Commun. math. Phys. 24, 107—132 (1972)

On the Directional Dependence of Composite Field Operators*

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Received July 26, 1971

Abstract. The Wilson expansion of the field operator product $A_1(x_1)\,A_2(x_2)$ may be used to define composite operators which are local with respect to $\frac{1}{2}(x_1+x_2)$ and depend in addition on a vector η proportional to the distance x_1-x_2. It is proved that the composite operators are polynomials in η, for fixed $\eta^2 \neq 0$, and that their dependence on η^2 only involves powers of η^2 and $\lg\eta^2$.

1. Introduction

The composite operators associated with the formal product $A_1(x)\,A_2(x)$ of two fields may be conveniently defined as the operators C_j appearing in the Wilson expansion

$$A_1(x_1)\,A_2(x_2) = \sum_{j=1}^{k} f_j(\varrho)\,C_j(x,\eta) + P_{k+1}(x,\eta,\varrho)\,, \qquad x_1 = x+\varrho\eta\,, \quad x_2 = x-\varrho\eta\,, \quad \varrho>0\,, \tag{1.1}$$

where the coefficients f_j satisfy

$$\lim_{\varrho\to 0}\frac{f_{j+1}(\varrho)}{f_j(\varrho)} = 0\,, \qquad \lim_{\varrho\to 0}\frac{P_{k+1}(x,\eta,\varrho)}{f_k(\varrho)} = 0\,. \tag{1.2}$$

In a recent paper [1] the expansion (1.1) was derived from general assumptions, and the operators C_j were shown to be local in x.

The operators C_j depend on the vector x of the center-of-mass point and an additional four vector η proportional to the distance of the arguments x_1 and x_2. The dependence on η is related to the directional dependence of composite field operators. This can be seen by setting

$$\eta = \frac{\xi}{\sqrt{-\xi^2}}\,, \qquad \varrho = \sqrt{-\xi^2}$$

* This work was supported in part by the National Science Foundation Grant No. GP-25609.

in (1.1). The Wilson expansion then takes the more familiar form

$$A_1(x+\xi)\,A_2(x-\xi)=\sum_{j=1}^{k} f_j(\sqrt{-\xi^2})\,C_j\left(x,\frac{\xi}{\sqrt{-\xi^2}}\right)+P_{k+1} \tag{1.3}$$

where the composite operators C_j depend on x and the direction $\eta=\xi/\sqrt{-\xi^2}$ of the difference vector ξ.

The purpose of this paper is to completely characterize the η-dependence of the operators C_j, using only the Wightman postulates [2] together with the additional assumptions of Ref. [1]. In particular, it will be shown that for fixed $\eta^2 \neq 0$ the operators C_j are polynomials in η, as one should expect from renormalized perturbation theory. Moreover we find that the dependence on η^2 only involves powers of η^2 and $\lg\eta^2$.

In this introduction we shortly sketch the proof of the main theorem for the composite operators which appear in the expansion of $A(x+\varrho\eta)$ $A(x-\varrho\eta)$. The rigorous treatment, as well as the generalization to the product of two different operators, will be given in Section 2–7.

The local operators C_j are recursively constructed by [1]

$$\begin{aligned}
&P_1(x,\eta,\varrho)=A(x+\varrho\eta)\,A(x-\varrho\eta)\\
&P_j(x,\eta,\varrho)=f_j(\varrho)\,C_j(x,\eta)+P_{j+1}(x,\eta,\varrho)\\
&C_j(x,\eta)=\lim_{\varrho\to 0}\frac{P_j(x,\eta,\varrho)}{f_j(\varrho)}\,.
\end{aligned} \tag{1.4}$$

For the Fourier transforms

$$\begin{aligned}
&\tilde P_j(x,u,\varrho)=\frac{1}{(2\pi)^2}\int d\eta\, e^{i\eta u}\,P_j(x,\eta,\varrho)\\
&\tilde C_j(x,u)=\frac{1}{(2\pi)^2}\int d\eta\, e^{i\eta u}\,C_j(x,\eta)
\end{aligned} \tag{1.5}$$

we have

$$\begin{aligned}
&\tilde P_j(x,u,\varrho)=f_j(\varrho)\,\tilde C_j(x,u)+\tilde P_{j+1}(x,u,\varrho)\\
&\tilde C_j(x,u)=\lim_{\varrho\to 0}\frac{\tilde P_j(x,u,\varrho)}{f_j(\varrho)}\,.
\end{aligned} \tag{1.6}$$

Let Φ_ϱ, Ψ_q be eigenvectors of the energy-momentum operator with eigenvalues p_μ or q_μ respectively. The energy-momentum eigenvalues r of an intermediate state is related to p, q, u by

$$r=\frac{w}{2\varrho}\,,\qquad w=u+\varrho(p+q)\,.$$

Therefore

$$(\Phi_p, \tilde{P}_1(x, u, \varrho)\,\Psi_q) = 0 \quad \text{unless} \quad w^2 \geqq 0\,, \quad w_0 \geqq 0\,. \tag{1.7}$$

Dividing (1.7) by $f_1(\varrho)$ and taking the limit $\varrho \to 0$ we find

$$(\Phi_p, \tilde{C}_1(x, u)\,\Psi_q) = 0 \quad \text{unless} \quad u^2 \geqq 0\,, \quad u_0 \geqq 0\,. \tag{1.8}$$

Repeated application of this argument to the recursion formulae (1.6) leads to

$$(\Phi_p, \tilde{C}_k(x, u)\,\Psi_q) = 0 \quad \text{unless} \quad u^2 \geqq 0\,, \quad u_0 \geqq 0\,.$$

By linear superposition of the vectors Φ_p and Ψ_q we obtain

$$(\Phi, \tilde{C}_k(x, u)\,\Psi) = 0 \quad \text{unless} \quad u^2 \geqq 0\,, \quad u_0 \geqq 0 \tag{1.9}$$

for arbitrary matrix elements. (For a rigorous formulation see Theorem 2 and Corollary of Section 3.)

The causality condition

$$[A(x_1), A(x_2)] = 0 \quad \text{if} \quad (x_1 - x_2)^2 < 0$$

implies

$$P_1(x, -\eta, \varrho) = P_1(x, \eta, \varrho) \quad \text{if} \quad \eta^2 < 0\,. \tag{1.10}$$

Dividing (1.10) by $f_1(\varrho)$ and taking the limit $\varrho \to 0$ we get

$$C_1(x, -\eta) = C_1(x, \eta) \quad \text{if} \quad \eta^2 < 0\,.$$

Using the recursion formulae (1.6) the relation

$$C_j(x, -\eta) = C_j(x, \eta) \quad \text{for} \quad \eta^2 < 0 \tag{1.11}$$

follows by induction. This implies that the matrix element

$$f(\eta) = (\Phi, (C_j(x, \eta) - C_j(x, -\eta))\,\Psi) \tag{1.12}$$

vanishes for spacelike η,

$$f(\eta) = 0 \quad \text{if} \quad \eta^2 < 0\,. \tag{1.13}$$

The Fourier transform

$$\tilde{f}(u) = (\Phi, (\tilde{C}_j(x, u) - \tilde{C}_j(x, -u))\,\Psi) \tag{1.14}$$

vanishes for spacelike u

$$\tilde{f}(u) = 0 \quad \text{if} \quad u^2 < 0 \tag{1.15}$$

as follows from (1.9). Because of (1.13) the Jost-Lehmann-Dyson representation [4] may be used to write

$$\tilde{f}(u) = \int_0^\infty d\kappa^2 \int du' \sigma(x, u - u', \kappa^2)\, \tilde{\Delta}(u', \kappa^2)\,. \tag{1.16}$$

