) \, (\Omega, L^*_{mns} L_{mns} \Omega)_A \,.$$

Ω is the Fock vacuum and $(\ )_A$ denotes the sum over those Wick terms whose graph consists of $A_{ii,1}$ and $A_{ii,2}$ components only.

By Lemma 4.1, $\| T_{rn\tau}\psi \|^2 \exp - A_{n\tau}$ is uniformly bounded in n and τ. It remains to show $C_{mns} \to 0$ as $m \to \infty$, uniformly in n and s. By definition

$$C_{mns} = \sum^{A}_{\gamma_m, \ldots, \gamma_{s-1} \geqq 0} \left\{ \prod_{j=m}^{s-1} \left[\exp_j A_{jj,1} \exp_j A_{jj,2} \right]_{\gamma_j} \exp - A_j \right\},$$

where $\sum^A$ extends over the set $\{\gamma_j \geqq j$, for at least one $j\}$. We bound

$$|C_{mns}| \leqq \sum_{i=m}^{s-1} \prod_{\substack{j=m \\ j \neq i}}^{s-1} \{\exp(|A_{jj,1}| + |A_{jj,2}|) \exp - A_j\}$$
$$\cdot \sum_{\gamma=i}^{\infty} ((|A_{ii,1}| + |A_{ii,2}|)^\gamma / \gamma!) \exp - A_i$$
$$\leqq C(m) \sum_{i=m}^{s-1} \sum_{\gamma=i}^{\infty} \frac{A_i^\gamma}{\gamma!} \exp - A_i \,,$$

by Lemma 4.6; note also that $C(m) \to 1$ as $m \to \infty$.

Now for $A > 0$, $\left(\exp A - \underset{j}{\exp} A\right) \exp - A < A^{j+1}/(j+1)!$, so we finally find, for large m,

$$|C_{mns}| \leqq 2 \cdot \sum_{i=m}^{s-1} \frac{A_i^{i+1}}{(i+1)!}$$

$$\leqq 2 \cdot \sum_{\gamma=m+1}^{\infty} \frac{\lambda_0^\gamma}{\gamma!},$$

and this goes to zero as $m \to \infty$. We have used $|A_i| \leqq \lambda_0$, Eq. (2.16). The uniformity in n follows as before. This completes the proof that $(4.32.1)^2 \to 0$ as $m \to \infty$, uniformly in n.

In order to bound (4.32.2), we proceed similarly.

$$|(4.32.2)|^2 = \| \hat{T}_{mns} T_{rn\tau} (T_{mns}^{\Delta} - 1) \psi \|^2 \exp - A_{n\sigma} .$$

There is no exact cancellation of the type (4.35), but instead each $V(g_{\Delta_j})$ has a kernel with small L_2-norm, by Lemma 4.6. We do not present the detailed arguments. If one writes $T_{mns}^{\Delta} = 1 + T_{mns}'$, then all contributions from T_{mns}' are small for large m, whether their vertices occur in Δ-components or in skeletons, by Lemma 4.6. The 1 cancels, and we get the assertion for (4.32.2); the proof of Lemma 4.5 is complete.

Part 2. Proof of the Main Theorem

In this second part, we shall prove Theorem 4.4. The proof given is rather technical but the main ideas are fairly simple. We want to use an expansion of $C \in \mathfrak{A}_0(B)$ in terms of creation and annihilation operators, and we require that this expansion of C should converge inside the functionals ω_{mns}. By Fabrey [8], one knows that $\lim_{\sigma \to \infty} \omega_{mn\sigma}(N)$, where N is the number operator, is infinite. This suggests that the usual expansion of C in terms of creation and annihilation operators, as given e.g. in [12, Eq. (4.17)], cannot converge inside $\omega_{mn\sigma}$ as $\sigma \to \infty$. However, if $K = \int A^*(x) A(y) k(x, y) \, dx \, dy$, where k is smooth and falls off at infinity, one can show that $\omega_{mn\sigma}(K)$ stays bounded as $\sigma \to \infty$ (Theorem 8.9). We take advantage of this fact by using a more specialized expansion of C in which smooth kernels similar to k will occur; indeed we shall expand C *only* in the region $(\sim Q)$, and it will be seen that such an expansion converges because $B \subset Q$. This construction induces the following complication in our bounds for functionals: We shall see that one can use the Wick expansion of the functional only in variables which are "localized" (in the Newton-Wigner sense) outside of Q. This has made necessary our choice of $\omega_{mn\sigma}$ in Sect. IV. We shall now describe the expansion (Sect. V), prove its convergence (Sect. VI) and then prove Theorem 4.4 (Sect. VII). Sect. VIII contains two main estimates: An analysis of the behaviour of $v_4(g)$ outside the support of g and bounds for $\omega_{mn\sigma}$ on operators with smooth kernels.

V. The Particle Expansion of Operators in $\mathfrak{A}_0(B)$

In this section, we use an expansion of operators $C \in \mathfrak{A}_0(B)$ in order to exhibit the fact that such operators, considered as sums of Newton-Wigner localized Wick monomials, have smooth kernels which rapidly decay outside the region $Q \supset B$ in position space.

We first describe an approximation construction which is similar to one devised by Glimm and Jaffe in [12].

Let $C \in \mathfrak{A}_0(B)$, i.e.

$$C = \sum_{j=1}^{J} \alpha'_j \exp i\,(\phi(f_j) + \pi(h_j))\,. \tag{5.1}$$

The test functions f_j and h_j are smooth and their support is contained in B; $\alpha'_j \in \mathbb{C}$. We want to go to Wick-ordered quantities. We define, for $\tau \in \mathbb{R}$, $f \in \mathscr{S}(\mathbb{R}^2)$, an operator $\mu_\tau : \mathscr{S}(\mathbb{R}^2) \to \mathscr{S}(\mathbb{R}^2)$, by

$$(\mu_\tau f)(x) = (2\pi)^{-1} \int \mu^\tau(k) \tilde{f}(k) e^{-ikx} dk\,, \tag{5.2}$$

where $\sim$ denotes Fourier transform. Now, for real $f \in \mathscr{S}(\mathbb{R}^2)$,

$$\phi(f) = 2^{-1/2} \int dx\, A^*(x)\,(\mu_{-1/2} f)(x) + 2^{-1/2} \int dx\, A(x)\,(\mu_{-1/2} f)(x) \tag{5.3}$$

and

$$\pi(f) = i 2^{-1/2} \int dx\, A^*(x)\,(\mu_{+1/2} f)(x) - i 2^{-1/2} \int dx\, A(x)\,(\mu_{+1/2} f)(x)\,. \tag{5.4}$$

Using the commutation relations, we find that the general element of $\mathfrak{A}_0(B)$ can be written as

$$C = \sum_{j=1}^{J} \alpha_j \exp A^*(c_{j+}) \exp A(c_{j-})\,, \tag{5.5}$$

where $\alpha_j \in \mathbb{C}$ and

$$c_{j+}(x) = i(2^{-1/2}(\mu_{-1/2} f_j)(x) + i 2^{-1/2}(\mu_{+1/2} h_j)(x))\,, \tag{5.6}$$

$$c_{j-}(x) = i(2^{-1/2}(\mu_{-1/2} f_j)(x) - i 2^{-1/2}(\mu_{+1/2} h_j)(x))\,. \tag{5.7}$$

One can invert the relations between $c_{j\pm}$ and f_j and h_j,

$$i f_j(x) = 2^{-1/2}(\mu_{+1/2} c_{j+})(x) + 2^{-1/2}(\mu_{+1/2} c_{j-})(x)\,, \tag{5.8}$$

$$i h_j(x) = -i 2^{-1/2}(\mu_{-1/2} c_{j+})(x) + i 2^{-1/2}(\mu_{-1/2} c_{j-})(x)\,. \tag{5.9}$$

It is at this point that we make use of the fact that the functions f_j and h_j have support contained in B. We introduce a function $\xi:\mathbb{R}^2\to\mathbb{R}$ with the following properties:

$$\xi\in C^\infty,\ 0\leqq\xi(x)\leqq 1,\ \xi(x)=1 \quad\text{if}\quad x\in B,\ \xi(x)=0 \quad\text{if}\quad x\notin B_{d/4}. \tag{5.10}$$

We can rewrite (5.8) and (5.9):

$$if_j(x)=2^{-1/2}\xi(x)\,(\mu_{+1/2}c_{j+})\,(x)+2^{-1/2}\xi(x)\,(\mu_{+1/2}c_{j-})\,(x)\,, \tag{5.11}$$

$$ih_j(x)=-i2^{-1/2}\xi(x)\,(\mu_{-1/2}c_{j+})\,(x)+i2^{-1/2}\xi(x)\,(\mu_{-1/2}c_{j-})\,(x)\,. \tag{5.12}$$

Inserting (5.11/12) into (5.6/7) we get

$$c_{j+}(x)=K_+c_{j+}(x)+K_-c_{j-}(x)\,, \tag{5.13}$$

$$c_{j-}(x)=K_-c_{j+}(x)+K_+c_{j-}(x)\,, \tag{5.14}$$

$$\text{where } K_\pm=1/2\ \mu_{-1/2}\xi\mu_{+1/2}\pm 1/2\ \mu_{+1/2}\xi\mu_{-1/2}\,. \tag{5.15}$$

We shall denote by $k_\pm(x,z)$ the distribution kernels of $K_\pm$;

$$k_\pm(x,z)$$
$$=1/2\int dy\,\mu_{-1/2}(x-y)\xi(y)\mu_{+1/2}(y-z)\pm 1/2\int dy\,\mu_{+1/2}(x-y)\xi(y)\mu_{-1/2}(y-z).$$

Let Q be the polygonally bounded region defined in Sect. IV, $Q\supset B_d$ for some $d>0$. We define an expansion

$$C=\sum_{m,m'=0}^{\infty}C_{mm'}\,, \tag{5.16}$$

in which, loosely speaking, $C_{mm'}$ is that term of C which creates and annihilates m and m' particles respectively in the region $(\sim Q)$, and an indefinite number of particles in Q. It is convenient to consider first the special case

$$C=\exp A^*(c_+)\exp A(c_-)\,. \tag{5.17}$$

In that case,

$$\begin{aligned}C_{mm'}=&\int_{x_i,x_j'\notin Q}dx_1\dots dx_m dx_1'\dots dx_{m'}'A^*(x_1)\dots A^*(x_m)A(x_1')\dots A(x_{m'}')\\ &\cdot\left\{\sum_{\alpha,\alpha'=0}^{\infty}\binom{m+\alpha}{\alpha}\binom{m'+\alpha'}{\alpha'}\frac{1}{(m+\alpha)!\,(m'+\alpha')!}\prod_{j=1}^{m}c_+(x_j)\prod_{j=1}^{m'}c_-(x_j')\right.\\ &\cdot\left.\int_{y_j,y_j'\in Q}dy_1\dots dy_\alpha dy_1'\dots dy_{\alpha'}'\prod_{j=1}^{\alpha}(A^*(y_j)\,c_+(y_j))\prod_{j=1}^{\alpha'}(A(y_j')\,c_-(y_j'))\right\}.\end{aligned} \tag{5.18}$$

The bracket { } is an operator valued kernel $c_{mm'}(x, x')$, and the operator acts on

$$\mathscr{F}_Q = \mathbb{C} \oplus \left(\bigoplus_{n=1}^{\infty} L_2(Q)^{\otimes_s n} \right), \tag{5.19}$$

where $\bigotimes_s$ denotes the symmetrized tensor product.

Let now $\mathring{\chi}_Q$ be the characteristic function of Q in $\mathbb{R}^2$ in one variable. We define

$$k_{\pm i}(x, y) = k_{\pm}(x, y)\, \mathring{\chi}_Q(y), \tag{5.20}$$

$$k_{\pm o}(x, y) = k_{\pm}(x, y)\,(1 - \mathring{\chi}_Q(y)). \tag{5.21}$$

The subscripts i and o stand for "inside" and "outside" respectively. Let $K_{\pm i}$, $K_{\pm o}$ be the operators with kernels $k_{\pm i}$, $k_{\pm o}$, resp. We write

$$\begin{aligned} c_{\pm}(x) = & \int_{\sim Q} dy\, k_{+o}(x, y)\, c_{\pm}(y) + \int_Q dy\, k_{+i}(x, y)\, c_{\pm}(y) \\ & + \int_{\sim Q} dy\, k_{-o}(x, y)\, c_{\mp}(y) + \int_Q dy\, k_{-i}(x, y)\, c_{\mp}(y). \end{aligned} \tag{5.22}$$

Introducing (5.22) into each factor of

$$\prod_{j=1}^{m} c_+(x_j) \prod_{j=1}^{m'} c_-(x'_j) \quad \text{of} \quad (5.18),$$

we get

$$\begin{aligned} & c_{mm'}(x_1 \ldots x_m, x'_1 \ldots x'_{m'}) \\ & = \sum_{\alpha, \alpha'=0}^{\infty} \sum_{\substack{j_1+j_2+j_3+j_4=m \\ j'_1+j'_2+j'_3+j'_4=m'}} \prod_{k=1}^{4} (j_k!\, j'_k!)^{-1} (\alpha!\, \alpha'!)^{-1} \\ & \cdot \int dz_1 \ldots dz_{j_1+j'_2}\, dz'_1 \ldots dz'_{j'_1+j_2}\, d\eta_1 \ldots d\eta_{j_3+j'_4}\, d\eta'_1 \ldots d\eta'_{j'_3+j_4} \\ & \cdot \prod_{n=1}^{j_1} k_{+o}(x_n, z_n)\, c_+(z_n) \prod_{n=1}^{j_2} k_{-o}(x_{j_1+n}, z'_{j'_1+n})\, c_-(z'_{j'_1+n}) \\ & \cdot \prod_{n=1}^{j_3} k_{+i}(x_{j_1+j_2+n}, \eta_n)\, c_+(\eta_n) \prod_{n=1}^{j_4} k_{-i}(x_{j_1+j_2+j_3+n}, \eta'_{j'_3+n})\, c_-(\eta'_{j'_3+n}) \\ & \cdot \prod_{n=1}^{j'_1} k_{+o}(x'_n, z'_n)\, c_-(z'_n) \prod_{n=1}^{j'_2} k_{-o}(x'_{j'_1+n}, z_{j_1+n})\, c_+(z_{j_1+n}) \\ & \cdot \prod_{n=1}^{j'_3} k_{+i}(x'_{j'_1+j'_2+n}, \eta'_n)\, c_-(\eta'_n) \prod_{n=1}^{j'_4} k_{-i}(x'_{j'_1+j'_2+j'_3+n}, \eta_{j_3+n})\, c_+(\eta_{j_3+n}) \\ & \cdot \int_{y_j, y'_j \in Q} \prod_{j=1}^{\alpha} dy_j A^*(y_j)\, c_+(y_j) \prod_{j=1}^{\alpha'} dy'_j\, A(y'_j)\, c_-(y'_j). \end{aligned} \tag{5.23}$$

Furthermore, we write, e.g.

$$\int d\eta\, k_{+i}(x, \eta)\, c_+(\eta) = \int dy\, d\eta\, k_{+i}(x, \eta)\, A(\eta) \smile A^*(y)\, c_+(y),$$

where $\smile$ means "contracted". Therefore,

$$
\begin{aligned}
&c_{mm'}(x_1 \ldots x_m, x'_1 \ldots x'_{m'}) \\
&= \sum_{\alpha,\alpha'=0}^{\infty} \sum_{\substack{j_1+\ldots+j_4=m \\ j'_1+\ldots+j'_4=m'}} (\alpha!\,\alpha'!)^{-1} \prod_{k=1}^{4} (j_k!\,j'_k!)^{-1} \int dz_1 \ldots dz_{j_1+j'_2} dz'_1 \ldots dz'_{j'_1+j_2} \\
&\cdot \prod_{n=1}^{j_1} k_{+o}(x_n, z_n) \prod_{n=1}^{j_2} k_{-o}(x_{j_1+n}, z'_{j'_1+n}) \\
&\cdot \left[\prod_{n=1}^{j_3} \int_Q d\tau A(\tau)\, k_{+i}(x_{j_1+j_2+n}, \tau) \prod_{n=1}^{j'_4} \int_Q d\tau A(\tau)\, k_{-i}(x'_{j'_1+j'_2+j'_3+n}, \tau) \right] \smile \\
&\smile \int \left(\prod_{j=1}^{\alpha+j_3+j'_4} dy_j A^*(y_j)\, c_+(y_j) \prod_{j=1}^{\alpha'+j'_3+j_4} dy'_j A(y'_j)\, c_-(y'_j) \right) \qquad (5.24) \\
&\cdot \prod_{n=1}^{j_1+j'_2} c_+(z_n) \prod_{n=1}^{j'_1+j_2} c_-(z'_n) \smile \\
&\smile \left[\prod_{n=1}^{j'_3} \int_Q d\tau A^*(\tau)\, k_{+i}(x'_{j'_1+j'_2+n}, \tau) \prod_{n=1}^{j_4} \int_Q d\tau A^*(\tau)\, k_{-i}(x_{j_1+j_2+j_3+n}, \tau) \right] \\
&\cdot \prod_{n=1}^{j'_1} k_{+o}(x'_n, z'_n) \prod_{n=1}^{j'_2} k_{-o}(x'_{j'_1+n}, z_{j_1+n}) \\
&\cdot ((\alpha+j_3+j'_4)!\,(\alpha'+j'_3+j_4)!)^{-1}\, \alpha!\,\alpha'!\,j_3!\,j'_3!\,j_4!\,j'_4!\,.
\end{aligned}
$$

The last combinatorial factor compensates for the number of contractions.