The spectral function $\sigma(x, v, \kappa^2)$ vanishes unless the hyperboloid $(u-v)^2=\kappa^2$ lies in the region $u^2 \geqq 0$ (where $\tilde{f}$ may be different from zero). Hence σ is non-vanishing only at the origin $v=0$, and so

$$\sigma = \sum_{m=1}^{M} a^{\mu_1 \dots \mu_m}(x, \kappa^2)\, \partial_{\mu_1} \dots \partial_{\mu_m} \delta(v)\,. \tag{1.17}$$

With this result $\tilde{f}$ becomes

$$\tilde{f}(u) = \sum_{m=1}^{M} \int_0^\infty d\kappa^2\, a^{\mu_1 \dots \mu_m}(x, \kappa^2)\, \partial_{\mu_1} \dots \partial_{\mu_m} \tilde{\Delta}(u, \kappa^2)\,. \tag{1.18}$$

For $u_0 > 0$ we have

$$(\Phi, \tilde{C}_j(x, u)\, \Psi) = \sum_{m=1}^{M} \int_0^\infty d\kappa^2\, a^{\mu_1 \dots \mu_m}(x, \kappa^2)\, \partial_{\mu_1} \dots \partial_{\mu_m} \tilde{\Delta}^+(u\kappa^2) \tag{1.19}$$

since

$$\tilde{f}(u) = (\Phi, \tilde{C}_j(x, u)\, \Psi) \quad \text{if} \quad u_0 > 0\,. \tag{1.20}$$

For $u_0 < 0$ or $u^2 < 0$ both sides of (1.19) vanish. Hence (1.19) is valid for all u except $u=0$. Therefore

$$\begin{aligned}(\Phi, \tilde{C}_j(x, u)\, \Psi) = &\sum_{m=1}^{M} \int_0^\infty d\kappa^2\, a^{\mu_1 \dots \mu_m}(x, \kappa^2)\, \partial_{\mu_1} \dots \partial_{\mu_m} \tilde{\Delta}^+(u, \kappa^2) \\ &+ \sum_{n=1}^{N} b^{\nu_1 \dots \nu_N}(x, \kappa^2)\, \partial_{\nu_1} \dots \partial_{\nu_N} \delta(u)\,.\end{aligned} \tag{1.21}$$

The Fourier transform of (1.21) with respect to u yields

$$(\Phi, C_j(x, \eta)\, \Psi) = \sum_{r=1}^{R} \eta_{\varrho_1} \dots \eta_{\varrho_r} t^{\varrho_1 \dots \varrho_r}(x, \eta^2 - i\varepsilon\eta_0)\,, \qquad \varepsilon \to +0\,. \tag{1.22}$$

This is the statement that the matrix elements of the composite operators C_j are polynomials in η, for fixed $\eta^2 \neq 0$. It should be noted that the proof just outlined goes through under much weaker conditions. It is not necessary to assume relation (2.15) of Ref. [1] which excludes oscillations for $\varrho \to 0$. Instead the hypotheses of Ref. [1] may be formulated in reference to a particular sequence ϱ_n with

$$\lim_{n \to \infty} \varrho_n = 0\,.$$

The Wilson expansion and the recursive construction of composite operators then hold with respect to this sequence which suffices for the derivation of (1.22).

In Section 6b it will be proved that the degree of the polynomial (1.22) stays bounded when the states Φ and Ψ are varied. This shows that also the operator C_j is a polynomial in η, for $\eta^2 \neq 0$ given.

Finally we examine the dependence of C_j on η^2. A characteristic property of the composite operators is that they obey simple transformation laws under the scaling transformation

$$\eta \to \sigma\eta \,.$$

In order to obtain the scaling law we write the expansion (1.1) in the equivalent form

$$\begin{aligned} A(x+\varrho\eta)\,A(x-\varrho\eta) &= A\left(x, +\frac{\varrho}{\sigma}\sigma\eta\right) A\left(x, -\frac{\varrho}{\sigma}\sigma\eta\right) \\ &= \sum_{j=1}^{k} f_j\left(\frac{\varrho}{\sigma}\right) C_j(x,\sigma\eta) + P_{k+1}\left(x,\sigma\eta,\frac{\varrho}{\sigma}\right). \end{aligned} \tag{1.23}$$

According to the uniqueness theorem the operators $C_j(x,\eta)$ and $C_j(x,\sigma\eta)$ must be related by a transformation of the form [5]

$$C_j(x,\sigma\eta) = \sum_{j'=1}^{j} s_{jj'}(\sigma)\,C_{j'}(x,\eta)\,. \tag{1.24}$$

The triangular, real $k \times k$ matrices

$$s(\sigma) = \|s_{jj'}(\sigma)\| \qquad j, j' = 1, \ldots, k$$

satisfy the multiplication rule

$$s(\sigma\tau) = s(\sigma)\,s(\tau) \tag{1.25}$$

as follows from the identity

$$\sum_{j'} s_{jj'}(\sigma\tau)\,C_{j'}(x,\eta) = C_j(x,\sigma\tau\eta) = \sum_{j'l} s_{jl}(\sigma)\,s_{lj'}(\tau)\,C_{j'}(x,\eta)\,.$$

Hence the matrices $s(\sigma)$ form a k-dimensional representation of the multiplicative group of the real numbers. These representations are well known, a normal form is given in Section 5. Here we only indicate the general situation for $k=1,2$. The scaling law of C_1 is

$$C_1(x,\sigma\eta) = \sigma^{c_1} C_1(x,\eta)\,. \tag{1.26}$$

After a suitable transformation of C_1, C_2 (by a triangular 2×2 matrix) the scaling law of C_2 becomes either

$$C_2(x,\sigma\eta) = \sigma^{c_1}(\lg\sigma\, C_1(x,\eta) + C_2(x,\eta)) \tag{1.27}$$

or

$$C_2(x\,\sigma\eta) = \sigma^{c_2} C_2(x\,\eta)\,. \tag{1.28}$$

In case of relation (1.27) C_2 can be written as a linear combination of C_1 and another operator Q which satisfies a power scaling law

$$C_2(x, \eta) = \lg\sqrt{-\eta^2 + i\varepsilon\eta_0}\, C_1(x, \eta) + Q(x, \eta), \qquad \varepsilon \to +0. \tag{1.29}$$

The operator Q as defined by (1.29) indeed satisfies

$$Q(x, \sigma\eta) = \sigma^{c_1} Q(x, \eta). \tag{1.30}$$

We further have

$$Q(x, -\eta) = -Q(x, \eta) \quad \text{if} \quad \eta^2 < 0 \tag{1.31}$$

and

$$\tilde{Q}(x, u) = 0 \quad \text{unless} \quad u^2 \geqq 0, \quad u_0 \geqq 0 \tag{1.32}$$

where $\tilde{Q}$ denotes the Fourier transform with respect to η. (1.32) follows from (1.9) and the corresponding property of the Fourier transform of $\lg\sqrt{-\eta^2 + i\varepsilon\eta_0}$.

It will be shown in general (Section 5) that after a suitable equivalence transformation the C_j are of the form

$$C_j(x, \eta) = \sum_{n=0}^{N} (\lg\sqrt{-\eta^2 + i\varepsilon\eta_0})^n\, Q_j^{(n)}(x, \eta) \qquad \varepsilon \to +0 \tag{1.33}$$

where the operators $Q_j^{(n)}$ satisfy a power scaling law

$$Q_j^{(n)}(x, \sigma\eta) = \sigma^c Q_j^{(n)}(x, \eta) \tag{1.34}$$

and the conditions

$$\begin{aligned} Q_j^{(n)}(x, -\eta) &= Q_j^{(n)}(x, \eta) \quad \text{if} \quad \eta^2 < 0 \\ \tilde{Q}_j^{(n)}(x, u) &= 0 \quad \text{unless} \quad u^2 \geqq 0, \quad u_0 \geqq 0. \end{aligned} \tag{1.35}$$

As in the case of the operators C_j the conditions (1.35) imply that the $Q_j^{(n)}$ are polynomials in η, for fixed $\eta^2 \neq 0$.

$$Q_j^{(n)}(x, \eta) = \sum_{r=1}^{R} \eta_{\varrho_1} \cdots \eta_{\varrho_r} T^{\varrho_1 \cdots \varrho_r}(x, \eta^2 - i\varepsilon\eta_0). \tag{1.36}$$

Due to the scaling law (1.34) the $T^{\varrho_1 \cdots \varrho_r}$ must be homogeneous in $\sqrt{-\eta^2}$ of degree $c - r$. Therefore

$$Q_j^{(n)}(x, \eta) = (\sqrt{-\eta^2 + i\varepsilon\eta_0})^c\, \Pi_j^{(n)}(x, \zeta) \qquad \varepsilon \to +0 \tag{1.37}$$

where $\Pi_j^{(n)}$ is a polynomial in the components of

$$\zeta = \frac{\eta}{\sqrt{-\eta^2 + i\varepsilon\eta_0}} \tag{1.38}$$

(1.33) and (1.37) state the final result that the η^2-dependence of the composite operators only involves powers of η^2 and $\lg\eta^2$.

The following sections contain a detailed and rigorous derivation of this result, generalized to the product of two different fields. Causality is used in Section 2 to show that the same functions f_j may be used in expanding $A_1 A_2$ and $A_2 A_1$. This leads to a locality relation of the composite operators for spacelike η. Analytic properties of the composite operators in η are derived in Section 4 which follow from the support properties of the Fourier transformations (Section 3). After a discussion of the scaling law (Section 5) the η-dependence is derived in Section 6 by an alternative method which does not make use of the Jost-Lehmann-Dyson representation.

2. Locality

The general assumption and notations of Ref. [1] will be used throughout the work that follows. A_1 and A_2 denote linear combinations of the basic fields $O_1, \ldots, O_c$. In addition to Wightman's postulates Hypothesis 3 of Ref. [1] will be assumed which implies that the operator product $A_1 A_2$ of two local field operators A_1, A_2 has the Wilson expansion

$$A_1(x+\varrho\eta)A_2(x-\varrho\eta)=\sum_{j=1}^{k} f_j(\varrho)C_j^{12}(x,\eta)+P_{k+1}^{12}(x,\eta,\varrho) \text{ in } \mathscr{S}'_{x\eta}(D_0). \quad (2.1)$$

The functions f_j may be chosen to be real, they satisfy

$$\begin{aligned} &\lim_{\varrho\to 0} \frac{f_{j+1}(\varrho)}{f_j(\varrho)}=0 \\ &\lim_{\varrho\to 0} \frac{(\Phi, P_{k+1}^{12}(x,\eta,\varrho)\,\Psi)}{f_k(\varrho)}=0 \quad \text{in} \quad \mathscr{S}'_{x\eta} \quad \text{for} \quad \Phi, \Psi\in D_0 . \end{aligned} \quad (2.2)$$

The operators $C_j^{12}(x,\eta)$ are local in x for given η. Another consequence of locality can be derived from

$$A_1(x+\varrho\eta)\,A_2(x-\varrho\eta)=\pm A_2(x-\varrho\eta)\,A_1(x+\varrho\eta) \quad (2.3)$$

valid in

$$\mathscr{S}'_x(D_0) \quad \text{for} \quad \eta^2<0 .$$

To this end we compare (2.1) with the expansion of $A_2 A_1$

$$A_2(x+\varrho\eta)A_1(x-\varrho\eta)=\sum_{j=1}^{k} f'_j(\varrho)C'^{21}_j(x,\eta)+P'^{21}_{k+1}(x,\eta,\varrho) \text{ in } \mathscr{S}'_{x\eta}(D_0). \quad (2.4)$$

For spacelike η (2.1) and (2.3) yield

$$A_2(x+\varrho\eta)\,A_1(x-\varrho\eta)=\sum_{j=1}^{k} f_j(\varrho)\,(\pm 1)\,C_j^{12}(x,-\eta)\pm P_{k+1}^{12}(x,-\eta,\varrho)$$
$$\text{in}\quad \mathscr{S}'_{x\eta}(D_0)\,,\qquad \eta^2<0 \tag{2.5}$$

which is equivalent to (2.4). According to the uniqueness theorem there must be an equivalence transformation

$$\pm C_j^{12}(x,-\eta)=\sum_{j'=1}^{j} a_{jj'}\,C_j'^{21}(x,\eta)$$
$$\pm P_{k+1}^{12}(x,-\eta,\varrho)=P_{k+1}'^{21}(x,\eta,\varrho)+\sum_{j=1}^{k} h_j(\varrho)C_j'^{21}(x,\eta) \tag{2.6}$$
$$a_{jj}\neq 0,\quad \lim_{\varrho\to 0}\frac{h_j(\varrho)}{f_j(\varrho)}=0$$

valid in $\mathscr{S}'_{x\eta}(D_0)$ for $\eta^2<0$. The functions f_j and f'_j are related by the asymptotic expansion

$$f_j(\varrho)=\sum_{j'=j}^{\infty} a_{j'j}f'_{j'}(\varrho)+h_j(\varrho)\,. \tag{2.7}$$

An equivalence transformation for (2.4) which is valid for all η may be set up by using (2.7) and defining

$$C_j^{21}(x,\eta)=\sum_{j'=1}^{j} a_{jj'}\,C_j'^{21}(x,\eta)$$
$$P_{k+1}^{21}(x,\eta,\varrho)=P_{k+1}'^{21}(x,\eta,\varrho)+\sum_{j=1}^{k} k_j(\varrho)\,C_j'^{21}(x,\eta) \tag{2.8}$$
$$\text{in}\quad \mathscr{S}'_{x\eta}(D_0)\,.$$

Applying (2.7–2.8) to (2.4) we find

$$A_2(x+\varrho\eta)\,A_1(x-\varrho\eta)=\sum_{j=1}^{k} f_j(\varrho)\,C_j^{21}(x,\eta)+P_k^{21}(x,\eta\varrho)\ \text{in}\ \mathscr{S}'_{x\eta}(D_0) \tag{2.9}$$

as an equivalent form of (2.4). We thus have

Theorem 1. *As a consequence of causality the operator products A_1A_2 and A_2A_1 may be expanded with the same coefficients f_j*

$$A_a(x+\varrho\eta)\,A_b(x-\varrho\eta)=\sum_{j=1}^{k} f_j(\varrho)\,C_j^{ab}(x,\eta)+P_{k+1}^{ab}(x,\eta\varrho)\ \text{in}\ \mathscr{S}'_{x\eta}(D_0)$$
$$(a,b=1,2)\,.$$

For spacelike η the operators C_j^{12} and C_j^{21} are related by

$$C_j^{21}(x, -\eta) = \pm C_j^{12}(x, \eta) \quad \textit{in} \quad \mathscr{S}'_{x\eta}, \quad \eta^2 < 0\,. \tag{2.10}$$