We reorder terms, setting

$$n = j_1 + j'_2,\ n' = j'_1 + j_2,\ \beta = \alpha + j_3 + j'_4,\ \beta' = \alpha' + j'_3 + j_4\,. \qquad (5.25)$$

Then (5.24) simplifies to

$$
\begin{aligned}
&c_{mm'}(x_1, \ldots, x_m; x'_1, \ldots, x'_{m'}) \\
&= \sum_{n+n' \leqq m+m'} \sum_{\substack{j_1+j'_2=n \\ j'_1+j_2=n' \\ j_1+j_2+j_3+j_4=m \\ j'_1+j'_2+j'_3+j'_4=m'}} \binom{n}{j_1} \binom{n'}{j'_1} \cdot \int_{z_i, z'_i \notin Q} dz_1 \ldots dz_n dz'_1 \ldots dz'_n \\
&\cdot \prod_{k=1}^{j_1} k_{+o}(x_k, z_k) \prod_{k=1}^{j_2} k_{-o}(x_{j_1+k}, z'_{j'_1+k}) \\
&\cdot \left[\prod_{k=1}^{j_3} \int_Q d\tau A(\tau)\, k_{+i}(x_{j_1+j_2+k}, \tau) \prod_{k=1}^{j'_4} \int_Q d\tau A(\tau)\, k_{-i}(x'_{j'_1+j'_2+j'_3+k}, \tau) \right] \smile \qquad (5.26) \\
&\smile c_{nn'}(z_1, \ldots, z_n; z'_1, \ldots, z'_{n'}) \smile \\
&\smile \left[\prod_{k=1}^{j'_3} \int_Q d\tau A^*(\tau)\, k_{+i}(x'_{j'_1+j'_2+k}, \tau) \prod_{k=1}^{j_4} \int_Q d\tau A^*(\tau)\, k_{-i}(x_{j_1+j_2+j_3+k}, \tau) \right] \\
&\cdot \prod_{k=1}^{j'_1} k_{+o}(x'_k, z'_k) \prod_{k=1}^{j'_2} k_{-o}(x'_{j'_1+k}, z_{j_1+k})\,.
\end{aligned}
$$

Two remarks are in order. So far, our calculations have been formal manipulations of power series. We shall show below the convergence of this series. The expansion (5.26) is a multi-linear expansion of $c_{mm'}$; indeed

$$c_{mm'} = L(\{c_{nn'}, n+n' \leqq m+m'\}), \tag{5.27}$$

where L is linear. Since expressions of the form (5.5) are finite linear combinations of expressions of the form (5.17), it follows from (5.27) that the expansion (5.26) is valid for the general case (5.5) and not only for (5.17). $C_{mm'}$ then denotes that part of $C \in \mathfrak{A}_0(B)$ in which exactly m (resp. m') particles are created (resp. annihilated) in the region $(\sim Q)$, and in which an indefinite number of particles is created and annihilated in Q.

VI. Convergence of the Particle Expansion of Operators in $\mathfrak{A}_0(B)$

In Sect. V, we defined an expansion (5.16)

$$C = \sum_{m,m'=0}^{\infty} C_{mm'}, \tag{6.1}$$

of a bounded operator $C \in \mathfrak{A}_0(B)$ in terms of unbounded operators $C_{mm'}$. These unbounded operators can be written as

$$\begin{aligned} C_{mm'} = \int_{\substack{x_i \notin Q \\ x_i' \notin Q}} & dx_1 \ldots dx_m dx_1' \ldots dx_{m'}' A^*(x_1) \ldots A^*(x_m) A(x_1') \ldots A(x_{m'}') \\ & c_{mm'}(x_1, \ldots, x_m; x_1', \ldots, x_{m'}') . \end{aligned} \tag{6.2}$$

We call $c_{mm'}(x_1, \ldots, x_m; x_1', \ldots, x_{m'}')$ the kernel of $C_{mm'}$; we shall see below that its value is a bounded operator which acts on $\mathscr{F}_Q$ (cf. also (5.19)).

It is useful to consider two Fock spaces $\mathscr{F}_Q$ and $\mathscr{F}_{\sim Q}$ as defined in (5.19) and to identify the Fock space $\mathscr{F}$ via a unitary transformation θ with $\mathscr{F}_{\sim Q} \otimes \mathscr{F}_Q$.

Let $(\mathscr{F}_{\sim Q})_n$, $(\mathscr{F}_Q)_n$ be the n-particle components of $\mathscr{F}_{\sim Q}$ and $\mathscr{F}_Q$ respectively. Then θ is defined by

$$\begin{aligned} & \theta : (\mathscr{F}_{\sim Q})_m \otimes (\mathscr{F}_Q)_n \to \mathscr{F}_{m+n}; \\ & \theta(\psi_m'' \otimes \psi_n'')(p_1, \ldots, p_{m+n}) = \binom{m+n}{m}^{1/2} \\ & \cdot \operatorname{sym} \psi_m''(p_1, \ldots, p_m) \psi_n''(p_{m+1}, \ldots, p_{m+n}) . \end{aligned} \tag{6.3}$$

Here, $\psi_m'' \in (\mathscr{F}_{\sim Q})_m$, $\psi_n'' \in (\mathscr{F}_Q)_n$ and sym is symmetrization in the $m+n$ variables.

Let now $\psi_m, \psi'_m \in \mathscr{F}_{m,Q} \equiv \theta((\mathscr{F}_{\sim Q})_m \otimes \mathscr{F}_Q)$. By the definition (6.1) and (6.2) of C_{mn}, we have

$$(\psi_0, C_{00}\psi'_0) = (\psi_0, C\psi'_0) \tag{6.4}$$

and

$$(\psi_m, C_{mn}\psi'_n) = (\psi_m, C\psi'_n) - \sum_{k=1}^{\min(m,n)} (\psi_m, C_{m-k,n-k}\psi'_n). \tag{6.5}$$

We now view $c_{mn}(x_1, \ldots, x_{m+n})$ as the kernel of an operator c_{mn} from $\mathscr{F}_{n,Q}$ to $\mathscr{F}_{m,Q}$ with norm $\|c_{mn}\|_Q$: For $\psi_m \in \mathscr{F}_{m,Q}$

$$\|c_{mn}\|_Q = \sup_{\substack{\|\psi_m\|=1 \\ \|\psi_n\|=1}} |(\psi_m, c_{mn}\psi_n)|.$$

From Glimm and Jaffe [12], we get

Lemma 6.1 ([12], Lemma 4.2).
Let $C \in \mathfrak{A}_0(B)$ and let c_{mn} be defined as above. Then

$$\|c_{mn}\|_Q \leqq 2^{m+n}\|C\|. \tag{6.6}$$

Proof. By definition of c_{mn},

$$\|c_{mn}\|_Q = \sup_{\substack{\|\psi_m\|=1 \\ \|\psi'_n\|=1}} |(\psi_m, C_{mn}\psi'_n)|(m!\,n!)^{-1/2}, \tag{6.7}$$

and also, for $0 \leqq k \leqq \min\{m, n\}$,

$$|(\psi_m, C_{m-k,n-k}\psi'_n)| \leqq \|\psi_m\|\;\|\psi'_n\|\;\|c_{m-k,n-k}\|_Q \left(\frac{m!}{k!}\frac{n!}{k!}\right)^{1/2}. \tag{6.8}$$

We now use (6.4) and (6.5) to get

$$\|c_{mn}\|_Q \leqq (m!\,n!)^{-1/2}\left(\|C\| + \sum_{k=1}^{\min\{m,n\}} \frac{(m!\,n!)^{1/2}}{k!}\|c_{m-k,n-k}\|_Q\right).$$

The assertion follows now by induction on $\min\{m, n\}$.

Our next step towards estimating $\omega_{mn\sigma}(C)$ is an estimate on the kernels $k_\pm$, defined in section V, (5.15). Due to the particular localization properties, they will be very well-behaved.
We describe properties of $\mu_\tau(x)$ and $k_\pm(x, y)$ in the following

Lemma 6.2. *Let $\mu_\tau(x)$ be the kernel of the operator defined in* (5.2), *let $\tau > -2$. Then*

$$\mu_\tau(x) = 2^{1/2}\Gamma(-\tau/2)^{-1}\left(\frac{2m_0}{|x|}\right)^{\frac{\tau+2}{2}} K_{-\frac{(\tau+2)}{2}}(m_0|x|), \quad \text{if} \quad x \neq 0, \tag{6.9}$$

where K_ν is a modified Bessel function, and

$$\mu_\tau(x) \quad \text{is} \quad C^\infty \quad \text{for} \quad x \in \mathbb{R}^2 - \{0\}. \tag{6.10}$$

For all $n_1, n_2 \in \mathbb{N} \cup \{0\}$, $x_0 > 0$, $\tau > -2$ one has

$$\left| \frac{d^{n_1+n_2}}{dx_1^{n_1}\, dx_2^{n_2}} \mu_\tau(x_1, x_2) \right| < C_{n_1+n_2, x_0, \tau} \exp(-m_0|x|), \quad \text{if} \quad |x| \geqq x_0. \tag{6.11}$$

Here

$$|x| = (x_1^2 + x_2^2)^{1/2}. \tag{6.12}$$

Let $k_\pm(x, y)$ be the kernel of the operator $K_\pm$ defined in Eq. (5.15) and let $x \notin Q$.

Then $k_\pm(x, y)$ is a C^∞ function on $(\sim Q) \times \mathbb{R}^2$. (6.13)

There is a constant K such that for $x \notin Q$,

$$|k_\pm(x, y)| \leqq K \exp(-m_0|x| - m_0|y|). \tag{6.14}$$

Proof. Eq. (6.9) is obtained by a simple combination of formulas (1.3.7) and (1.13.45) in Bateman [2], if $\tau \neq 0, 2, 4, \ldots$. If $\tau = 0, 2, 4, \ldots$, $\mu_\tau(x) = 0$ if $x \neq 0$. Eq. (6.10) and Eq. (6.11) follow from the properties of the modified Bessel functions K_ν (see e.g. Jahnke-Emde [17]). To prove (6.13), we use the fact that $B_{d/4}$ is a fixed compact region, and we use (6.11) with $x_0 + d/12$. We consider only the term $\mu_{-1/2}\xi\mu_{+1/2}$, the estimate for the other term of $K_\pm$ is similar. Now

$$(\mu_{-1/2}\xi\mu_{+1/2})(x, y) = \int \mu_{-1/2}(x - z)\xi(z)\,\mu_{+1/2}(z - y)\,dz\,.$$

Since $\xi(z) = 0$ if $z \notin B_{d/4}$, and since $x \notin Q$, $\mu_{-1/2}(x - z)\xi(z)$ is a C^∞ function of x and z, and therefore $(\mu_{-1/2}\xi\mu_{+1/2})(x, z)$ is a C^∞ function of x and z on $(\sim Q) \times \mathbb{R}^2$, because $\mu_{+1/2}(z)$ has a pole which is of finite order and falls off otherwise.

To prove (6.14), we consider $(x, y) \in (\sim Q) \times (\sim Q)$ and $(x, y) \in (\sim Q) \times Q$ separately. On $(\sim Q) \times (\sim Q)$, by (6.11) and (5.10),

$$\begin{aligned} |k_\pm(x, y)| &\leqq C_{0, d/2, -1/2} \exp(-m_0 \operatorname{dist}(x, B_{d/2})) \cdot \sup_z |\xi(z)| \\ &\cdot C_{0, d/2, +1/2} \exp(-m_0 \operatorname{dist}(y, B_{d/2})) \leqq \text{const} \exp(-m_0(|x| + |y|)), \end{aligned}$$

since $B_{d/2}$ is compact. On $(\sim Q) \times Q$, by (6.11) and (5.10), we get a bound const $\exp(-m_0|x|)$, by integration by parts, and since $y \in Q$, and Q is compact; the assertion (6.13) follows, the lemma is proved.

We now define a decomposition of $C_{mm'}$ which is designed to make extensive use of the properties of $k_{\pm io}$ derived in Lemma 6.2.

We rewrite Eq. (5.26).

$$C_{mm'} = \sum_{n+n' \leqq m+m'} \sum_{\substack{j_1+j_2'=n \\ j_1'+j_2=n' \\ j_1+j_2+j_3+j_4=m \\ j_1'+j_2'+j_3'+j_4'=m'}} \binom{n}{j_1}\binom{n'}{j_1'}$$

$$\cdot \int_{x_i, x_i' \notin Q} dx_1 \ldots dx_m dx_1' \ldots dx_m'$$

$$\prod_{k=1}^{m} A^*(x_k) \left[\prod_{k=1}^{j_3} \int_Q d\tau\, k_{+i}(x_{j_1+j_2+k}, \tau) A(\tau) \right.$$

$$\left. \cdot \prod_{k=1}^{j_4'} \int_Q d\tau\, k_{-i}(x'_{j_1'+j_2'+j_3'+k}, \tau) A(\tau) \right] \smile$$

$$\smile \left\{ \int_{z_i, z_i' \notin Q} dz_1 \ldots dz_n dz_1' \ldots dz_{n'}' \prod_{k=1}^{j_1} k_{+o}(x_k, z_k) \prod_{k=1}^{j_2} k_{-o}(x_{j_1+k}, z'_{j_1'+k}) \right.$$

$$\left. \cdot\, c_{nn'}(z_1, \ldots, z_n; z_1', \ldots, z_{n'}') \prod_{k=1}^{j_1'} k_{+o}(x_k', z_k') \prod_{k=1}^{j_2'} k_{-o}(x'_{j_1'+k}, z_{j_1+k}) \right\} \smile$$

$$\smile \left[\prod_{k=1}^{j_3'} \int_Q d\tau\, k_{+i}(x'_{j_1'+j_2'+k}, \tau) A^*(\tau) \right. \tag{6.15}$$

$$\left. \cdot \prod_{k=1}^{j_4} \int_Q d\tau\, k_{-i}(x_{j_1+j_2+j_3+k}, z) A^*(\tau) \right] \prod_{k=1}^{m'} A(x_k') .$$

We shall estimate later $\omega_{mn\sigma}(C_{mm'})$ by estimating each term of the sum (6.15).