3. Support Properties

We begin with the derivation of some support properties for the composite operator C_1. For the time being we consider matrix elements of $A_a(x+\varrho\eta)\,A_b(x-\varrho\eta)$ and $C_1^{ab}(x\eta)$ between vectors Φ, $\Psi \in B$. B denotes the domain of all vectors which can be obtained by applying polynomials of the basic fields $O_j(f)$ to the vacuum where the Fourier transform of the test functions f has compact support. Since such vectors are of bounded energy-momentum it will be possible to establish support properties in momentum space. It is sufficient to take vectors of the special form

$$\begin{aligned} \Phi &= A'_1(f_1)\,\ldots\,A'_n(f_n)\,\Omega \in B \\ \Psi &= A''_1(g_1)\,\ldots\,A''_m(g_m)\,\Omega \in B \end{aligned} \tag{3.1}$$

where A'_j, A''_j denote basic fields $O_1, \ldots, O_c$. We first express

$$\phi^{ab}(x, \eta\varrho) = (\Phi, A_a(x+\varrho\eta)\,A_b(x-\varrho\eta)\,\Psi) \qquad a, b = 1, 2$$

by a Wightman function in momentum space. To this end we form the Fourier integral

$$\begin{aligned} \phi^{ab}(x, \eta, \varrho) &= \frac{1}{(2\pi)^4} \int dr_1\, dr_2\, e^{-ix(r_1+r_2)-i\varrho\eta(r_1-r_2)} (\Phi, \tilde{A}_a(r_1)\,\tilde{A}_b(r_2)\,\Psi) \\ &= \frac{1}{(4\pi\varrho)^4} \int du\, dv\, e^{-i\eta u - ixv}\, (\Phi, \tilde{A}_a(r_1)\,\tilde{A}_b(r_2)\,\Psi) \end{aligned} \tag{3.2}$$

with

$$\begin{aligned} &v = r_1 + r_2, \qquad u = \varrho(r_1 - r_2) \\ &r_1 = \frac{v}{2} + \frac{u}{2\varrho}, \qquad r_2 = \frac{v}{2} - \frac{u}{2\varrho}\,. \end{aligned} \tag{3.3}$$

The Fourier transform of ϕ^{ab} with respect to η becomes

$$\begin{aligned} \tilde{\phi}^{ab}(x, u, \varrho) &= \frac{1}{(2\pi)^2} \int d\eta\, e^{i\eta u}\, \phi^{ab}(x, \eta\varrho) \\ &= \frac{1}{(2\varrho)^4 (2\pi)^2} \int dv\, e^{-ixv} (\Phi, \tilde{A}_a(r_1)\,\tilde{A}_b(r_2)\,\Psi) \end{aligned} \tag{3.4}$$

inserting the state vectors (3.1) we obtain

$$\tilde{\phi}^{ab}(x,u,\varrho)=\frac{1}{(2\varrho)^4(2\pi)^2}\int dp\,dq\,e^{-ix(Q-P)}\tilde{f}_1(p_1)\ldots\tilde{f}_n(p_n)\,\tilde{g}_1(q_1)\ldots\tilde{g}_m(q_m)\cdot W(S_1,\ldots,S_{n+m+1}) \tag{3.5}$$

with

$$\begin{aligned}&dp=dp_1\ldots dp_n,\quad dq=dq_1\ldots dq_m,\quad P=-\sum_{j=1}^{n}p_j,\quad Q=\sum_{j=1}^{m}q_j\\&S_1=p_n,\quad S_2=-p_n-p_{n-1},\ldots,S_n=P\\&S_{n+1}=\frac{P+Q}{2}+\frac{u}{2\varrho}\\&S_{n+2}=Q,\quad S_{n+3}=q_2-\cdots-q_m,\ldots,S_{n+m+1}=-q_1\,.\end{aligned} \tag{3.6}$$

The Wightman function W is defined by

$$\begin{aligned}&\langle\tilde{A}_n'^*(-p_n)\ldots\tilde{A}_1'^*(-p_1)\,\tilde{A}_a(r_1)\,\tilde{A}_b(r_2)\,\tilde{A}_1''(q_1)\ldots A_m''(q_m)\rangle\\&=\delta(P+v-Q)\,W(S_1,\ldots,S_{n+m+1})\end{aligned} \tag{3.7}$$

where translation invariance was used for separating $\delta(P+v-Q)$ [6]. $\tilde{A}_j'^*$ denotes the Fourier transform of the adjoint $A_j^*(x)$, i.e.,

$$A_j'^*(-p_j)=\tilde{A}_j'(p_j)^*\,. \tag{3.8}$$

According to (3.5–3.7), $\tilde{\phi}$ is a tempered distribution in u for given x. Hence ϕ is a tempered distribution in η. In a similar way translation invariance can be used to show that

$$g^{ab}(x,\eta)=(\Phi,C_1^{ab}(x,\eta)\,\Psi)\qquad\Phi,\Psi\in B \tag{3.9}$$

is a tempered distribution in η only. The relation

$$(\Phi,C_1^{ab}(x,\eta)\,\Psi)=\lim_{\varrho\to0}\frac{(\Phi,A_a(x+\varrho\eta)\,A_b(x-\varrho\eta)\,\Psi)}{f_1(\varrho)}\quad\text{in }\mathcal{S}'_{x\eta}\ \text{ for }\Phi,\Psi\in B \tag{3.10}$$

may therefore be interpreted as a limit relation of tempered distributions in η at any value of x

$$g^{ab}(x,\eta)=\lim_{\varrho\to0}\frac{\phi^{ab}(x,\eta\varrho)}{f_1(\varrho)}\quad\text{in}\quad\mathcal{S}'_\eta\,. \tag{3.11}$$

For the Fourier transforms $\tilde{\phi}$ and

$$\tilde{g}^{ab}(x,u)=\frac{1}{(2\pi)^2}\int d\eta\,e^{i\eta u}g^{ab}(x,\eta) \tag{3.12}$$

we obtain the relation

$$\tilde{g}^{ab}(x,u) = \lim_{\varrho\to 0} \frac{\tilde{\Phi}^{ab}(x,u,\varrho)}{f_1(\varrho)} \quad \text{in} \quad \mathscr{S}'_u\,. \tag{3.13}$$

The support properties of $\tilde{\phi}$ follow from the support properties of the Wightman function (3.7). We obtain

$$\tilde{\phi}^{ab}(x,u,\varrho) = 0 \tag{3.14}$$

unless

$$u + \varrho(P+Q) \in \bar{V}_+ \tag{3.15}$$

for at least one vector

$$P+Q \in C\,. \tag{3.16}$$

The compact set C is given by the conditions

$$\begin{aligned} &P = -\Sigma p_j, &&Q = +\Sigma q_j\\ &p_j \in \operatorname{supp} \tilde{f}_j, &&q_j \in \operatorname{supp} \tilde{q}_j\\ &-p_1 \in \bar{V}_+, &&-p_1 - p_2 \in \bar{V}_+, \ldots, P \in \bar{V}_+\\ &-q_1 \in \bar{V}_+, &&-q_1 - q_2 \in \bar{V}_+, \ldots, Q \in \bar{V}_+ \end{aligned} \tag{3.17}$$

(3.13–3.17) then implies that

$$\tilde{g}^{ab}(x,u) = 0 \quad \text{unless} \quad u \in \bar{V}_+\,. \tag{3.18}$$

For if $u \notin \bar{V}_+$ we can find a value $\varepsilon > 0$ such that

$$u + \varrho(P+Q) \notin \bar{V}_+ \quad \text{for any} \quad P+Q \in C$$

provided $\varrho \leqq \varepsilon$.

The result (3.18) can easily be carried over to the general case of the operator C_j^{ab}. To this end we form matrix elements of P_j^{ab} and C_j^{ab} between vectors Φ, Ψ of the form (3.1)

$$\begin{aligned} \phi_j^{ab}(x,\eta\varrho) &= (\Phi, P_j^{ab}(x,\eta\varrho)\,\Psi)\\ g_j^{ab}(x,\eta) &= (\Phi, C_j^{ab}(x,\eta)\,\Psi)\\ \tilde{\phi}_j^{ab}(x,u,\varrho) &= \frac{1}{(2\pi)^2}\int d\eta\, e^{i\eta u}\,\phi_j^{ab}(x,\eta\varrho)\\ \tilde{g}_j^{ab}(x,u) &= \frac{1}{(2\pi)^2}\int d\eta\, e^{i\eta u}\,g_j^{ab}(x,\eta)\,. \end{aligned} \tag{3.19}$$

Using the recursion formulae

$$\begin{aligned} \tilde{\phi}_j^{ab}(x,u,\varrho) &= f_j(\varrho)\,\tilde{g}_j^{ab}(x,u) + \tilde{\phi}_{j+1}^{ab}(x,u,\varrho)\\ \tilde{g}_j^{ab}(x,u) &= \lim_{\varrho\to 0} \frac{\tilde{\phi}_j^{ab}(x,u,\varrho)}{f_j(\varrho)} \end{aligned} \tag{3.20}$$

we obtain by induction that $\tilde{\phi}_j^{ab}$, $\tilde{g}_j^{ab}$ are distributions in $\mathscr{S}_u'$ with

$$\tilde{g}_j^{ab}(x,u)=0 \quad \text{unless} \quad u\in \overline{V}_+ \tag{3.21}$$

and

$$\tilde{\phi}_j^{ab}(x,u,\varrho)=0 \tag{3.22}$$

unless u satisfies (3.15–3.17) or $u\in \overline{V}_+$. We summarize the results by the following

Theorem 2. *For given x and vectors Φ, $\Psi\in B$ the matrix element*

$$g_j^{ab}(x,\eta)=(\Phi, C_j^{ab}(x,\eta)\,\Psi) \qquad a,b=1,2 \tag{3.23}$$

is a tempered distribution in η. Its Fourier transform with respect to η vanishes unless $u^2\geqq 0$, $u_0\geqq 0$.

Using continuity in Φ and Ψ we obtain the following

Corollary. *The Fourier transform $\tilde{g}_k^{ab}(x,u)$ of*

$$\begin{aligned} g_k^{ab}(x,\eta)&=(\Phi, C_k^{ab}(x,\eta)\,\Psi)\in \mathscr{S}_x' \\ a,b&=1,2, \qquad \Phi\in H,\ \Psi\in D_0 \end{aligned} \tag{3.24}$$

with respect to η vanishes unless $u^2\geqq 0$, $u_0\geqq 0$.

Proof. For $\Phi\in B$ given, (3.24) is continuous in Ψ. Hence the statement follows for all vectors $\Phi\in B$, $\Psi\in H$. Keeping now $\Psi\in D_0$ fixed and using continuity in Φ the theorem follows.

4. Analyticity

In this section we establish some analytic properties of the composite operators in the variable η. We first introduce linear combinations of C_j^{12}, C_j^{21} which are even or odd respectively for spacelike η

$$\begin{aligned} C_j^{\text{even}}(x,\eta)&=C_j^{12}(x,\eta)\pm C_j^{21}(x,\eta) \\ C_j^{\text{odd}}(x,\eta)&=C_j^{12}(x,\eta)\mp C_j^{21}(x,\eta)\,. \end{aligned} \tag{4.1}$$

The signs are chosen corresponding to the signs in Eq. (2.9) and (2.10). Matrix elements between vectors (3.1) are denoted by

$$\begin{aligned} g_j^{\text{even}}(x,\eta)&=(\Phi, C_j^{\text{even}}(x,\eta)\,\Psi)=g_j^{12}(x,\eta)\pm g_j^{21}(x,\eta) \\ g_j^{\text{odd}}(x,\eta)&=(\Phi, C_j^{\text{odd}}(x,\eta)\,\Psi)=g_j^{12}(x,\eta)\mp g_j^{21}(x,\eta). \end{aligned} \tag{4.2}$$

For spacelike η

$$\begin{aligned} C_j^{\text{even}}(x,-\eta)&=C_j^{\text{even}}(x,\eta), \qquad g_j^{\text{even}}(x,-\eta)=g_j^{\text{even}}(x,\eta) \\ C_j^{\text{odd}}(x,-\eta)&=-C_j^{\text{odd}}(x,\eta), \qquad g^{\text{odd}}(x,-\eta)=-g_j^{\text{odd}}(x,\eta) \quad \text{if} \quad \eta^2<0\,. \end{aligned} \tag{4.3}$$

g_j^{even} and g_j^{odd} have the support properties of Theorem 2 which imply the following theorem on the analytic continuation of $g_j^{(\ )}(x,\eta)$ in η (see for instance Ref. 2, Section 2–3). $^{(\ )}$ stands for the superscript even or odd.

Theorem 3. *The matrix element*

$$g_j^{(\ )}(x,\eta)=(\Phi, C_j^{(\ )}(x,\eta)\,\Psi) \quad \Phi,\ \Psi\in B$$

is the boundary value

$$g_j^{(\ )}(x,\eta_1)=\lim_{\eta_2\to+0} G_j^{(\ )}(x,\eta_1-i\eta_2) \quad in \quad \mathcal{S}'_{\eta_1} \tag{4.4}$$

of an analytic function $G_j^{(\ )}(x,\eta)$ *which is regular and bounded by a polynomial inside the cone* $-\operatorname{Im}\eta\in V_+$.

The functions

$$\hat G_j^{\text{even}}(x,\eta)=G_j^{\text{even}}(x,-\eta)\,; \qquad \hat G_j^{\text{odd}}(x,\eta)=-G_j^{\text{odd}}(x,-\eta)$$

are analytic and bounded by a polynomial inside the cone $\operatorname{Im}\eta\in V_+$.

$$\begin{aligned}\lim_{-\eta_2\to+0} \hat G_j^{(\ )}(x,\eta_1-i\eta_2)&=\pm g_j^{(\ )}(x,-\eta_1)\\ &=g_j^{(\ )}(x,\eta_1) \quad for \quad \eta_1^2<0.\end{aligned} \tag{4.5}$$

$G_j^{(\ )}(x,\eta)$ *and* $\hat G_j^{(\ )}(x,\eta)$ *are analytic at spacelike* η *and continuations of each other*

$$G_j^{(\ )}(x,\eta)=\hat G_j^{(\ )}(x,\eta) \quad for \quad \eta \ real\ and\ \eta^2<0\,. \tag{4.6}$$

Remark. In the preceding theorem and the work that follows an analytic function $F(\eta)$ which is regular in $\mp\operatorname{Im}\eta\in V_+$ is called bounded by a polynomial inside this region if there exists a polynomial $P_u(\eta)$ depending on a four vector $u\in V_+$ such that

$$|F(\eta)|\leqq|P_u(\eta)|$$

for any $\operatorname{Re}\eta$ and $\mp\operatorname{Im}\eta=u+v$ with $u\in V_+$ fixed, $v\in V_+$ arbitrary.

As a consequence of the Corollary to Theorem 2 we have

Theorem 4. *Let* $w(\eta)$ *be a function which is the boundary value*

$$w(\eta_1)=\lim_{\eta_2\to+0} W(\eta_1-i\eta_2)$$

of an analytic function $W(\eta)$, *regular and bounded by a polynomial inside the cone* $-\operatorname{Im}\eta\in V_+$. *Then*

$$w(\eta)\,C_j^{ab}(x,\eta)$$

defines an operator in $\mathcal{S}'_{x\eta}(D_0)$.

Proof. The matrix element

$$m(\eta) = \int dx\, s(x)\, (\Phi, C_k^{ab}(x, \eta)\, \Psi); \qquad \Phi \in H, \ \Psi \in D_0, \ s \in \mathscr{S}(R_4) \quad (4.7)$$

is the boundary value

$$m(\eta_1) = \lim_{\eta_2 \to +0} M(\eta_1 - i\eta_2)$$

of an analytic function $M(\eta)$ which is regular and bounded by a polynomial inside the cone $-\operatorname{Im}\eta \in V_+$. Therefore the product $w(\eta)\, m(\eta)$ is well defined by

$$w(\eta_1)\, m(\eta_1) = \lim_{\eta_2 \to +0} W(\eta_1 - i\eta_2)\, M(\eta_1 - i\eta_2).$$

Since (4.7) is linear and continuous in Φ there exists a vector $\hat{\Psi}$ with

$$\int d\eta\, u(\eta)\, W(\eta)\, m(\eta) = (\Phi, \hat{\Psi}).$$

The equation

$$\int dx\, d\eta\, s(x)\, u(\eta)\, C_k^{ab}(x, \eta)\, \Psi = \hat{\Psi} \qquad u \in \mathscr{S}(R_4)$$

defines a linear operator in $\mathscr{S}'_{x\eta}(D_0)$.