We now use the following technical device. If D is a bounded operator and if $\mu^\varepsilon f_i$, $i=1, \ldots, n$ is square integrable then at least as bilinear form, on $\mathscr{D}(A(f_i)) \times \mathscr{D}(A(f_i)) \subset \mathscr{F}(g) \times \mathscr{F}(g)$,

$$A(f_i) \smile D = A(f_i)\, D - D\, A(f_i)$$

and it is straightforward to calculate that

$$\prod_{j=1}^{k} A(f_j) \smile D \smile \prod_{j=k+1}^{n} A^*(f_j)$$

$$= [\ldots \underbrace{[A(f_k), [\ldots [A(f_1), D] \ldots]}_{k \text{ commutators}}, A^*(f_{k+1})], \ldots], A^*(f_n)] . \tag{6.16}$$

This means that [ ]$\smile${ }$\smile$[ ] of (6.15) can be written as a sum of terms of the form

$$\pm \int_{x_i, x_i' \notin Q} dx_1 \dots dx_m \, dx_1' \dots dx_{m'}'$$
$$\cdot \prod_{k=1}^{m} A^*(x_k) \prod_{k=1}^{j_3'''} \int_Q d\tau \, k_{+i}(x_{j_1'+j_2'+k}', \tau) \, A^*(\tau)$$
$$\cdot \prod_{k=1}^{j_4''} \int_Q d\tau \, k_{-i}(x_{j_1+j_2+j_3+k}, \tau) \, A^*(\tau)$$
$$\cdot \prod_{k=j_3''+1}^{j_3} \int_Q d\tau \, k_{+i}(x_{j_1+j_2+k}, \tau) \, A(\tau) \cdot \prod_{k=j_4'''+1}^{j_4'} \int_Q d\tau \, k_{-i}(x_{j_1'+j_2'+j_3'+k}', \tau) A(\tau)$$
$$\cdot \{ \ \} \tag{6.17}$$
$$\cdot \prod_{k=1}^{j_3''} \int_Q d\tau \, k_{+i}(x_{j_1+j_2+k}, \tau) \, A(\tau) \prod_{k=1}^{j_4'''} \int_Q d\tau \, k_{-i}(x_{j_1'+j_2'+j_3'+k}', \tau) \, A(\tau)$$
$$\cdot \prod_{k=j_3'''+1}^{j_3'} \int_Q d\tau \, k_{+i}(x_{j_1'+j_2'+k}', \tau) \, A^*(\tau) \prod_{k=j_4''+1}^{j_4} \int_Q d\tau \, k_{-i}(x_{j_1+j_2+j_3+k}, \tau) \, A^*(\tau)$$
$$\cdot \prod_{k=1}^{m'} A(x_k') \, .$$

$C_{mm'}$ is a sum of at most $8^{m+m'}$ terms of the form (6.17).
As a next step, we Wick-order the A's and the A^*'s to the right of { } in (6.17). The number of terms thus obtained is bounded by $(m+m')!$. We have thus rewritten $C_{mm'}$ as a sum of at most $(m+m')! \, 8^{m+m'}$ terms which are of the form

$$\pm \int dx_1 \dots dx_{j_1+j_2'} \, dx_1' \dots dx_{j_1'+j_2}'$$
$$\cdot \prod_{k=1}^{j_1+j_2'} A^*(x_k) \, \mathscr{W}((Y_\square)^p \{.\}) \prod_{k=1}^{j_1'+j_2} A(x_k') \, , \tag{6.18}$$

where $\{.\}$ is as in (6.15), each $Y_\square$ is one of the quadratic creation-annihilation operators described below, $p \leqq m + m' - (j_1 + j_2 + j_1' + j_2')$ and $\mathscr{W}$ is an ordering of creation and annihilation operators to the left and the right of $c_{nn'}$ which are inside the { } brackets, subject to the following rules:

1. All creation and annihilation operators to the left of $c_{nn'}$ are Wick-ordered.

2. All creation and annihilation operators to the right of $c_{nn'}$ are Wick-ordered.

By construction, Y_4, Y_6 does not appear to the left of $c_{nn'}$, and Y_3, Y_5 does not appear to the right of $c_{nn'}$.

And this is the list of operators Y_i which can occur as a $Y_{\square}$. (The variables x, x' are always integrated over $\sim Q$, τ is integrated over Q).

$$
\begin{aligned}
Y_1 &= \int A^*(x)\, A(\tau)\, k_{+i}(x,\tau)\, dx\, d\tau\,, \\
Y_2 &= \int A^*(\tau)\, A(x)\, k_{+i}(x,\tau)\, dx\, d\tau\,, \\
Y_3 &= \int A^*(x)\, A^*(\tau)\, k_{-i}(x,\tau)\, dx\, d\tau\,, \\
Y_4 &= \int A(x)\, A(\tau)\, k_{-i}(x,\tau)\, dx\, d\tau\,, \\
Y_5 &= \int A^*(x)\, A^*(x')\, k_{+i}(x,\tau)\, k_{-i}(x',\tau)\, dx dx' d\tau\,, \\
Y_6 &= \int A(x)\, A(x')\, k_{+i}(x,\tau)\, k_{-i}(x',\tau)\, dx dx' d\tau\,, \\
Y_7 &= \int A^*(x)\, A(x')\, k_{+i}(x,\tau)\, k_{+i}(x',\tau)\, dx dx' d\tau\,, \\
Y_8 &= \int A^*(x)\, A(x')\, k_{-i}(x,\tau)\, k_{-i}(x',\tau)\, dx dx' d\tau\,.
\end{aligned}
$$

One can draw a graphical picture to show the allowed positions of the Y's after $\mathscr{W}$-ordering. We let $\underline{A}^* = \int A^*(x) \ldots$, $A^* = \int A^*\ (\tau) \ldots$, etc. Then the allowed terms are:

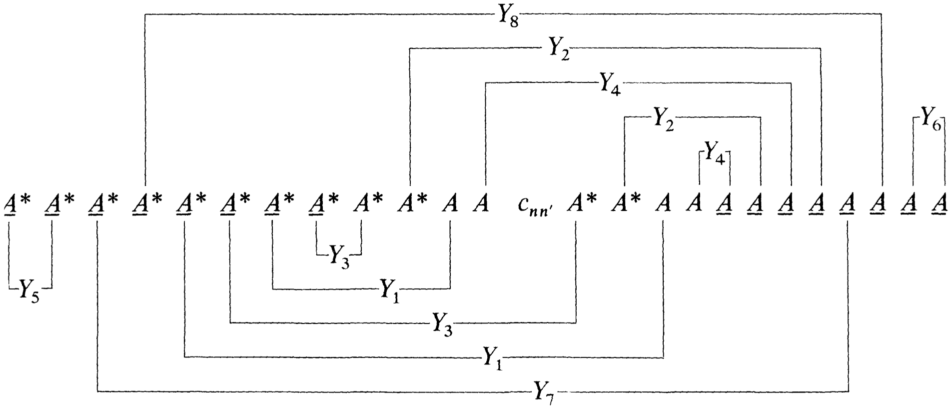

Fig. 3

We note that the kernels f of Y_i, $i=1,\ldots,8$ and of $K_{\pm {i \atop o}}$ are of the form $f(x,y)=\chi_1(x)\, f'(x,y)\, \chi_2(y)$, where $f'(x,y)$ is C^∞ and falls off exponentially at infinity with all its derivatives. χ_α, $\alpha=1,2$ is the characteristic function of one of the regions Q, $\sim Q$ or $\mathbb{R}^2$.

We expand all Y_i's and all expressions $\int A^{(*)}(x)\, k_{\pm {i \atop o}}(x,z)$ as sums of operators $:A^{(*)}(f'_{i_1}\cdot\chi_1)\, A^{(*)}(f'_{i_2}\cdot\chi_2):$ and $A^{(*)}(f'_{i_1}\cdot\chi_1)\, f'_{i_2}(z)\, \chi_2(z)$ respectively. Here, f'_i is the i^{th} Hermite function and $A(f)=\int A(x)\, f(x)\, dx$. We shall write $f_i \equiv f'_i \cdot \chi$ without keeping track of the support of χ (which is always one of the regions Q, $\sim Q$, or $\mathbb{R}^2$).

Then (6.18) takes the form

$$\sum_{\substack{i_1,\ldots,i_q=0\\ j_1,\ldots,j_{n+n'}=0}}^{\infty} \alpha_{i_1\ldots i_q j_1\ldots j_{n+n'}} \prod_{j=1}^{a_1} A^*(f_{i_j}) \prod_{j=a_1+1}^{a_1+a_2} A(f_{i_j})$$
$$\cdot \left\{ \int \prod_{k=1}^{n} f_{j_k}(x_k)dx_k \prod_{k=1}^{n'} f_{j_{k+n}}(x'_k)dx'_k\, c_{nn'}(x_1,\ldots,x_n,x'_1,\ldots,x'_{n'}) \right\} \tag{6.19}$$
$$\cdot \prod_{j=a_1+a_2+1}^{a_1+a_2+a_3} A^*(f_{i_j}) \prod_{j=a_1+a_2+a_3+1}^{q} A(f_{i_j}).$$

By construction, $q \leqq 2\ (m+m')$ and (by bounding the maximal number of commutators in going from (6.17) to (6.18)) $q \geqq m+m'$. The smoothness of the $k_{\pm}$ will insure the convergence of (6.19).

Our next theorem uses only the decomposition (5.18) and none of the more sophisticated decompositions derived later. Let $C \in \mathfrak{A}_0(B)$ and let $C_{mm'}$ be defined as in (5.16). Let $C_{mm'} = \sum_{\alpha,\alpha'=0}^{\infty} C_{mm',\alpha\alpha'}$ be the decomposition of $C_{mm'}$ into Wick monomials as described by (5.18), that is

$$C_{mm',\alpha\alpha'} = \sum_{j=1}^{J} \beta_j (m!\, m'!\, \alpha!\, \alpha'!)^{-1} A^*(\mathring{\chi}_{\sim Q} c_{j+})^m A^*(\mathring{\chi}_Q c_{j+})^\alpha$$
$$\cdot A(\mathring{\chi}_Q c_{j-})^{\alpha'} A(\mathring{\chi}_{\sim Q} c_{j-})^{m'}. \tag{6.20}$$

Let $\psi'_1, \psi'_2 \in \mathscr{D}_0$, $r_1, r_2 \in \mathbb{N}$. For $n \in \mathbb{N}$, $\sigma = 2^s$, $s \in \mathbb{N}$, we define

$$\omega_{n\sigma}(C) = (\hat{T}_{r_1 ns}\psi'_1, C\, \hat{T}_{r_2 ns}\psi'_2)\exp(-\Lambda_{n\sigma}). \tag{6.21}$$

Our control over such expressions is collected in

Theorem 6.3. *For fixed $\psi'_1, \psi'_2, r_1, r_2$ and for every $C \in \mathfrak{A}_0(B)$ there exists a constant K such that for all M, M', n, σ,*

$$\sum_{m,m'=0}^{M} \sum_{\alpha,\alpha'=0}^{M'} |\omega_{n\sigma}(C_{mm',\alpha\alpha'})| \leqq K.$$

There exists a constant $M(C,\varepsilon) = M(C,\varepsilon,\psi'_1,\psi'_2,r_1,r_2)$ such that for all $M > M(C,\varepsilon)$,

$$\left| \omega_{n\sigma}(C) - \omega_{n\sigma}\left(\sum_{m,m',\alpha,\alpha'=0}^{M} C_{mm',\alpha\alpha'} \right) \right| < \varepsilon$$

uniformly in n and σ.

Proof. It is evidently sufficient to prove the assertion for a C of the form

$$C = \exp A^*(c_+) \exp A(c_-),$$

see also (5.5). We use Theorem 8.9 and we get

$$\omega_{n\sigma}(C_{mm'\alpha\alpha'}) \leqq |\beta| C_1 C_2^{-(m+m'+\alpha+\alpha')(1+C_3/2)}$$

$$\cdot \|c_+\|_{\sim}^{m+\alpha} \|c_-\|_{\sim}^{m'+\alpha'} \quad \text{with} \quad C_1, C_3 > 0, C_2 > 1\,, \tag{6.22}$$

where (as will be seen in Sect. VIII) $\|c_+\|_\sim$ and $\|c_-\|_\sim$ are finite. Note that this bound is not uniform in $\|C\|$. It is just one of the main problems of this paper to show that $|\omega_{n\sigma}(C_{mm'}) - \omega_{n'\sigma}(C_{mm'})| < \varepsilon \|C\|$ as $n, n' > n_0$. The assertions of the theorem follow now from (6.22) and from the fact that for finite σ, only a finite number of $C_{mm'\alpha\alpha'}$'s give non zero contributions to $\omega_{n\sigma}(C_{mm'\alpha\alpha'})$. This theorem also proves the validity of Eq. (5.24) and Eq. (5.26) since they are obtained by splitting each term in (5.18) into a finite number of terms.

We come back to the *proof of Theorem 4.3*. In section IV, we have already proved the existence of the following limit: Let $\omega_{mn\sigma}$ be defined by (4.10). For $n \geqq m$, by using the definition (6.20), we find that with $\psi'_\alpha = T_{r_\alpha n \tau} \psi_\alpha$, $\alpha = 1, 2$, $\tau = 2^m$, the functionals $\omega_{mn\sigma}$ and $\omega_{n\sigma}$ are equal. So Theorem 6.3 shows that

$$\left| \omega_{mn\sigma}(C) - \omega_{mn\sigma}\left(\sum_{k,k',\alpha,\alpha'=0}^{M} C_{kk',\alpha\alpha'} \right) \right| < \varepsilon\,, \tag{6.23}$$

for large M. By applying the arguments for the proof of the existence of $\lim_{\sigma\to\infty} \omega_{mn\sigma}(\mathbf{1})$ to expressions of the form

$$\omega_{mn\sigma}\left(\prod_{j=1}^{k} A^*(f_j) \prod_{j=1}^{k'} A(f'_j) \right), \text{ one can easily see that}$$

$$\lim_{\sigma\to\infty} \omega_{mn\sigma}(C_{kk'\alpha\alpha'}) \quad \text{exists,} \tag{6.24}$$

since the kernels $c_\pm$ fall off fast enough. The existence of $\lim_{\sigma\to\infty} \omega_{mn\sigma}(C)$ follows then at once from (6.23) and (6.24).

We now come to an estimate which makes use of the expansion (6.19).

Theorem 6.4. *Let* $C \in \mathfrak{A}_0(B)$. *Then the decomposition* (6.19) *of* $C_{mm'}$ *is bounded in the sense that*

$$\sum_{i_1,\ldots,i_{q+n+n'}} |\alpha_{i_1\ldots i_{q+n+n'}}|$$

$$\cdot \left| \omega_{n\sigma}\left(\prod_{j=1}^{a_1} A^*(f_{i_j}) \prod_{j=a_1+1}^{a_1+a_2} A(f_{i_j}) \{.\} \prod_{j=a_1+a_2+1}^{a_1+a_2+a_3} A^*(f_{i_j}) \prod_{j=a_1+a_2+a_3+1}^{q} A(f_{i_j}) \right) \right|$$

$$\leqq K \cdot \|C\| \, \gamma^{-(m+m')^\tau}, \quad \textit{with} \quad \gamma, \tau > 1\,. \tag{6.25}$$

Proof. By construction of (6.19), there is to every A to the left of $\{.\}$ at least one A to the right of $\{.\}$ or an A^* to the left of $\{.\}$; a symmetric statement holds for A^*: for every A^* to the right of $\{.\}$ there is at least one A^* to the left of $\{.\}$ or an A to the right of $\{.\}$, see also Fig. 3. Since there are q operators A and A^*, and by the above argument, at least $q/2$ of the factors $A^{(*)}$ are either an A to the right of $\{.\}$ or an A^* to the left of $\{.\}$. Note that $q \geqq (m+m')$. We use the Schwarz inequality and apply Theorem 8.9. Then we get

$$\left|\omega_{n\sigma}\left(\prod_{j=1}^{a_1} A^*(f_{i_j}) \prod_{j=a_1+1}^{a_1+a_2} A(f_{i_j})\{.\} \prod_{j=a_1+a_2+1}^{a_1+a_2+a_3} A^*(f_{i_j}) \prod_{j=a_1+a_2+a_3+1}^{q} A(f_{i_j})\right)\right|$$

$$\leqq \left\| \prod_{j=a_1+1}^{a_1+a_2} A^*(f_{i_j}) \prod_{j=1}^{a_1} A(f_{i_j})\, \hat{T}_{mns}\psi'_1 \right\| e^{-\Lambda_{n\sigma}/2}$$

$$\cdot \left\| \prod_{j=a_1+a_2+1}^{a_1+a_2+a_3} A^*(f_{i_j}) \prod_{j=a_1+a_2+a_3+1}^{q} A(f_{i_j})\, \hat{T}_{mns}\psi'_2 \right\| e^{-\Lambda_{n\sigma}/2}$$

$$\cdot \|\{\ \}\|_Q .$$

Here $\|\{\ \}\|_Q \leqq \|c_{nn'}\|_Q \prod_{k=1}^{n+n'} \|f_{j_k}\|_2$, and this bound comes from the observation that

$$\int \prod_{k=1}^{n+n'} f_{j_k}(x_k)\, c_{nn'}(x_1, \ldots, x_{n+n'})\, dx_1 \ldots dx_{n+n'}$$

defines in a natural way an operator from $\mathscr{F}_{0,Q}$ to $\mathscr{F}_{0,Q}$, and hence an operator from $\mathscr{F}_Q$ to $\mathscr{F}_Q$, or from $\mathscr{F}$ to $\mathscr{F}$ (by tensoring with $\mathbf{1}$). This explains the above bound.

By Lemma 6.1 and the fact that $f_{j_i} = f'_{j_i} \cdot \chi$, where the f'_{j_i} are Hermite functions and χ a characteristic function, we get

$$\|c_{nn'}\|_Q \prod_{k=1}^{n+n'} \|f'_{j_k}\|_2 \leqq \|C\| \cdot K^{m+m'} \prod_{k=1}^{n+n'} (j_k)^{3/2} .$$

Assume that ψ'_1 has at most r particles. We want to apply Theorem 8.9 to

$$\left\| \prod_{j=a_1+1}^{a_1+a_2} \mathrm{A}^*(f_{i_j}) \prod_{j=1}^{a_1} A(f_{i_j})\, \hat{T}_{mns}\psi'_1 \right\|^2 \exp - \Lambda_{n\sigma}$$

$$= \omega_{\tau\sigma}\left(\prod_{j=1}^{a_1} A^*(f_{i_j}) \prod_{j=a_1+1}^{a_1+a_2} A(f_{i_j}) \prod_{j=a_1+1}^{a_1+a_2} A^*(f_{i_j}) \prod_{j=1}^{a_1} A(f_{i_j})\right)$$

$$= \omega_{\tau\sigma}(R^* R) .$$

We expand $R^* R$ as a sum of Wick ordered terms, $R^* R = \Sigma\, W_{ll'}$, and it is easy to see that the number of terms in the sum is bounded by $a_2!$. The kernel $\bar{w}_{ll'}$ of $W_{ll'}$ is of the form

$$\bar{w}_{ll'} = \text{const} \prod_k f_{i_k}(x_k) \prod_{k'} f_{i_{k'}}(x_{k'}) ,$$

where k, k' run over $1, \dots, a_1$ and some subset of $\{a_1+1, \dots, a_1+a_2\}$. The constant is of the form $\Pi'(f_{i_j}, f_{i_{j'}})$, where the product runs over a pairing of the indices which do not occur in Π_k and $\Pi_{k'}$ respectively. We bound the constant by

$$|\Pi'(f_{i_j}, f_{i_{j'}})| \leqq \Pi' \|f_{i_j}\|_2 \|f_{i_{j'}}\|_2 \leqq \Pi' \|f'_{i_j}\|_2 \|f'_{i_{j'}}\|_2 \leqq \Pi' \|f'_{i_j}\|_\sim \|f'_{i_{j'}}\|_\sim .$$

We now apply Theorem 8.9 to $\omega_{\tau\sigma}(W_{ll'})$, using the kernel

$$\bar{w}_{ll'} = \prod_k \chi^{(\beta_k)} \prod_{k'} \chi^{(\beta_{k'})} \; w_{ll'} = \prod_k \chi^{(\beta_k)} f'_{i_k} \prod_{k'} \chi^{(\beta_{k'})} f'_{i_{k'}} .$$

Note that $l > a_1$ and $l' > a_1$. With these preliminaries, we find, using in particular Eqs. (8.24) and (8.25), a bound

$$\left\| \prod_{j=a_1+1}^{a_1+a_2} A^*(f_{i_j}) \prod_{j=1}^{a_1} A(f_{i_j}) \, \hat{T}_{mns} \psi'_1 \right\|^2 \exp - A_{n\sigma}$$

$$\leqq C(\psi'_1) \sum_{p,p'=0}^{\infty} ((4p+a_1+a_2)!\,(4p'+a_1+a_2)!)^{1/2} \prod_{j=1}^{a_1+a_2} \|f'_{i_j}\|^2_\sim$$

$$\cdot K^{p+p'+a_1+a_2} \,. \sum_{\substack{j_1,\dots,j_p \in J_{a_1-r} \\ j'_1,\dots,j'_p \in J_{a_1-r}}} \prod_{j=1}^{p} \gamma^{-j_k} \prod_{j=1}^{p'} \gamma^{-j'_k}$$

$$\leqq C(\psi'_1) \sum_{p,p'=0}^{\infty} ((4p+a_1+a_2)!\,(4p'+a_1+a_2)!)^{1/2} \prod_{j=1}^{a_1+a_2} \|f'_{i_j}\|^2_\sim$$

$$\cdot K^{p+q+a_1+a_2} \,.\, C_1^{-a_1^\tau - p^\tau - p'^\tau} .$$

Here $j_1, \dots, j_p \in J_a \equiv \{j_i | j_i \geqq i^{1/2}, p \geqq \max(a, 0)\}$, and γ, C_1, $\tau > 1$. We use the bound $\|f'_i\|_\sim \leqq K i^3$ for the $i-$th Hermite function. Summing over p and p', we get a bound

$$C(\psi'_1) \cdot \prod_{j=1}^{a_1+a_2} (i_j)^6 \, C_1^{-a_1^\tau} (a_1+a_2)! \cdot K^{a_1+a_2} .$$

By the Schwarz inequality, the L.H.S. of (6.25) is therefore bounded by

$$C(\psi'_1) \sum_{i_1,\dots,i_{q+n+n'}} |\alpha_{i_1 \dots i_{q+n+n'}}| \prod_{j=1}^{q+n+n'} (i_j)^6 \|C\| \cdot K^{m+m'} \tag{6.26}$$

$$\cdot C_1^{-1/2(a_1^\tau + a_3^\tau)} (a_1+a_2+a_3+a_4)! ,$$

where $a_1 + \dots + a_4 = q$. By the remarks made at the beginning of this proof, $a_1 + a_3 \geqq q/2$ and $q \geqq m + m'$, and so we get the bound

$$(6.26) \leqq C(\psi'_1) \cdot C_3^{-(m+m')^\tau} \|C\| , \qquad \text{for some } C_3 > 1, \tau > 1 ,$$

and where we have used $n+n' \leqq m+m'$ and

$$\sum_{i_1,\dots,i_{q+n+n'}=0}^{\infty} |\alpha_{i_1\dots i_{q+n+n'}}| \prod_{j=1}^{q+n+n'} (i_j)^6 \leqq K^{q+n+n'} \leqq K^{3(m+m')},$$

since the Hermite expansion $\Sigma\, \alpha_{i_1\dots i_k} f'_{i_1}(x_1)\dots f'_{i_k}(x_k)$ of a smooth function f has the property that

$$\Sigma |\alpha_{i_1\dots i_k}| \prod_{l=1}^{k} i_l^N < \|f\|_s \quad \text{(see [13])},$$

with a Schwartz-norm $\| \ \|_s$. Theorem 6.4 is proved.

VII. Proof of Theorem 4.4

We shall prove in this section the bound

Theorem 7.1. *For any $\psi_1, \psi_2 \in \mathscr{D}_0$, $r_1, r_2 \in \mathbb{N}$ and every $\varepsilon > 0$ there exists a constant $N = N(\varepsilon, \psi_1, \psi_2, r_1, r_2)$ such that for all $t', t > N$, uniformly in s, with $\sigma = 2^s$*

$$\begin{aligned} &|(\hat{T}_{rts}\psi_1, C_{mm'} \hat{T}_{rts}\psi_2) \exp(-\Lambda_{t\sigma}) \\ &- (\hat{T}_{rt's}\psi_1, C_{mm'} \hat{T}_{rt's}\psi_2) \exp(-\Lambda_{t'\sigma})| \leqq \varepsilon \|C\| \cdot \gamma^{-m-m'} \end{aligned} \tag{7.1}$$

for all $C \in \mathfrak{A}_0(B)$, with $\gamma > 1$.

It is immediate (sum over m, m', and use Theorem 6.3) that Theorem 7.1 proves Theorem 4.4.

Proof of Theorem 7.1. We shall use a partial Wick expansion (in $\sim Q$) of (7.1). By our lengthy construction, $V_{R_j}^{(*)}$ is an operator on $\mathscr{F}_{\sim Q}$, and we can therefore expand in terms of contractions between the $V_{R_j}^{(*)}$ and the other "legs" in $\sim Q$ coming from $C_{mm'}$ (only $m+m'$ legs!) and from $V_{Q_j}^{(*)}$ (at most 3 legs in $\sim Q$!) or from the vectors ψ_1, ψ_2 (compact support in momentum space!). The above remarks should indicate why this expansion should nicely converge "on $\mathscr{F}_{\sim Q}$". Our elaborations in Sects. V and VI have shown how convergence is enforced "on $\mathscr{F}_Q$".

We now use a variable n, which we choose very large, and we assume $t, t' > n$. For a given ε in (7.1) we shall find a minimal n for which the proof will furnish the required bound, and that n will define $N(\varepsilon, \psi_1, \psi_2, r_1, r_2)$. We apply the term "$R_n$-skeleton" to a Wick term whose graph contains no $V_{R_j}^* \smile_4 V_{R_i}$ components, $i \geqq n, j \geqq n$. The combinatorial argument of Sect. IV, Eq. (4.16) shows that one can decompose the lefthand side of (7.1) into a sum of R_n-skeletons:

Let $n < t < t'$, let $V^{(*)}_{R_j,t} = V'^{(*)}_{R_j}$ if $j < t$, $V^{(*)}_{R_j,t} = V^{(*)}_{R_j}$ if $j \geqq t$. Then

$$\begin{aligned}
&(\hat{T}_{r_1ts}\psi_1, C_{mm'}\hat{T}_{r_2ts}\psi_2)\exp - \Lambda_{t\sigma} - (\hat{T}_{r_1t's}\psi_1, C_{mm'}\hat{T}_{r_2t's}\psi_2)\exp - \Lambda_{t'\sigma} \\
&= \Sigma'\left(\psi_1, \prod_{j=r_1}^{s-1}\left(\frac{V^{p_j}_{R_j,t}}{p_j!}\right)^*\left(\prod_{j=r_1}^{s-1}\exp V_{Q_j}\right)^* C_{mm'}\left(\prod_{j=r_2}^{s-1}\exp V_{Q_j}\right)\right. \\
&\quad\left.\cdot\left(\prod_{j=r_2}^{s-1}\frac{V^{p'_j}_{R_j,t}}{p'_j!}\right)\psi_2\right)_{R_n\text{-skel}}\Sigma''\left\{\prod_{i,j=n}^{s-1}\frac{\Lambda^{k_{ij}}_{ij,1}}{k_{ij}!}\right\}\exp - \Lambda_{t\sigma} \\
&\quad - \Sigma'\left(\psi_1, \prod_{j=r_1}^{s-1}\left(\frac{V^{p_j}_{R_j,t'}}{p_j!}\right)^*\left(\prod_{j=r_1}^{s-1}\exp V_{Q_j}\right)^* C_{mm'}\left(\prod_{j=r_2}^{s-1}\exp V_{Q_j}\right)\right. \\
&\quad\left.\cdot\left(\prod_{j=r_2}^{s-1}\frac{V^{p'_j}_{R_j,t'}}{p'_j!}\right)\psi_2\right)_{R_n\text{-skel}}\Sigma''\left\{\prod_{i,j=n}^{s-1}\frac{\Lambda'^{k_{ij}}_{ij,1}}{k_{ij}!}\right\}\exp - \Lambda_{t'\sigma}\,.
\end{aligned} \tag{7.2}$$

Here Σ' runs over the set $\{0 \leqq p_j \leqq j, 0 \leqq p'_j \leqq j\}$ and Σ'' runs over $\left\{k_{ij}; 0 \leqq \sum_i k_{ij} \leqq j - p'_j,\ 0 \leqq \sum_j k_{ij} \leqq i - p_i\right\}$. We write (7.2) as a sum of two terms $I_{1,n} + I_{2,n}$, similar to the proof of Theorem 4.3. In $I_{1,n}$, $p_j, p'_j = 0$ for all $j \geqq n$; $I_{2,n}$ is the remainder.