5. Scaling Law

We will make use of the fact that

$$P^{ab}(x, \eta, \varrho) = A_a(x + \varrho\eta)\, A_b(x - \varrho\eta)$$

depends on ϱ and η through the product $\varrho\eta$ only. For every test function $t \in \mathscr{S}_{x\eta}$ we have

$$P^{ab}(t, \varrho) = P^{ab}\left(t_\sigma, \frac{\varrho}{\sigma}\right) \quad \text{on} \quad D_0, \tag{5.1}$$

where t_σ is the test function

$$t_\sigma(x, \eta) = \sigma^{-4}\, t\left(x, \frac{\eta}{\sigma}\right). \tag{5.2}$$

(5.1) implies

$$P^{ab}(t, \varrho) = \sum_{j=1}^{k} f_j\left(\frac{\varrho}{\sigma}\right) C_{\sigma j}^{ab}(t) + P_{k+1}^{ab}\left(t_\sigma, \frac{\varrho}{\sigma}\right) \tag{5.3}$$

with

$$C_{\sigma j}^{ab}(t) = C_j^{ab}(t_\sigma). \tag{5.4}$$

Since (5.3) is an equivalent form of the Wilson expansion the uniqueness theorem implies the scaling law

$$C_{\sigma j}^{ab}(t) = \sum_{j'=1}^{j} s_{jj'}(\sigma)\, C_{j'}^{ab}(t) \tag{5.5}$$

or

$$C_j^{ab}(\sigma\eta) = \sum_{j'=1}^{j} s_{jj'}(\sigma)\, C_{j'}^{ab}(\eta) \tag{5.6}$$

with real coefficients $s_{jj'}$. Since the matrix elements of C_j^{ab} between vectors $\Phi, \Psi \in B$ are analytic for spacelike η the functions $s_{jj'}$ must be differentiable.

For the moment we restrict ourselves to the first k composite operators. We write the first k equations of (5.6) in matrix form

$$C^{ab}(\sigma\eta) = s(\sigma)\, C^{ab}(\eta) \tag{5.7}$$

with

$$s = \begin{pmatrix} s_{11} & 0 & 0 \;\dots & 0 \\ s_{21} & s_{22} & 0 \;\dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ s_{k1} & s_{k2} & s_{k3} \;\dots & s_{kk} \end{pmatrix}, \quad C^{ab} = \begin{pmatrix} C_1^{ab} \\ C_2^{ab} \\ \cdot \\ \cdot \\ \cdot \\ C_k^{ab} \end{pmatrix}. \tag{5.8}$$

The multiplication law

$$s(\sigma\tau) = s(\sigma)\, s(\tau) \tag{5.9}$$

follows from the identity

$$s(\sigma\tau)\, C(\eta) = C(\sigma\tau\eta) = s(\sigma)\, s(\tau)\, C(\eta)\,.$$

Accordingly $s(\sigma)$ is a k-dimensional, real and differentiable representation of the multiplicative group of the real numbers. Introducing new composite operators

$$\begin{gathered} C' = TC \qquad T_{jj'} = 0 \quad \text{for} \quad j < j' \\ C'(\sigma\eta) = s'(\sigma)\, C'(\eta) \end{gathered} \tag{5.10}$$

by a suitable triangular matrix T the matrices s can be transformed into [7]

$$s' = \begin{pmatrix} \begin{array}{c|c} s_1 & 0 \\ \hline 0 & s_2 \\ \hline 0 & 0 \end{array} & & \\ & \ddots & 0 \\ & 0 & \boxed{s_n} \end{pmatrix} \tag{5.11}$$

where each s_j is an $N_j \times N_j$ matrix of the normal form

$$= \sigma^{c_j} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lg\sigma & 1 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2}\lg^2\sigma & \lg\sigma & 1 & \cdots & 0 & 0 & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ & & & & 1 & 0 & 0 \\ & & & & \lg\sigma & 1 & 0 \\ \frac{\lg^{N_j-1}\sigma}{(N_j-1)!} & & & \cdots & \frac{1}{2}\lg^2\sigma & \lg\sigma & 1 \end{pmatrix} \tag{5.12}$$

with real exponents c_j.

It is convenient to relabel the operators C_j' according to the decomposition of s' into submatrices s_j

$$C_{11}, \ldots, C_{1N_1}; \quad C_{21} \cdots \quad ; \quad C_{n1}, \ldots, C_{nN_n} \tag{5.13}$$

(5.17) may then be broken up into relations

$$C_{(j)}(\sigma\eta) = s_j(\sigma)\, C_{(j)}(\eta) \tag{5.14}$$

with

$$C_{(j)} = \begin{pmatrix} C_{j1} \\ \vdots \\ C_{jN_j} \end{pmatrix}.$$

The scaling law (5.14) can be used to continue the matrix element

$$g_{jk}^{ab}(x, \eta) = (\Phi, C_{jk}^{ab}(x, \eta)\, \Psi) \qquad \Phi, \Psi \in B \tag{5.15}$$

beyond the domain given in Theorem 3. Since both sides of

$$g_{jl}^{ab}(x, \sigma\eta) = \sum_{l'=1}^{l} s_{jll'}(\sigma)\, g_{jl'}^{ab}(x, \eta)$$

are boundary values of analytic functions in η a similar relation holds for their continuations

$$G_{jl}^{ab}(x, \sigma\eta) = \sum_{l'=1}^{l} s_{jll'}(\sigma)\, G_{jl'}^{ab}(x, \eta) \quad \text{if} \quad -\operatorname{Im}\eta \in V_+ . \tag{5.16}$$

For given η both sides of (5.16) are analytic in σ. Hence we find that (5.16) holds for all values η and σ with

$$-\operatorname{Im}\eta \in V_+ , \qquad -\operatorname{Im}\sigma\eta \in V_+ . \tag{5.17}$$

This allows defining the further continuation

$$G_{jl}^{ab}(x, \eta) = \sum_{l'=1}^{l} s_{jll'}(\sigma)\, G_{j'}^{ab}\left(x, \frac{\eta}{\sigma}\right) \quad \text{if} \quad -\operatorname{Im}\frac{\eta}{\sigma} \in V_+ . \tag{5.18}$$

According to (5.12–5.14) the first operator of each subsequence satisfies a power scaling law

$$C_{j1}(x, \sigma\eta) = \sigma^{c_j} C_{j1}(x, \eta) . \tag{5.19}$$

We will show in general that every composite operator can be reduced to operators satisfying power scaling laws. To this end we introduce new operators Q_{jk}^{ab} by

$$Q_{(j)}^{ab}(x, \eta) = t_j(\eta)\, C_{(j)}^{ab}(x, \eta) \tag{5.20}$$

where $t_j(\eta)$ denotes the matrix

$$\begin{aligned} t_j(\eta) &= \lim_{\varepsilon \to +0} (\sqrt{-\eta^2 + i\varepsilon\eta_0})^{c_j}\, s_j\left(\frac{1}{\sqrt{-\eta^2 + i\varepsilon\eta_0}}\right) \\ &= \lim_{u \to +0} (\sqrt{-(\eta - iu)^2})^{c_j}\, s_j\left(\frac{1}{\sqrt{-(\eta - iu)^2}}\right). \end{aligned} \tag{5.21}$$

$\sqrt{-(\eta - iu)^2}$ is defined in $u \in V_+$ by continuing from spacelike η with $\sqrt{-\eta^2} > 0$. According to Theorem 4 (5.20) defines an operator in $\mathscr{S}'_{x\eta}(D_0)$. The scaling law of (5.20) immediately follows from (5.9) and (5.14)

$$Q_{(j)}^{ab}(x, \sigma\eta) = \sigma^{c_j} Q_{(j)}^{ab}(x, \eta) . \tag{5.22}$$

The operators C_{jl}^{ab} can easily be expressed as linear combinations of the Q_{jl}^{ab}:

$$C_{(j)}^{ab}(x, \eta) = \lim_{\varepsilon \to +0} (\sqrt{-\eta^2 + i\varepsilon\eta_0})^{-c_j}\, s_j(\sqrt{-\eta^2 + i\varepsilon\eta_0})\, Q_{(j)}^{ab}(x, \eta) .$$

Combining the power scaling law (5.22) with analyticity we will determine the general form of the operators Q_{jl}^{ab} in the following section.