For the bound on term $I_{1,n}$ we bound first

$$\begin{aligned}
\Delta(n) = \Bigg|\Sigma''\left\{\prod_{i,j=n}^{s-1}\left(\frac{\Lambda^{k_{ij}}_{ij,1}}{k_{ij}!}\right)\exp(-\Lambda_{ij,1})\right\} \\
- \Sigma''\left\{\prod_{i,j=n}^{s-1}\left(\frac{\Lambda'^{k_{ij}}_{ij,1}}{k_{ij}!}\right)\exp(-\Lambda'_{ij,1})\right\}\Bigg|\,.
\end{aligned} \tag{7.3}$$

Here Σ'' runs over the set

$$\left\{k_{ij}, \sum_i k_{ij} \leqq j, \sum_j k_{ij} \leqq i\right\}, \tag{7.4}$$

and we show

$$\lim_{n\to\infty}\Sigma''\left\{\prod_{i,j=n}^{s-1}\left(\frac{\Lambda^{k_{ij}}_{ij,1}}{k_{ij}!}\right)\exp(-\Lambda_{ij,1})\right\} = 1\,. \tag{7.5}$$

We have already seen in Sect. IV, Eq. (4.27), that $\Sigma''\{\ \}$ is uniformly bounded for all n. Now the sum of all terms in which $k_{ij} \neq 0$ for some $i \neq j$ goes to zero like γ^{-n} since $(\Lambda_{ij,1}) \leqq \text{const}\,\gamma^{-i-j}$ by (4.23). We therefore have to show

$$\lim_{n\to\infty}\prod_{j=n}^{s-1}\exp_j \Lambda_{jj,1}\exp(-\Lambda_{jj,1})\prod_{\substack{i,j=n\\ i\neq j}}^{s-1}\exp(-\Lambda_{ij,1}) = 1$$

and this follows at once from (4.25), and the fact that

$$1 - \prod_j \exp_j A_j \exp(-A_j) \leqq \sum_j A_j^{j+1}/(j+1)!\,.$$

This shows that for any $\varepsilon > 0$ there is an $N < \infty$ such that for all $n > N$, $t, t' > n$ the expression (7.3) is bounded by ε. Note that by (4.25)

$$\prod_{i=n}^{s-1} \exp(-\Lambda_{ij,1} - \Lambda_{ij,2}) \prod_{i=n}^{s-1} \exp \Lambda_{i\sigma}(g^{(t)}) \to 1$$

as $n \to \infty$, which justifies our replacing

$$\exp(-\Lambda_{t\sigma}) \quad \text{by} \quad \prod_{j=0}^{n-1} \exp(-\Lambda_{j\sigma}(g)) \cdot \prod_{i,j=n}^{s-1} \exp(-\Lambda_{ij,1} - \Lambda_{ij,2}).$$

By construction, the contribution $I_{1,n}$ can be arbitrarily approximated by

$$\begin{aligned} &\left[\left(\psi_1, \left(\prod_{j=r_1}^{n-1} \exp V'_{R_j} \prod_{j=r_1}^{s-1} \exp V_{Q_j}\right)^* C_{mm'} \left(\prod_{j=r_2}^{s-1} \exp V_{Q_j} \prod_{j=r_2}^{n-1} \exp V'_{R_j}\right) \psi_2\right)\right. \\ &\quad \left. \cdot \prod_{i,j=0}^{n-1} \exp - \Lambda_{ij,1} \prod_{i,j=0}^{s-1} \exp - \Lambda_{ij,2}\right] \prod_{i,j}{}'' \exp - \Lambda_{ij,1} \Delta(n), \end{aligned} \tag{7.6}$$

where Π'' runs over the set $\{i,j; (i < n, j \geqq n) \text{ or } (i \geqq n, j < n)\}$. The contribution [ ] is uniformly bounded in n, s by const. $\|C\| \cdot \gamma^{-m-m'}$, by Theorem 6.4, and the product $\prod_{i,j}''$ is bounded by a constant, uniformly in n, s, since $\lim_{n\to\infty} \prod_{i,j} \exp(-\Lambda_{ij,1}) = 1$. Finally we have seen that $\Delta(n) \to 0$ as $n \to \infty$ and so the assertion of Theorem 7.1 is proved for the contribution I_1.

In I_2, we shall not need any cancellation between the terms coming from

$$(\hat{T}_{r_1 ts} \psi_1, C_{mm'} \hat{T}_{r_2 ts} \psi_2) \exp(-\Lambda_{t2^s})$$

and

$$(\hat{T}_{r_1 t's} \psi_1, C_{mm'} \hat{T}_{r_2 t's} \psi_2) \exp(-\Lambda_{t'2^s}),$$

but we shall show that the contribution from each of these two terms is small, as n is large.

We call "R-Wick expansion" the Wick expansion in terms of operators $A^*(x)$, $A(x)$, $x \notin Q$. By construction of I_1 and I_2, each R-Wick term of $(\hat{T}_{r_1 ts} \psi_1, C_{mm'} \hat{T}_{r_2 ts} \psi_2) \exp - \Lambda_{t2^s}$ which contributes to I_2 must contain at least one of the following contractions:

1) $V^*_{R_j,t} \underset{k}{\smile} V_{R_i}$, $k = 1, 2, 3$, $j \geqq n$, $i \geqq n$; or $k = 1, 2, 3, 4$, $j \geqq n$, $i < n$,
2) $V^*_{R_j,t} \underset{k}{\smile} V_{Q_i}$, $k = 1, 2, 3$, $j \geqq n$,
3) $V^*_{R_j,t} \underset{k}{\smile} C_{mm'}$, $k = 1, 2, 3, 4$, $j \geqq n$,
4) $V^*_{R_j,t} \smile \psi_2$, $k = 1, 2, 3, 4$, $j \geqq n$,
5) "adjoint" versions of 1), 2), 3), and 4).

The smallness of the contributions to I_2 follows from the presence of these contractions. We can expand this in the following way:

Let $$\hat{T}_{rts,i} \equiv \prod_{j=r}^{s-1} \exp_{j-\delta_{ij}} V_{R_{j,t}} \prod_{j=r}^{s-1} \exp_j V_{Q_j}.$$

Let $V^*_{R_j \searrow}$ denote "$V^*_{R_{j,t}}$ contracted in one or several of the fashions 1) to 5)"; let $\swarrow V_{R_j}$ denote the corresponding statement for $V_{R_{j,t}}$.

Then

$$\begin{aligned}
&[(\hat{T}_{r_1 ts}\psi_1, C_{mm'}\hat{T}_{r_2 ts}\psi_2)]_{I_2} \exp - A_{t2^s} \\
&= \sum_{j=n}^{s-1} (\hat{T}_{r_1 ts,j}\psi_1, V^*_{R_j \searrow} C_{mm'} \hat{T}_{r_2 ts}\psi_2) \exp - A_{t2^s} \\
&\quad + \sum_{j=n}^{s-1} (\hat{T}_{r_1 ts}\psi_1, C_{mm' \swarrow} V_{R_j} \hat{T}_{r_2 ts,j}\psi_2) \exp - A_{t2^s} \qquad (7.6)\\
&\quad - \sum_{j,j'=n}^{s-1} (\hat{T}_{r_1 ts,j}\psi_1, V^*_{R_j \searrow} C_{mm' \swarrow} V_{R_{j'}}, \hat{T}_{r_2 ts,j'}\psi_2) \exp - A_{t2^s}.
\end{aligned}$$

$\pm$ further terms of third and higher order (sign alternates with number of $V's$).

We only discuss the first three terms of (7.6), the other terms are bounded in a similar fashion. We bound the absolute value of each term. In each of these three terms there is a finite number of different Wick-terms involving $V^*_{R_j}$ or V_{R_j}. Each of these terms contains at least one contraction which is "small" as n is large. We write, e.g.

$$V^*_{R_j \searrow} C_{mm'} = \sum_{k=1}^{4} :V^*_{R_j \searrow_k} C_{mm'}:_R + C_{mm'} V^*_{R_j \searrow}. \qquad (7.7)$$

Here $:\,:_R$ is Wick ordering of $A^*(x)A(x)$, with $x \notin Q$. Now $V^*_{R_j \searrow_k} C_{mm'}$ contains a kernel $\bar{v}_{R_j} k_{\pm \iota \atop \pm o}$, with at least two variables identified, and we have seen that the Hilbert Schmidt norm of this kernel is small as j is large.

Indeed the decomposition

$$\bar{v}_{R_j} k_{\pm \iota \atop \pm o} = \bar{v}_{R_j} \mu^{-\tau} \cdot \mu^{\tau} k_{\pm \iota \atop \pm o}$$

shows that the contribution from the first four terms of (7.7), summed over $j \geqq t$ goes to zero as $t \to \infty$, uniformly in $\|C\|$, using the bounds of Sect. VI.

The last term in (7.7) yields a small contribution since there are at most 3 contractions between $V^*_{R_j}$ and $V_{R_{j'}}$, $j, j' \geqq t$, and so in this and all other possible cases one has a factor $\sum_{j \geqq t} \gamma^{-j-j'} \leqq (\gamma-1)^{-1} \gamma^{-t-j'}$ due to (4.21)–(4.23). This proves that the first two terms in (7.6) go to zero in absolute value, uniformly in $\|C\|$. In the third term the argument is similar but there occur more terms.

We expand

$$V^*_{R_j} C_{mm'} V_{R_{j'}} = \sum_{\substack{k_1, k_2, k_3 \\ k_3 \neq 4}} :V^*_{R_j} C_{mm'} V_{R_j}:_R ,$$

and then we repeat the above arguments.

In higher order terms, the argument is similar. It is easy to verify that for a term of order p in $V^{(*)}_{R_j}$, one will get a bound

$$\sum_{\substack{j_1 \cdots j_p = n \\ j_1 \cdots j_p \in J}}^{s-1} (\hat{T}_{r_1 ts}\psi_1, V^*_{R_{j_1}} \cdots V^*_{R_{j_k}} C_{mm'} V_{R_{j_{k+1}}} \cdots V_{R_{j_p}} \hat{T}_{r_2 ts}\psi_2) \exp - A_{t2^s}$$

$$\leqq \text{const.} \sum_{j_i \geqq (i+n)^{1/2}} \prod_{i=1}^{p/2} \gamma^{-(j_i)^{1/2}} \gamma^{-m-m'} \|C\|$$

$$\leqq \text{const.}\, \gamma^{-(p+n)^\tau} \gamma^{-m-m'} \|C\| ,$$

where $\gamma, \tau > 1$, $J \equiv \{j_i \mid j_i \geqq (n+i)^{1/2}\}$.

Summing over p, we get the bound $\varepsilon(n) \|C\| \gamma^{-m-m'}$ for I_2, with $\varepsilon \to 0$ as $n \to \infty$, and hence Theorem 7.1 is proved.

VIII. Estimates

In this section, we present the technical estimates we needed in the previous sections.

The estimates are roughly of two types: estimates on certain numerical kernels and estimates of functionals $\omega_{n\sigma}$ acting on Wick monomials with smooth kernels.

We start with the estimates on the numerical kernels $v(g)$, defined in Sect. II. Our two basic estimates describe the following facts: Lemma 8.1 shows that $\|\chi_j v(g)\|_2 \sim \mathcal{O}(\text{Vol}(\text{supp} g))$ if g is smooth enough and if its support is essentially a disk in $\mathbb{R}^2$. Lemma 8.3 shows that if at least one of the four variables of $v(g)$ is projected outside (supp$g$) by a projection P then $\|P\chi_j v(g)\|_2^2 \sim \mathcal{O}(\gamma^{-j})$ for some $\gamma > 1$. This means that $v(g)$ is essentially concentrated on (supp$g$), with a square integrable tail outside of (supp$g$).

We shall use the following notation. Let f be any function, then we write

$$D_n(f) \equiv \sum_{l=0}^{n} \sup_{k \in \mathbb{R}^2} |k|^l \, |f(k)| . \tag{8.1}$$

If the Fourier transform $\tilde{f}$ of f exists and is n times differentiable, then there is a constant K_n, independent of f, such that

$$\begin{aligned} D_n(f) \leqq K_n \sum_{l=0}^{n} \max_{|\varrho| \leqq l} \sup_{x \in \mathbb{R}^2} \left| \frac{\partial^{|\varrho|}}{\partial x^\varrho} \tilde{f}(x) \right| \\ \cdot \operatorname{Vol}\left(\bigcup_{|\varrho|=l} \left(\operatorname{supp} \frac{\partial^{|\varrho|}}{\partial x^\varrho} \tilde{f} \right) \right) \equiv E_n(\tilde{f}) . \end{aligned} \tag{8.2}$$

Here,
$$\frac{\partial^{|\varrho|}}{\partial x^\varrho} = \frac{\partial^{\varrho_1 + \varrho_2}}{\partial x_1^{\varrho_1} \partial x_2^{\varrho_2}}, \quad |\varrho| = \varrho_1 + \varrho_2 .$$

Symbols $C(a, b, \ldots)$ will denote constants which depend on $a, b, \ldots$; the symbol may take different values in different lines of an equation. C is any constant. The absence of an argument in $C(a, b, \ldots)$ does not imply that the constant is independent of this argument.

Lemma 8.1. *Let g be a smooth function with compact support. Let $v(g)$ be defined by* (2.8) *and let χ_j be defined by* (2.5). *Then there is a constant λ_0, independent of g, such that*

$$\|\chi_j v(g)\|_2 \leqq \lambda_0 \sup_{k \in \mathbb{R}^2} (1 + |k|) \, |\tilde{g}(k)| \leqq \lambda_0 D_1(\tilde{g}) .$$

Proof. By definition

$$\begin{aligned} \|\chi_j v(g)\|_2^2 = \int dk_1 \ldots dk_4 \chi_j(k_1, \ldots, k_4)^2 \\ \cdot \prod_{i=1}^{4} \mu(k_i)^{-1} \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-2} |\tilde{g}(k_1 + \cdots + k_4)|^2 . \end{aligned} \tag{8.3}$$

We bound (8.3) by

$$\begin{aligned} & D_1(\tilde{g})^2 \int dk_1 \ldots dk_4 \chi_j(k_1, \ldots, k_4)^2 \\ & \quad \cdot \prod_{i=1}^{4} \mu(k_i)^{-1} \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-2} (1 + |k_1 + \cdots + k_4|)^{-2} \\ & \leqq D_1(\tilde{g})^2 \, C(\chi_j)^2 , \end{aligned}$$

by power counting [22], [6].

Note. $C(\chi_j)$ depends on the scaling of χ_j in the following manner:

$$C(\chi_j) = C(m_0) \ln \left(\max_{k \in \operatorname{supp} \chi_j} \max_{i=1 \ldots 4} |k_i| \Big/ \min_{k \in \operatorname{supp} \chi_j} \max_{i=1 \ldots 4} |k_i| \right), \tag{8.4}$$

where m_0 is the mass in $\mu : \mu(k) = (m_0^2 + k^2)^{1/2}$.

Lemma 8.2. *Let ζ_j be a smoothed version of χ_j, defined by*

$$\zeta_j(k_1, \ldots, k_4) = \int dl_1 \ldots dl_4 \alpha^{-8j} \prod_{i=1}^{4} \tilde{\varphi}(\alpha^{-j} l_i) \, \chi_j(k_1 - l_1, \ldots, k_4 - l_4) ,$$

where $\varphi \in \mathcal{S}$, $\int \tilde{\varphi}(k)\,dk = 1$, $\tilde{\varphi}(k) \geqq 0$, $1 < \alpha < 2$. *There exists a constant* λ_0 *and a constant* $\gamma > 1$ *such that*

$$\|\zeta_j v(g) - \chi_j v(g)\|_2 \leqq \lambda_0 \left(\frac{2}{\alpha}\right)^{-j/2} D_1(\tilde{g}), \quad \textit{for all } j\,. \tag{8.5}$$

Proof. By Lemma 8.1, it is sufficient to bound

$$\begin{aligned} &\int dk_1 \ldots dk_4 (\zeta_j - \chi_j)^2 (k_1, \ldots, k_4) \\ &\quad \cdot \prod_{i=1}^{4} \mu(k_i)^{-1} \left(\sum_{i=1}^{4} \mu(k_i)\right)^{-2} (1 + |k_1 + \cdots + k_4|)^{-2}\,. \end{aligned} \tag{8.6}$$

We divide $(\mathbb{R}^2)^4$ into 3 disjoint regions R_{1j}, R_{2j}, R_{3j}: Let $\beta = (2\alpha)^{1/2}$; we define

$$R_{1j} = \left\{(k_1, \ldots, k_4);\ \max_{i=1 \ldots 4} |k_i| \in [2^j + \beta^j, 2^{j+1} - \beta^j]\right\},$$

$$R_{2j} = \left\{(k_1, \ldots, k_4);\ \max_{i=1 \ldots 4} |k_i| \leqq 2^j - \beta^j \text{ or } \max_{i=1 \ldots 4} |k_i| \geqq 2^{j+1} + \beta^j\right\},$$

$$R_{3j} = \left\{(k_1, \ldots, k_4);\ \max_{i=1 \ldots 4} |k_i| \in (2^j - \beta^j, 2^j + \beta^j) \cup (2^{j+1} - \beta^j, 2^{j+1} + \beta^j)\right\}.$$

We bound $\int_{R_{1j}}$ by observing that if $k \in R_{1j}$, each k_i is at least at a distance β^j from the boundary of $\operatorname{supp}\chi_j$. Therefore the integration in ζ_j extends at least over the set $|l_i| \leqq \beta^j$, and

$$\begin{aligned} \zeta_j(k_1, \ldots, k_4) &\geqq \left(\int_{|l| \leqq \beta^j} \alpha^{-2j} \tilde{\varphi}(\alpha^{-j} l)\right)^4 \\ &= \left(\int_{|l| \leqq 2^{j/2} \alpha^{-j/2}} dl\, \tilde{\varphi}(l)\right)^4 \geqq 1 - C(\gamma_1, \varphi) \gamma_1^{-j} \end{aligned} \tag{8.7}$$

for any $\gamma_1 > 1$ and some finite $C(\gamma_1, \varphi)$ since $\tilde{\varphi}$ decreases faster than any polynomial. From (8.7), we conclude that

$$\begin{aligned} \int_{R_{1j}} &\leqq \int dk_1 \ldots dk_4 \chi_j(k_1 \ldots k_4) \\ &\quad \cdot \prod_{i=1}^{4} \mu(k_1)^{-1} \left(\sum_{i=1}^{4} \mu(k_1)\right)^{-2} (1 + |k_1 + \cdots + k_4|)^{-2} \cdot C(\gamma_1, \varphi)^2 \cdot \gamma_1^{-2j} \end{aligned}$$

and so, the assertion (8.5) holds for this part. A similar argument can be used for R_{2j}. Finally the contribution of R_{3j} can be bounded by using (8.4). We get as bound for large j: $D_1(\tilde{g}) \cdot C \ln((2^j + \beta^j)/(2^j - \beta^j)) \leqq D_1(\tilde{g}) \cdot C \ln(1 + 3\beta^j/2^j) \leqq C \cdot D_1(\tilde{g}) \beta^j/2^j$. This completes the proof of Lemma 8.2. Lemma 8.2 will allow us to smooth freely the function χ_j.

Lemma 8.3. *Let g be a smooth function with compact support, and let ζ be a smooth function whose support satisfies* $\operatorname{dist}(\operatorname{supp}\zeta, \operatorname{supp} g) > d$, *for some* $d > 0$. *For* $d \leqq 1$ *the following estimates hold. If* ζ *has compact support, then*

$$\|\chi_j^{(p)} \zeta^{(x)} v(g)\|_2 \leqq \lambda_0 E_3(\zeta) E_1(g) d^{-3} 2^{-j/2}. \tag{8.8}$$

If $1-\zeta$ *has compact support then*

$$\|\chi_j^{(p)} \zeta^{(x)} v(g)\|_2 \leqq \lambda_0 \left(E_3(1-\zeta) + \sup_{x \in \mathbb{R}^2} \sup_{|\varrho| \leqq 2} \left| \frac{\partial^{|\varrho|}}{\partial x^\varrho} \zeta(x) \right| \right) E_1(g) d^{-3} 2^{-j/2} \tag{8.9}$$

for some λ_0 *independent of* ζ, g *and* j.

Proof. We write

$$\begin{aligned}
&\|\chi_j^{(p)} \zeta^{(x)} v(g)\|_2 \\
&= \left\{ \int \left| \int dl \chi_j(k_1, \dots, k_4) \tilde{\zeta}(k_1 - l) \tilde{g}(l + k_2 + k_3 + k_4) \mu(l)^{-1/2} \prod_{n=2}^{4} \mu(k_n)^{-1/2} \right.\right. \\
&\quad \left.\left. \cdot \left(\mu(l) + \sum_{n=2}^{4} \mu(k_n) \right)^{-1} \right|^2 dk_1 \dots dk_4 \right\}^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leqq \left\{ \int dk_1 \dots dk_4 \left| \int dl \tilde{\zeta}(k_1 - l) \tilde{g}(l + k_2 + k_3 + k_4) \mu(l)^{-1/2} \right.\right. \\
&\quad \left.\left. \cdot \chi_j(k_1, \dots, k_4) \prod_{i=2}^{4} \mu(k_i)^{-1/2} \left(\sum_{i=1}^{4} \mu(k_i) \right)^{-1} \right|^2 \right\}^{1/2} && (8.10.1) \\
&\quad + \left\{ \int dk_1 \dots dk_4 \left| \int \chi_j(k_1, \dots, k_4) \tilde{g}(l + k_2 + k_3 + k_4) \mu(l)^{-1/2} \prod_{i=2}^{4} \mu(k_i)^{-1/2} \right.\right. \\
&\quad \left.\left. \cdot \tilde{\zeta}_{reg}(k_1 - l) \left[\left(\mu(k_1) + \sum_{i=2}^{4} \mu(k_i) \right)^{-1} - \left(\mu(l) + \sum_{i=2}^{4} \mu(k_i) \right)^{-1} \right] dl \right|^2 \right\}^{1/2}. && (8.10.2)
\end{aligned}$$

The term $\tilde{\zeta}_{reg}$ is obtained as follows: If ζ has compact support then we set $\tilde{\zeta}_{reg} = \tilde{\zeta}$. If $\zeta - 1$ has compact support then $\tilde{\zeta}_{reg} = (1 - \zeta)^{\sim}$, in other words, we have omitted from (8.10.1) a contribution coming from the δ function $(= \tilde{1})$. But this contribution is zero since for $k_1 = l$, [ ] vanishes. One can view these remarks as a consequence of the fact that (8.10.2) is a commutator.

We estimate (8.10.1) and (8.10.2) separately. For the first term, we recall from Lemma 6.2 the bound

$$\left| \frac{\partial^{|\varrho|}}{\partial x^\varrho} \mu_{-1/2}(x) \right| \leqq C(\varrho) |x|^{-3-|\varrho|} e^{-m_0|x|}.$$

We now use the essential assumption that the supports of ζ and g are separated by a distance $d>0$. Therefore

$$|\int (1+|k|)^N \tilde{\zeta}(k-l)\, \mu(l)^{-1/2}\, \tilde{g}(l+p)\,(1+|p|)^{N'}\, dl|$$

$$\leqq C(N,N') \sum_{\substack{|\varrho|\leqq N\\ |\tau|\leqq N'}} \left| \int e^{i(kx+py)} \frac{\partial^{|\varrho|}}{\partial x^\varrho} \frac{\partial^{|\tau|}}{\partial y^\tau} \zeta(x)\, \mu_{-1/2}(x-y)\, g(y)\, dx\, dy \right|$$

$$\leqq C(N,N') \sum_{\substack{|\varrho|\leqq N\\ |\tau|\leqq N'}} \int \left| \frac{\partial^{|\varrho|}}{\partial x^\varrho} \frac{\partial^{|\tau|}}{\partial y^\tau} \zeta(x)\, \mu_{-1/2}(x-y)\, g(y) \right| dx\, dy\,.$$

To bound (8.9), we bound this quantity by

$$C(N,N') \sum_{\substack{|\varrho|\leqq N\\ |\tau|\leqq N'}} \left(\sup_{x\in\mathbb{R}^2} \sup_{|\varrho'|\leqq N} \left| \frac{\partial^{|\varrho'|}}{\partial x^{\varrho'}} \zeta(x) \right| \right)$$

$$\cdot \left(\sup_{y\in \operatorname{supp} g} \int dx\, d^{-3-N-N'} e^{-m_0|x-y|} \right) \cdot E_{N'}(g) \equiv B_1(N,N',d)\,.$$

We have assumed $d \leqq 1$.

To bound (8.8) we shall use instead the bound

$$C(N,N')\, E_N(\zeta) \cdot d^{-3-N-N'} E_{N'}(g) \equiv B_2(N,N',d)\,.$$

We therefore get

$$|(8.10.1)| \leqq \left(\int dk_1 \ldots dk_4 \left| \int \chi_j(k_1,\ldots,k_4)\,(|k_1|+1)^{-2}(|k_2+k_3+k_4|+1)^{-1} \right.\right.$$

$$\left.\left. \cdot B(2,1,d) \prod_{i=2}^4 \mu(k_i)^{-1/2} \left(\sum_{i=1}^4 \mu(k_i)\right)^{-1} \right|^2 \right)^{1/2} \leqq C(m_0)\cdot B(2,1,d)\, 2^{-j/2}$$

by power counting; $B=B_1$ or B_2.

To bound (8.10.2), we use the inequality

$$\left| \left(\mu(l) + \sum_{i=2}^4 \mu(k_i) \right)^{-1} - \left(\mu(k_1) + \sum_{i=2}^4 \mu(k_i) \right)^{-1} \right| |\tilde{\zeta}_{reg}(k_1-l)|$$

$$\leqq |\tilde{\zeta}_{reg}(k_1-l)\, \mu(k_1-l)| \cdot \left(\mu(l) + \sum_{i=2}^4 \mu(k_i) \right)^{-1} \left(\sum_{i=1}^4 \mu(k_i) \right)^{-1}.$$

Now we bound

$$|\tilde{\zeta}_{reg}(k)\, \mu(k)| \leqq (1+|k|)\, C(m_0) \cdot |\tilde{\zeta}_{reg}(k)| \leqq (1+|k|)^{-2}\, C(m_0) \cdot D_3(\tilde{\zeta}_{reg})\,.$$

Therefore

$$|(8.10.2)|^2$$

$$\leqq \int dk_1 \dots dk_4 \Big| \int dl_1\, dl_2\, \chi_j^2(k_1 \dots k_4)\, \tilde{g}(l_1+k_2+k_3+k_4)\, \tilde{g}(l_2+k_2+k_3+k_4)$$

$$\cdot \mu(l_1)^{-1/2}\, \mu(l_2)^{-1/2}\, \tilde{\zeta}_{reg}(k_1-l_1)\, \mu(k_1-l_1) \cdot \tilde{\zeta}_{reg}(k_1-l_2)\, \mu(k_1-l_2)$$

$$\cdot \prod_{n=2}^{4} \mu(k_n)^{-1} \left(\sum_{n=1}^{4} \mu(k_n)\right)^{-2} \left(\mu(l_1) + \sum_{n=2}^{4} \mu(k_n)\right)^{-1} \left(\mu(l_2) + \sum_{n=2}^{4} \mu(k_n)\right)^{-1} \Big|$$

$$\leqq C(m_0)^2\, D_3(\tilde{\zeta}_{reg})^2\, D_1(\tilde{g})^2$$

$$\cdot \int dk_1 \dots dk_4\, dl_1\, dl_2 (1+|k_1-l_1|)^{-2}\, (1+|k_1-l_2|)^{-2}$$

$$\cdot (1+|l_1+k_2+k_3+k_4|)^{-1} \cdot (1+|l_2+k_2+k_3+k_4|)^{-1}$$

$$\cdot \chi_j^2(k_1, \dots, k_4) \prod_{n=2}^{4} \mu(k_n)^{-1} \cdot \left(\sum_{n=1}^{4} \mu(k_n)\right)^{-2} \left(\mu(l_1) + \sum_{n=2}^{4} \mu(k_n)\right)^{-1}$$

$$\cdot \left(\mu(l_2) + \sum_{n=2}^{4} \mu(k_n)\right)^{-1}$$

$$\leqq C(m_0)\, D_3(\tilde{\zeta}_{reg})^2\, D_1(\tilde{g})^2\, 2^{-j},$$

by an explicit application of Weinberg's theorem [22]. This establishes our bound on (8.10.2) and proves Lemma 8.3.

Corollary 8.4. *Let g be a smooth function with compact support and let $R \subset \mathbb{R}^2$ be a region such that* $\mathrm{dist}(\mathrm{supp}\, g, R) > d$, *for some* $1 \geqq d > 0$. *Let P_R denote the projection onto R in one variable. If R is a compact region, then*

$$\|P_R \chi_j v(g)\|_2 \leqq C E_1(g) d^{-7} \{(2/\alpha)^{-j/2} + (2/\alpha^8)^{-j/2} \operatorname{Vol}(R_{+3\alpha^{-j}d})\} \tag{8.11}$$

for any $1 < \alpha^8 < 2$.