6. Composite Operators with Power Scaling Law

In the last section it was shown that composite operators either satisfy the power scaling law (5.19) in η or can be written as linear combinations of such operators. We summarize the properties of these operators Q_{jl}^{ab} which have been established so far. The subscripts j, l will be omitted throughout this section.

(i) $Q^{ab}(x, \eta)$ is an operator in $\mathscr{S}'_{x\eta}(D_0)$.

(ii) The even and odd parts.

$$Q^{\text{even}} = Q^{12} \pm Q^{21}, \quad Q^{\text{odd}} = Q^{12} \mp Q^{21} \tag{6.1}$$

satisfy

$$\begin{aligned} Q^{\text{even}}(x, -\eta) &= Q^{\text{even}}(x, \eta) \\ Q^{\text{odd}}(x, -\eta) &= -Q^{\text{odd}}(x, \eta). \end{aligned} \quad \text{if} \quad \eta^2 < 0 \tag{6.2}$$

(iii) The matrix element

$$g^{(\,)}(x, \eta) = (\Phi, Q^{(\,)}(x, \eta)\, \Psi) \qquad \Phi, \Psi \in B \tag{6.3}$$

satisfies the analytic properties of Theorem 3.

(iv) A power scaling law holds for the operator

$$Q^{(\,)}(x, \sigma\eta) = \sigma^c Q^{(\,)}(x, \eta) \tag{6.4}$$

and the continuations of the matrix element (6.3)

$$\begin{aligned} G^{(\,)}(x, \sigma\eta) &= \sigma^c G^{(\,)}(x, \eta) \qquad -\operatorname{Im}\eta \in V_+, \quad -\operatorname{Im}\sigma\eta \in V_+. \\ \hat{G}^{(\,)}(x, \sigma\eta) &= \sigma^c \hat{G}(x, \eta) \qquad \operatorname{Im}\eta \in V_+, \quad \operatorname{Im}\sigma\eta \in V_+. \end{aligned} \tag{6.5}$$

Using this information we will first investigate the η-dependence of the matrix elements (6.3).

6a. *Directional Dependence of Matrix Elements*

The following theorem states that the function G^{ab} is a polynomial in η apart from a power of η^2.

Theorem 5. *The function G^{ab} is of the general form*

$$G^{ab}(x, \eta) = (\sqrt{-\eta^2})^c\, \Pi^{ab}\left(x, \frac{\eta}{\sqrt{-\eta^2}}\right) \tag{6.6}$$

where $\Pi^{ab}(x, \zeta)$ is a polynomial in the components of the four vector ζ.

For the proof we first list some properties of the function

$$D_f^{(\,)}(x, \eta) = (-\eta^2)^f\, G^{(\,)}(x, \eta) \tag{6.7}$$

which immediately follow from the corresponding properties of $G^{(\,)}$. Here f is a real number, $(-\eta^2)^f$ denotes the continuation of the positive values at spacelike η into the regions $-\operatorname{Im}\eta \in V_+$ and $\operatorname{Im}\eta \in V_+$.

(i) $D_f^{(\,)}$ is analytic and bounded by a polynomial in $-\operatorname{Im}\eta \in V_+$.

(ii) The functions

$$\hat{D}_f^{\substack{\text{even}\\ \text{odd}}}(x,\eta) = \pm D_f^{\substack{\text{even}\\ \text{odd}}}(x,-\eta) = (-\eta^2)^f \hat{G}^{\substack{\text{even}\\ \text{odd}}}(x,\eta) \tag{6.8}$$

are analytic and bounded by a polynomial in $\mathrm{Im}\,\eta \in V_+$.

(iii) For spacelike η the functions D_f and $\hat{D}_f$ coincide

$$D_f^{(\,)}(x,\eta) = \hat{D}_f^{(\,)}(x,\eta) \quad \text{if } \eta \text{ real and } \eta^2 < 0\,. \tag{6.9}$$

From property (i) we obtain (Theorem 2–8 of Ref. [2]) that $D_f^{(\,)}$ is a Laplace transform

$$D_f^{(\,)}(x,\eta_1 - i\eta_2) = \frac{1}{(2\pi)^2}\int e^{-ip(\eta_1 - i\eta_2)}\,\tilde{d}_{f+}^{(\,)}(x,p)\,dp \tag{6.10}$$

provided $\eta_2 \in V_+$. From (ii) we find

$$\hat{D}_f^{(\,)}(x,\eta_1 + i\eta_2) = \frac{1}{(2\pi)^2}\int e^{-ip(\eta_1 + i\eta_2)}\,\tilde{d}_{f-}^{(\,)}(x,p)\,dp \tag{6.11}$$

with $\eta_2 \in V_+$. $\tilde{d}_{f\pm}^{(\,)}(x,p)$ are tempered distributions in $\mathscr{S}'p$ and have the support properties

$$\begin{aligned} \tilde{d}_{f+}^{(\,)}(x,p) &= 0 \quad \text{unless} \quad p \in \bar{V}_+ \\ \tilde{d}_{f-}^{(\,)}(x,p) &= 0 \quad \text{unless} \quad p \in \bar{V}_-\,. \end{aligned} \tag{6.12}$$

According to Theorem 2–9 of Ref. [2] the limits of (6.10) and (6.11) for $\eta_2 \to +0$ are

$$\begin{aligned} &\lim_{\eta_2 \to +0} D_f^{(\,)}(x,\eta_1 - i\eta_2) = d_{f+}^{(\,)}(x,\eta_1) \\ &= \frac{1}{(2\pi)^2}\int e^{-ip\eta_1}\,\tilde{d}_{f+}^{(\,)}(x,p)\,dp \quad \text{in} \quad \mathscr{S}'_{\eta_1} \\ &\lim_{\eta_2 \to +0} \hat{D}_f^{(\,)}(x,\eta_1 + i\eta_2) = d_{f-}^{(\,)}(x,\eta_1) \\ &= \frac{1}{(2\pi)^2}\int e^{-ip\eta_1}\,\tilde{d}_{f-}^{(\,)}(x,p)\,dp \quad \text{in} \quad \mathscr{S}'_{\eta_1}\,. \end{aligned} \tag{6.13}$$

Our aim is to choose f such that $D_f^{(\,)} = \hat{D}_f^{(\,)}$ for η real and $\eta^2 \neq 0$. To this end we form

$$\begin{aligned} H_{\mathrm{I}}(x,\eta) &= D^{\text{even}}_{-\frac{c}{2}}(x,\eta) = \frac{G^{\text{even}}(x,\eta)}{(\sqrt{-\eta^2})^c} \\ H_{\mathrm{II}}(x,\eta) &= D^{\text{odd}}_{-\frac{c+1}{2}}(x,\eta) = \frac{G^{\text{odd}}(x,\eta)}{(\sqrt{-\eta^2})^{c+1}} \end{aligned} \tag{6.14}$$

$$\hat{H}_{\mathrm{I}}(\eta) = H_{\mathrm{I}}(-\eta)\,, \qquad \hat{H}_{\mathrm{II}}(\eta) = -H_{\mathrm{II}}(-\eta) \tag{6.15}$$

with the boundary values

$$h_{\mathrm{I}\pm} = d^{\mathrm{even}}_{-\frac{c}{2},\pm}\,, \qquad h_{\mathrm{II},\pm} = d^{\mathrm{odd}}_{-\frac{c+1}{2},\pm}\,. \tag{6.16}$$