If R is the complement of a compact region then

$$\begin{aligned} \|P_R \chi_j v(g)\|_2 &\leqq C \cdot E_1(g) \cdot d^{-7} \\ &\quad \cdot \{(2/\alpha)^{-j/2} + (2/\alpha^8)^{-j/2} (\operatorname{Vol}(\mathbb{R}^2 - R) + 1)\}\,. \end{aligned} \tag{8.12}$$

Proof. We first replace χ_j by a smooth function ζ_j, as in the proof of Lemma 8.2. Let $\varphi(k) \geqq 0$; $\tilde{\varphi}(x) = 0$ if $|x| \geqq 1/2$ and suppose

$$\int \varphi(k)\, dk = 1, \qquad \int_{|k|<l} \varphi(k)\, dk \geqq 1 - C_N l^{-N}.$$

We set

$$\begin{aligned} \zeta_j(k_1, \dots, k_4) &\equiv \int dl_1 \dots dl_4 (\alpha^{-j} d)^8 \\ &\quad \cdot \prod_{n=1}^{4} \varphi(\alpha^{-j} l_n d) \cdot \chi_j(k_1 - l_1, \dots, k_4 - l_4)\,. \end{aligned} \tag{8.13}$$

We shall fix α later. Then

$$P_R^{(x)}\zeta_j^{(p)} = P_R^{(x)}\zeta_j^{(p)}\xi^{(x)}$$

where we choose ξ to be smooth, $\xi=1$ on $(\mathrm{supp}P_R)_{+\alpha^{-j}d}$, $\xi=0$ outside $(\mathrm{supp}P_R)_{+3\alpha^{-j}d}$, and smoothly interpolated between 0 and 1 elsewhere. For large j, $\mathrm{dist}(\mathrm{supp}\xi, \mathrm{supp}g) > d/2$. We bound (8.11) by

$$\begin{aligned}\|P_R\chi_j v(g)\|_2 &\leqq \|P_R(\zeta_j-\chi_j)\,v(g)\|_2 + \|P_R\zeta_j\xi v(g)\|_2\\ &\leqq \|(\zeta_j-\chi_j)\,v(g)\|_2 + \|\zeta_j\xi v(g)\|_2\\ &\leqq \|(\zeta_j-\chi_j)\,v(g)\|_2 + \left\|\sum_{i=0}^{\infty}\chi_i\zeta_j\xi v(g)\right\|_2 .\end{aligned} \tag{8.14}$$

By Lemma 8.2, the first term in (8.14) is bounded by

$$\lambda_0\left(\frac{2}{\alpha}\right)^{-j/2} d^{-1/2}D_1(\tilde{g}) . \tag{8.15}$$

Let $\mathrm{dist}(k_1, \mathrm{supp}\chi_j) \geqq 2^p$. Then

$$\begin{aligned}\zeta_j(k_1, \ldots, k_4) &\leqq \int_{2^p\leqq|l_1|} dl_1(\alpha^{-j}d)^{-2}\,\varphi(\alpha^{-j}dl_1)\\ &= \int_{d2^p\alpha^{-j}\leqq|l|} dl\,\varphi(l) \leqq C_N(d2^p\alpha^{-j})^{-N} .\end{aligned}$$

Therefore we find

$$|\chi_i(k_1, \ldots, k_4)\,\zeta_j(k_1, \ldots, k_4)| \leqq C(N)\,\chi_i(k_1, \ldots, k_4)\,\alpha^{jN}2^{-|i-j|N}d^{-N} . \tag{8.16}$$

If R is a compact region, then by (8.8)

$$\begin{aligned}\|\chi_i\xi v(g)\|_2 &\leqq \lambda_0\cdot E_3(\xi)\,E_1(g)\,d^{-3}2^{-i/2}\\ &\leqq Cd^{-3}\alpha^{3j}\,\mathrm{Vol}(R_{+3d\alpha^{-j}})\,E_1(g)d^{-3}2^{-i/2} .\end{aligned}$$

The bound on the derivatives of ξ comes from the smooth construction of ξ. We therefore get the following bound on (8.14):

$$\begin{aligned}\|P_R\chi_j v(g)\|_2 &\leqq C\left(\frac{2}{\alpha}\right)^{-j/2} d^{-1/2}D_1(\tilde{g})\\ &\quad + C(N)\sum_{i=0}^{\infty} 2^{-|i-j|N}\alpha^{jN}d^{-N}\cdot d^{-3}\alpha^{3j}\mathrm{Vol}(R_{+3\alpha^{-j}d})\,E_1(g)d^{-3}2^{-i/2}\\ &\leqq C\cdot E_1(g)\,d^{-7}\{(2/\alpha)^{-j/2} + (2/\alpha^8)^{-j/2}\cdot\mathrm{Vol}(R_{+3\alpha^{-j}d})\} ;\end{aligned} \tag{8.16a}$$

by an obvious bound on $\sum_i$. This proves (8.11).

To prove (8.12), we use (8.9)

$$\|\chi_i \xi v(g)\|_2 \leqq \lambda_0 \left(E_3(1-\xi) + \sup_x \sup_{|\varrho| \leqq 2} \left| \frac{\partial^{|\varrho|}}{\partial x^\varrho} \xi(x) \right| \right) E_1(g) d^{-3} 2^{-i/2}$$

$$\leqq \lambda_0 (d^{-3} \alpha^{3j} \operatorname{Vol}(\mathbb{R}^2 - R) + d^{-2} \alpha^{2j}) E_1(g) d^{-3} 2^{-i/2};$$

and therefore

$$\|P_R \chi_j v(g)\|_2 \leqq C \left(\frac{2}{\alpha} \right)^{-j/2} d^{-1/2} D_1(\tilde{g})$$

$$+ C(N) \sum_{i=0}^{\infty} 2^{-|i-j|N} \alpha^{jN} d^{-N} \cdot d^{-3} \alpha^{3j} (\operatorname{Vol}(\mathbb{R}^2 - R) + 1) \cdot E_1(g) d^{-3} 2^{-i/2}$$

$$\leqq C \cdot E_1(g) \cdot d^{-7} \{ (2/\alpha)^{-j/2} + (2/\alpha^8)^{-j/2} (\operatorname{Vol}(\mathbb{R}^2 - R) + 1) \}.$$

The bound (8.12) is established, Corollary 8.4 is proved.

Lemma 8.5. *Let* $j \in \mathbb{N}$ *and let* $g_{\pm j}$ *be defined by* (4.4). *Then there is a constant* $\lambda_0 < \infty$ *such that*

$$\|\chi_j v(g_{\pm j})\|_2 < \lambda_0 \quad \text{for all } j \in \mathbb{N}.$$

Lemma 8.6. *Let* v_{R_j} *and* v_{Q_j} *be defined by* (4.8). *Then for* v *small enough,*

$$\|v_{R_j} - v_j(g_{+j})\|_2 + \|v_{Q_j} - v_j(g_{-j})\|_2 < \lambda_0 \gamma^{-j} \quad \text{for some } \gamma > 1.$$

Lemma 8.7. *Let* $g_{\pm j}$ *be defined by* (4.4). *Then, for* v *small enough*

$$\|v_j(g) - v_j(g_{+j}) - v_j(g_{-j})\|_2 < \lambda_0 \gamma^{-j}$$

for some $\gamma > 1$.

Lemma 8.8. *With the notations* (4.17), *if* $i \neq j$ *then*

$$\sum_{\alpha = 1,2,6,7,8,9} |\Lambda_{ij,\alpha}| \leqq C \cdot \gamma^{-i-j} \quad \text{for some } \gamma > 1.$$

Note that Lemmata 8.5–8.8 prove Lemma 4.5.

Proof of Lemma 8.5. This Lemma follows at once from Lemma 8.1. Note that $|\tilde{g}_{+j}(k)|$ is uniformly bounded in j by const. $\max_x g(x)$ $\cdot \operatorname{Vol}((\operatorname{supp} g)_{+4v})$. Note also that $|k|\,|\tilde{g}_{+j}(k)|$ is bounded uniformly in j by $\mathcal{O}$ ("length of boundary of Q" $\cdot\, 8\, v^{-j} \cdot v^j$).
The last factor comes from the derivative near the boundary of Q and the first two factors describe the volume over which this derivative is

not zero. Since g and g' have compact support and are smooth, the assertion follows.

Proof of Lemma 8.6. We prove $\|v_{R_j}-v_j(g_{+j})\|_2 < C\gamma^{-j}$ only, the proof for the other term is similar.
By definition,

$$\begin{aligned} \|v_{R_j}-v_j(g_{+j})\|_2 &= \|\chi^{(x)}_{\sim Q}\,\zeta^{(p)}_j\, v(g_{+j}) - \chi^{(p)}_j\, v(g_{+j})\|_2 \\ &= \|\chi^{(x)}_{\sim Q}\,\zeta^{(p)}_j\, v(g_{+j}) - \zeta^{(p)}_j\, v(g_{+j}) + \zeta^{(p)}_j\, v(g_{+j}) - \chi^{(p)}_j\, v(g_{+j})\|_2 \qquad (8.17) \\ &\leqq \|(1-\chi^{(x)}_{\sim Q})\, v^{(p)}_j\, v(g_{+j})\|_2 + \|(\zeta^{(p)}_j - \chi^{(p)}_j)\, v(g_{+j})\|_2\,. \end{aligned}$$

The first term is bounded as follows: By construction of $v(g_{+j})$, ζ_j and of $(1-\chi_{\sim Q})$ there is at least one of the four variables in which

$$(1-\chi^{(x)}_{\sim Q})\,\zeta^{(p)}_j\, v(g_{+j}) = (1-\chi^{(x)}_{\sim Q})\,\zeta^{(p)}_j\, \xi^{(x)}\, v(g_{+j})$$

with dist (supp ξ, supp g_{+j}) $\geqq \nu^{-j}$. By Lemma 8.3 we have

$$\|\chi^{(p)}_i\, \xi v(g_{+j})\|_2 \leqq C\,\nu^{3j}\cdot E_1(g_{+j})\cdot \nu^{3j} 2^{-i/2}\,.$$

Note that $E_1(g_{+j})$ is uniformly bounded in j (see proof of Lemma 8.5), and the assertion follows now by an argument similar to (8.16):

$$\sum_{i=0}^{\infty} \|(\chi_i\zeta_j)^{(p)}\,\xi\, v(g_{+j})\|_2 \leqq C\,\nu^{7j} 2^{-j/2}\,.$$

The second term in (8.17) is bounded by Lemma 8.2. This completes the proof of Lemma 8.6.

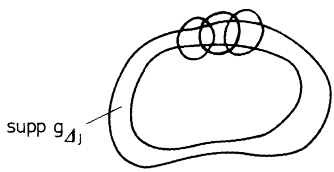

Fig. 4

Proof of Lemma 8.7. The proof is a combination of Lemma 8.1 and Corollary 8.4. Let $g_{\Delta j} = g - g_{+j} - g_{-j}$. We write $g_{\Delta j}$ as a sum of g_{ji}'s which have more or less circular support and whose derivatives are bounded by $\nu^{(1+\varepsilon)j}$, see Fig. 4.
The number of such g_{ji}'s necessary to "cover" $g_{\Delta j}$ is $\mathcal{O}(\nu^j)$. Then

$$\|\chi_j v(g_{\Delta j})\|_2^2 = \Big\|\sum_i \chi_j v(g_{ji})\Big\|_2^2 \leqq \sum_{i,i'} (\chi_j v(g_{ji}), \chi_j v(g_{ji'}))\,. \qquad (8.18)$$

By Corollary 8.4, if g_{ji} and $g_{ji'}$ are not nearest or next to nearest neighbors, then

$$
\begin{aligned}
|(\chi_j v(g_{ji}), \chi_j v(g_{ji'}))| &\leqq |(P_{(\operatorname{supp} g_{ji})_{+\nu^{-j}}} \chi_j v(g_{ji}), \chi_j v(g_{ji'}))| \\
&+ |(P_{\sim[(\operatorname{supp} g_{ji})_{+\nu^{-j}}]} \chi_j v(g_{ji}), \chi_j v(g_{ji'}))| \\
&\leqq \|\chi_j v(g_{ji})\|_2 \cdot \|P_{(\operatorname{supp} g_{ji})_{+\nu^{-j}}} \chi_j v(g_{ji'})\|_2 \qquad (8.19)\\
&+ \|P_{\sim[(\operatorname{supp} g_{ji})_{+\nu^{-j}}]} \chi_j v(g_{ji})\|_2 \, \|\chi_j v(g_{ji'})\|_2 \\
&\leqq C\, E_1(g_{ji})^2 \cdot \nu^{7j} \cdot \{(2/\alpha)^{-j/2} + (2/\alpha^8)^{-j/2}\} \\
&\leqq C\, \nu^{7j} \alpha^{4j} 2^{-j/2},
\end{aligned}
$$

and we get the desired bound if $\nu^9 \alpha^4 2^{-1/2} < 1$. This is our choice of α.

It is now easy to bound (8.18):

Let $\sum_{i,i'} = \sum'_{i,i'} + \sum''_{i,i'}$, where Σ' extends over $\{i = i'\}$, nearest neighbors and next nearest neighbors.

Then

$$
\begin{aligned}
(8.18) &\leqq \sum_{i,i'}{}' \max_{i''} \|\chi_j v(g_{ji''})\|_2^2 + \sum_{i,i'}{}'' C \cdot (\nu^7 \alpha^4 2^{-1/2})^j \\
&\leqq C \cdot 5 \nu^j \max E_1(g_{ji})^2 + C \cdot \nu^{2j} (\nu^7 \alpha^4 2^{-1/2})^j \\
&\leqq C\, 5 \nu^j \nu^{-2j+2\varepsilon} + C(\nu^9 \alpha^4 2^{-1/2})^j \leqq C \cdot \gamma^{-j}
\end{aligned}
$$

for small ε with some $\gamma > 1$. QED.

Proof of Lemma 8.8. We prove the assertion for $\Lambda_{ij,1}$ only; the proof for $\Lambda_{ij,\alpha}$, $\alpha = 2, 6, 7, 8, 9$, is similar. By definition, $\Lambda_{ij,1} = (\zeta_i v(g_{+i}), \chi_{\sim Q} \zeta_j v(g_{+j}))$. We assume $i > 3j$. Then let

$$
\xi_i(x_1, \dots, x_4) = \int dy_1 \dots dy_4 \, \beta^{+8i} \prod_{n=1}^{4} \varphi(\beta^{+i} y_n) \, \chi_{\sim Q}(x_1 - y_1, \dots, x_4 - y_4),
$$

where $\tilde{\varphi}(k)$ is smooth with compact support contained in

$$
|k| \leqq 1/2, \ \varphi(x) \geqq 0, \ \int \varphi(x)\, dx = 1 \quad \text{and} \quad \int_{|x| \leqq \chi} \varphi(x)\, dx \geqq 1 - C_N \chi^{-N};
$$

β will be fixed later. Now we rewrite

$$
\begin{aligned}
\Lambda_{ij,1} &= ((\chi_{\sim Q} - \xi_i)^{(x)} \zeta_i v(g_{+i}), \zeta_j v(g_{+j})) \\
&\quad + \sum_{k=0}^{\infty} (\chi_k \zeta_i v(g_{+i}), \chi_k \xi_i \zeta_j v(g_{+j})). \qquad (8.20)
\end{aligned}
$$

To bound the first term in (8.20), we use an argument similar to the proof of Lemma 8.2.

We consider 3 regions (in position space):

$R_{1i}=\{(x_1,\dots,x_4)\,|\,x_k\in Q$, and $\operatorname{dist}(x_k,\partial Q)\geqq\beta^{-i/2}$ for $k=1,\dots,4\}$,

$R_{2i}=\{(x_1,\dots,x_4)\,|\,x_k\notin Q$, and $\operatorname{dist}(x_k,\partial Q)\geqq\beta^{-i/2}$ for at least one $k\}$,

$R_{3i}=(\mathbb{R}^2)^4-R_{1i}-R_{2i}$.

Let $\chi_{R_{\delta i}}$ be the characteristic function of $R_{\delta i}$, $\delta=1,2,3$. Then

$$\|(\chi_{\sim Q}-\xi_i)^{(x)}\,\zeta_i^{(p)}\,v(g_{+i})\|_2\leqq\sum_{\delta=1}^{3}\|\chi^{(x)}_{R_{\delta i}}(\chi_{\sim Q}-\xi_i)^{(x)}\,\zeta_i^{(p)}\,v(g_{+i})\|_2\,.$$

For $\delta=1,2$, we bound

$$\begin{aligned}&\|\chi^{(x)}_{R_{\delta i}}(\chi_{\sim Q}-\xi_i)^{(x)}\,\zeta_i^{(p)}\,v(g_{+i})\|_2\\&\leqq\sup_{(x_1\dots x_4)\in R_{\delta i}}|(\chi_{\sim Q}-\xi_i)(x_1,\dots,x_4)|\cdot\|\chi^{(x)}_{R_{\delta i}}\zeta_i^{(p)}\,v(g_{+i})\|_2\leqq C\gamma^{-i}\end{aligned}$$

by an argument similar to (8.16) and by Lemma 8.5, and Lemma 8.2. The smallness of the term with $\delta=3$ will come from the bounded volume of R_{3i} and its distance from supp g_{+j}.

We use the second term of (8.16a), which is a bound on $\|\chi_R\zeta_j v(g)\|_2$. We get for $\beta>v$, i large,

$$\begin{aligned}&\|\chi_{R_{3i}}(\chi_{\sim Q}-\xi_i)^{(x)}\zeta_i^{(p)}v(g_{+i})\|_2\\&\leqq\|\chi_{R_{3i}}\zeta_i^{(p)}v(g_{+i})\|_2\leqq CE_1(g_{+i})\,v^{7i}\left(\frac{2}{v^8}\right)^{-i/2}.\end{aligned}$$

We shall therefore require $1<v^{22}<2$. This gives a bound

$$|((\chi_{\sim Q}-\xi_i)^{(x)}\zeta_i v(g_{+i}),\,\zeta_j v(g_{+j}))|\leqq C\gamma_1^{-i}\leqq C\gamma^{-(i+j)},$$

for some $\gamma>1$. To bound the second term in (8.20), we note that according to (8.16)

$$|\chi_k\zeta_i|\leqq C(N)\,\chi_k(k_1,\dots,k_4)\,v^{iN}2^{-|i-k|N}\,.$$

By construction, we also have

$$\chi_k^{(p)}\xi_i^{(x)}f=\chi_k^{(p)}\xi_i^{(x)}\vartheta^{(p)}f,\quad\text{where}$$

$$\vartheta(k_1,\dots,k_4)=\begin{cases}1 & \text{if } \max_n|k_n|\in[2^k-\beta^i,\,2^k+\beta^i)\\0 & \text{otherwise.}\end{cases}$$

Therefore

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}(\chi_k\zeta_i v(g_{+i}),\,\chi_k\xi_i\zeta_j v(g_{+j}))\right|\\&\leqq\sum_{k=0}^{\infty}C(N)\,v^{iN}2^{-|i-k|N}\cdot\|\xi_i\vartheta_k\zeta_j v(g_{+j})\|_2\,.\end{aligned}\tag{8.21}$$

Now again by (8.16),

$$|\vartheta_k \zeta_j| \leqq 1 \qquad \text{if} \quad k \leqq 2j < 2i/3$$

and $|\vartheta_k \zeta_j| \leqq C(N) \nu^{jN} 2^{-|k-j|N/2}$, since, for large k, $2^k - \beta^k > 2^{k/2}$. Therefore

$$\begin{aligned}(8.21) &\leqq C\left\{\sum_{k=0}^{2i/3} 2^{-iN} \nu^{iN} 2^{+kN}\right\} \\ &\quad + C\left\{\sum_{k=2i/3}^{\infty} \nu^{iN} 2^{-|i-k|N} \nu^{jN'} 2^{-|k-j|N'/2}\right\} \\ &\leqq C\gamma'^{-i} \leqq C\gamma^{-(i+j)} \quad \text{if } \nu \text{ is small enough.}\end{aligned}$$

This proves the assertion for the case $i > 3j$. The case $j > 3i$ follows by symmetry. The other cases follow from Lemma 8.6.

We now discuss the functionals $\omega_{mn\sigma}$ on Wick monomials with smooth kernels.

We shall use the following norm. Let $w_{mn}(x_1, \ldots, x_{m+n})$ be a function. We define

$$\|w_{mn}\|_{\sim} = \sup_{0 \leqq \nu_i \leqq 3, \forall i} \left\| \prod_{i=1}^{m+n} \mu(k_i)^{\nu_i} \tilde{w}_{mn}(k_1, \ldots, k_{m+n}) \right\|_2 .$$

Then one has the important bound

Theorem 8.9. *Let W_{mn} be a Wick monomial of the form*

$$\begin{aligned} W_{mn} = \int dx_1 \ldots dx_{m+n} \prod_{i=1}^{m+n} \chi^{(\beta_i)}(x_i)\, w_{mn}(x_1, \ldots, x_{m+n}) \\ \cdot A^*(x_1) \ldots A^*(x_m)\, A(x_{m+1}) \ldots A(x_{m+n}), \end{aligned}$$

with $\beta_i \in \{1, 2\}$, $\chi^{(1)} = \chi_Q$, $\chi^{(2)} = \chi_{\sim Q}$. Then uniformly in τ and σ,

$$|\omega_{\tau\sigma}(W_{mn})| \leqq C_1 \cdot \|w_{mn}\|_{\sim}\, C_2^{-m(1+C_3)-n(1+C_3)},$$

with $C_1 > 0$, $C_2 > 1$, $C_3 > 0$.

Proof. We look first at a contraction

$$\begin{aligned} &\int_{x \in Q} dx\, w_{mn}(x, \ldots)\, v_{Q_j}(x, \ldots) \\ &= \int_{\substack{x \in Q \\ z \in Q}} dx\, dy\, dz\, w_{mn}(x, \ldots)\, \mu_\nu(x-y)\, \mu_{-\nu}(y-z)\, v_{Q_j}(z, \ldots), \end{aligned}$$

with $\nu > 0$, and μ_ν is defined by (5.2). We now write

$$\begin{aligned} &\int dx\, w_{mn}(x, \ldots)\, \chi_Q(x)\, \mu_\nu(x-y) \\ &= \int dx dz dz'\, w_{mn}(x, \ldots)\, \mu_{+\nu+\varepsilon}(x-z) \cdot \mu_{-\nu-\varepsilon}(z-z')\, \chi_Q(z')\, \mu_\nu(z'-y). \end{aligned}$$

By Lemma 8.10, $\| \int \mu_{-\nu-\varepsilon}(z-z')\chi_Q(z')\mu_\nu(z'-y)dz' \|_2 = C_{\nu,\varepsilon} < \infty$ if $\varepsilon > 1$, $\frac{1}{2} > \nu > \frac{-1-2\varepsilon}{2}$. Similar statements hold with Q replaced by $\sim Q$, and for contractions to v_{R_j}.

We write, with $v_{S_j} \equiv v_{R_j}$ or v_{R_j},

$$\begin{aligned}
&\int_{x\notin Q} dx\, w_{mn}(x, \ldots)\, v_{S_j}(x, \ldots) \\
&= \int dxdydz\, w_{mn}(x, \ldots)\, \mu_\nu(x-y)\, \mu_{-\nu}(y-z)\, v_{S_j}(z, \ldots) \\
&- \int_{x\in Q} dxdydz\, w_{mn}(x, \ldots)\, \mu_\nu(x-y)\, \mu_{-\nu}(y-z)\, v_{S_j}(z, \ldots).
\end{aligned}$$

We now "absorb" μ_ν or $\mu_{\nu+\varepsilon}$ into w_{mn}:

$$\begin{aligned}
&\left\| \int dx_1 \ldots dx_{m+n}\, w_{mn}(x_1, \ldots, x_{m+n}) \prod_{\iota=1}^{m+n} (\chi^{(\beta_i)}(x_i)\mu_\nu(x_\iota - z_i)) \right\|_2 \\
&\leq \sup_I \left\| \int dx_1 \ldots dx_{m+n}\, w_{mn}(x_1, \ldots, x_{m+n}) \prod_{i\in I} \chi_Q(x_i)\, \mu_\nu(x_\iota - z_\iota) \prod_{i\notin I} \mu_\nu(x_i - z_i) \right\|_2 \\
&\leq \sup_I \left\| \int dx_1 \ldots dx_{m+n}\, w_{mn}(x_1, \ldots, x_{m+n}) \prod_{i\in I} \mu_{\nu+\varepsilon}(x_i - z_i) \prod_{i\notin I} \mu_\nu(x_i - z_\iota) \right\|_2 \\
&\cdot \prod_{\iota\in I} \| \int dz\, \mu_{-\nu-\varepsilon}(x-z)\chi_Q(z)\mu_\nu(z-y) \|_2 \\
&\leq \| w_{mn} \|_\sim (C'_{\nu,\varepsilon})^{m+n}.
\end{aligned}$$

Here $\sup_I$ extends over all subsets I of $\{1,\ldots,m+n\}$, $C'_{\nu,\varepsilon} = \max\{1, C_{\nu,\varepsilon}\}\cdot 2$, and all $\| \ \|_2$-norms are over the variables which are left after the integration over the variables inside $\| \ \|_2$.

As in Sect. IV, Eq. (4.28), we estimate $\omega_{\tau\sigma}(W_{mn})$ by giving a bound on

$$\sum_{p,q=0}^{\infty} \sum_{S_{pqmn}} (|\psi_1|, |S_{pqmn}|\, |\psi_2|), \tag{8.22}$$

where $\sum_{S_{pqmn}}$ extends over all Wick terms of $(V^*)^p W_{mn} V^q$ whose graph is a skeleton graph. We get

$$|(8.22)| \leq \sum_{p,q=0}^{\infty} \sum_{S_{pqmn}} \max (|\psi_1|, |S_{pqmn}|\, |\psi_2|)\, ((4p+n)!\, (4q+m)!)^{1/2} K^{p+q+m+n}. \tag{8.23}$$

Suppose ψ_1 and ψ_2 have at most r particles, (i.e. the k-particle components of ψ_1 and ψ_2 are zero if $k > r$). Then at least $m-r$ and $n-r$ contractions are between W_{mn} and some $V_{Q_j}^{(*)}$ or $V_{R_j}^{(*)}$, and in these contractions we apply the inequalities derived above. It follows at once that for $\nu > 0$

$$\begin{aligned}
\| \mu^{-\nu} v_{Q_j} \|_2 &\leq \| \mu^{-\nu} v_j(g_{-j}) \|_2 + m_0^{-\nu} \cdot \| v_j(g_{-j}) - v_{Q_j} \|_2 \\
&\leq \text{const.}\ \gamma^{-j} \text{ for some } \gamma > 1,
\end{aligned}$$

by Lemma 8.6. Therefore we can apply Lemma 2.2, and we get

$$\max_{S_{pqmn}} (|\psi_1|, |S_{pqmn}| |\psi_2|) \leqq C_{\psi_1\psi_2} \sum_{\substack{j_1 \dots j_p \in J_{m-r} \\ j'_1 \dots j'_q \in J_{n-r}}} \prod_{k=1}^{p} \gamma^{-j_k} \prod_{k=1}^{q} \gamma^{-j'_k} \cdot \|w_{mn}\|_\sim \tag{8.24}$$

$$\leqq C_{\psi_1\psi_2} \|w_{mn}\|_\sim (\gamma-1)^{-(p+q)} \gamma^{-1/2 \sum_{k=1}^{p} k^{1/2} - 1/2 \sum_{k=1}^{m-r} k^{1/2} - 1/2 \sum_{k=1}^{q} k^{1/2} - 1/2 \sum_{k=1}^{n-r} k^{1/2}}$$

$$\leqq C_{\psi_1\psi_2\varepsilon} \|w_{mn}\|_\sim \gamma'^{-m^{3/2-\varepsilon} - n^{3/2-\varepsilon}} \gamma''^{-p^{3/2-\varepsilon} - q^{3/2-\varepsilon}}, \tag{8.25}$$

for any $\varepsilon > 0$.

Here $j_1, \dots, j_p \in J_{m-r} \equiv \{j_i | j_i \geqq i^{1/2}$ (truncation), $p \geqq m-r$ (number of legs to connect)$\}$. The constants γ', γ'' are larger than 1.
The assertion of Theorem 8.9 follows now at once from (8.23) and (8.25).

Lemma 8.10. *Let Q be a polynomially bounded region in $\mathbb{R}^2$. Then*

$$\left\| \int dz\, \mu_{-\nu-\varepsilon}(x-z)\, \chi_Q(z)\, \mu_\nu(z-y) \right\|_2 < \infty$$

if $\dfrac{-1-2\varepsilon}{2} < \nu < 1/2$ *and* $\varepsilon > 1$.

Proof. We use the following statement, and then Lemma 8.10 follows by simple power counting.

Lemma 8.11. *Let Q be a polynomially bounded region in $\mathbb{R}^2$ and let χ_Q be the characteristic function of Q. Then the Fourier transform $\tilde{\chi}_Q(k)$ satisfies the following:*

For $|k| = 1$, $k = (k_1 \cos\varphi, k_1 \sin\varphi)$, k_1 fixed,

$$\tilde{\chi}_Q(\eta k) \leqq C \frac{1}{\eta+1} \quad \textit{for all } \varphi \textit{ and } \eta > 0$$

and $\tilde{\chi}_Q(\eta k) \leqq C' \dfrac{1}{(\eta+1)^2}$ *for almost all $\varphi \in [0, 2\pi)$, and $\eta > 0$.*

Proof. Let $\{\zeta_\alpha\}$ be a covering of $\mathbb{R}^2$ with smooth functions of compact support such that

1) $\zeta_\alpha \cdot \chi_Q \neq 0$ for a finite set of α's.

2) The boundary of the support of $\zeta_\alpha \cdot \chi_Q$ contains either a segment of two edges of the boundary of Q which meet in a corner, or no edge of the boundary of Q or a segment of one edge of the boundary of Q (Fig. 5).

We now prove the assertion of the Lemma by proving it for each $\zeta_\alpha \cdot \chi_Q$.

If $\zeta_\alpha \cdot \chi_Q = \zeta_\alpha$, then the assertion is trivial, by the smoothness of ζ_α. If $\zeta_\alpha \cdot \chi_Q$ contains one sharp edge, then $\zeta_\alpha \cdot \chi_Q = \zeta_\alpha \cdot \chi_{Q'}$, where Q' is a square.

Then $\tilde{\chi}_{Q'}$ has the properties which we want to prove for $\tilde{\chi}_Q$ (as can be seen by a direct calculation), and hence $(\zeta_\alpha \cdot \chi_Q)^\sim (k) = \int \tilde{\zeta}_\alpha(k-l)\, \tilde{\chi}_{Q'}(l)\, dl$ has these same properties since convolution by a function which falls off faster than any polynomial preserves the asymptotic behavior, (see [16] Lemma 4.3). The case with two edges follows in the same way, by letting Q' be a parallelogram.

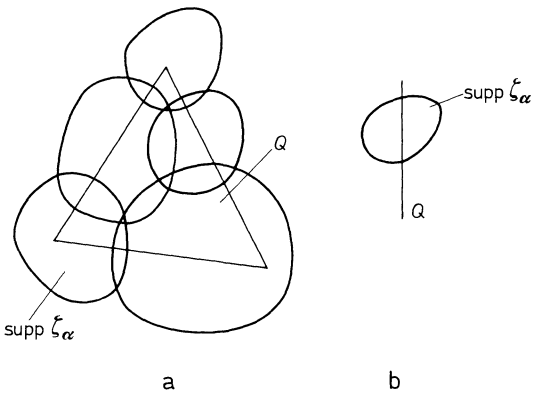

Fig. 5

Acknowledgements. I am deeply indebted to Prof. J. Glimm for proposing this problem and for constant encouragement without which this paper would never have been written. I am grateful to Dr. K. Osterwalder for reading critically the manuscript.

Most of this work was done at Brandeis University under Grant O.N.R. No. 14 – 70 – SC – 0418. Part of the work was supported by the Fonds National Suisse.

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J.-P. Eckmann
Département de Physique Théorique
Université de Genève
CH - 1211 Genève 4, Switzerland