The scaling law then takes the form

$$H_{\mathrm{I}}(x, \sigma\eta) = H_{\mathrm{I}}(x, \eta)\,, \tag{6.17}$$

$$H_{\mathrm{II}}(x, \sigma\eta) = \frac{1}{\sigma} H_{\mathrm{II}}(x, \eta)\,. \tag{6.18}$$

(6.17) implies that $H_{\mathrm{I}}(\eta)$ is analytic for timelike η and satisfies

$$H_{\mathrm{I}}(x, -\eta) = H_{\mathrm{I}}(x, \eta) \quad \text{if} \quad \eta \in V_{\pm}\,. \tag{6.19}$$

For the proof we form

$$H_{\mathrm{I}}(x, \sigma\eta) \tag{6.20}$$

with

$$\eta \in V_+\,, \qquad \sigma = re^{-i\varphi}, \qquad r > 0, \qquad 0 < \phi < \pi\,.$$

Then $\sigma\eta$ lies in the regularity domain

$$-\operatorname{Im}(\sigma\eta) \in V_+\,.$$

Due to (6.17) the expression (6.20) is independent of σ which shows that H_{I} is regular at $\eta \in V_{\pm}$. Taking the limit $\phi \to 0$ and $\phi \to \pi$ of (6.20) we find (6.19). Combining (6.9) and (6.19) we get

$$H_{\mathrm{I}}(x, \eta) = \hat{H}_{\mathrm{I}}(x, \eta) \quad \text{if } \eta \text{ real and } \eta^2 \neq 0\,. \tag{6.21}$$

Similarly for H_{II}

$$H_{\mathrm{II}}(x, \eta) = -H_{\mathrm{II}}(x, \eta) \quad \text{if} \quad \eta \in V_1\,, \tag{6.22}$$

$$H_{\mathrm{II}}(x, \eta) = \hat{H}_{\mathrm{II}}(x, \eta) \quad \text{if } \eta \text{ real and } \eta^2 \neq 0\,. \tag{6.23}$$

As a consequence of (6.21), (6.23) the boundary values from above and below agree for all real η except $\eta^2 = 0$:

$$\left.\begin{aligned} \Delta_{\mathrm{I}}(x, \eta) &= h_{\mathrm{I}+}(x, \eta) - h_{\mathrm{I}-}(x, \eta) = 0 \\ \Delta_{\mathrm{II}}(x, \eta) &= h_{\mathrm{II}+}(x, \eta) - h_{\mathrm{II}-}(x, \eta) = 0 \end{aligned}\right\} \quad \text{if} \quad \eta^2 \neq 0\,. \tag{6.24}$$

Now we use the following lemma which is proved in the appendix.

Lemma. *Let $D(\eta) \in \gamma'(R_4)$ be a distribution vanishing for $\eta^2 \neq 0$. Then there exists an integer k such that $(-\eta^2)^k D(\eta) = 0$. If $D(\eta)$ is bounded in the norm $\|\ \|_{rs}$ any integer $k > |s|$ will do.*

As an immediate consequence we have, finally

$$(-\eta^2)^{m_{\mathrm{I}}}\, \Delta_{\mathrm{I}}(x,\eta)=0$$
$$(-\eta^2)^{m_{\mathrm{II}}}\, \Delta_{\mathrm{II}}(x,\eta)=0 \tag{6.25}$$

for some positive integers $m_{\mathrm{I}}, m_{\mathrm{II}}$.

We next form

$$E_{\mathrm{I}}(x,\eta)=(-\eta^2)^{m_{\mathrm{I}}}\, H_{\mathrm{I}}(x,\eta)=D^{\mathrm{even}}_{m_{\mathrm{I}}-\frac{c}{2}}(x,\eta)$$
$$E_{\mathrm{II}}(x,\eta)=(-\eta^2)^{m_{\mathrm{II}}} H_{\mathrm{II}}(x,\eta)=D^{\mathrm{odd}}_{m_{\mathrm{II}}-\frac{c+1}{2}}(x,\eta)\,. \tag{6.26}$$

We denote the boundary values by

$$e_{\mathrm{I},\pm}=d^{\mathrm{even}}_{m_{\mathrm{I}}-\frac{c}{2},\,\pm}\,,\qquad e_{\mathrm{II},\pm}=d^{\mathrm{odd}}_{m_{\mathrm{II}}-\frac{c+1}{2},\,\pm}\,. \tag{6.27}$$

From (6.25) it follows

$$e_{\mathrm{I}+}(\eta)=e_{\mathrm{I}-}(\eta)$$
$$e_{\mathrm{II}+}(\eta)=e_{\mathrm{II}-}(\eta)\,. \tag{6.28}$$

Hence

$$\tilde{e}^{+}_{(\,)}(p)=\tilde{e}^{-}_{(\,)}(p)\,, \tag{6.29}$$

where $_{(\,)}$ stands for the subscript I or II. Since $e^{+}_{(\,)}$ has support in $\bar{V}_{+}$ and $e^{-}_{(\,)}$ has support in $\bar{V}_{-}$ we get

$$\tilde{e}^{\pm}_{(\,)}(p)=0 \quad \text{unless} \quad p=0\,. \tag{6.30}$$

Therefore, the Fourier transform $e_{(\,)}(x,\eta)$ must be a polynomial in η. On the other hand the scaling law

$$E_{\mathrm{I}}(x,\sigma\eta)=\sigma^{2m_{\mathrm{I}}}E_{\mathrm{I}}(x,\eta)$$
$$E_{\mathrm{II}}(x,\sigma\eta)=\sigma^{2m_{\mathrm{II}}-1}\,E_{\mathrm{II}}(x,\eta) \tag{6.31}$$

implies

$$E_{\mathrm{I}}(x,\eta)=(\sqrt{-\eta^2})^{2m_{\mathrm{I}}}\, E_{\mathrm{I}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right)$$
$$E_{\mathrm{II}}(x,\eta)=(\sqrt{-\eta^2})^{2m_{\mathrm{II}}} E_{\mathrm{II}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right). \tag{6.32}$$

Thus

$$H_{\mathrm{I}}(x,\eta)=E_{\mathrm{I}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right)$$
$$H_{\mathrm{II}}(x,\eta)=\frac{1}{\sqrt{-\eta^2}}\,E_{\mathrm{II}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right) \tag{6.33}$$

and

$$G^{\text{even}}(x,\eta)=(\sqrt{-\eta^2})^c\,E_{\mathrm{I}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right)$$
$$G^{\text{odd}}(x,\eta)=(\sqrt{-\eta^2})^c\,E_{\mathrm{II}}\left(x,\frac{\eta}{\sqrt{-\eta^2}}\right). \tag{6.34}$$

This leads to Eq. (6.6) of Theorem 4 with

$$\Pi^{12}=\tfrac{1}{2}(E_{\mathrm{I}}\pm E_{\mathrm{II}})\,,\qquad \Pi^{21}=\tfrac{1}{2}(E_{\mathrm{I}}\mp E_{\mathrm{II}})\,.$$

6b. Directional Dependence of the Operator

In the last section we have seen that every matrix element

$$(\Phi, Q^{ab}(x\eta)\,\Psi)\qquad \Phi,\Psi\in B$$

is a polynomial in $\frac{\eta}{\sqrt{-\eta^2}}$ apart from a factor $(\sqrt{-\eta^2})^c$. We proceed to establish a similar property for the operator Q^{ab}. To do this we must verify that the degree of the polynomial Π^{ab} is uniformly bounded for all x and $\Phi, \Psi\in B$.

The proof depends essentially on the following theorem, which is proved in the article [8] by Simon (Th. 11) and in Ref. [9].

Theorem *(uniform boundedness). Let $D(\alpha;\eta)$ be a family of distributions in $\mathscr{S}(\eta)$ parameterized by α, and such that for every testing function $u(\eta)$ there exists a real number $M(u)$ with*

$$|\textstyle\int d\eta\, D(\alpha;\eta)\,u(\eta)|\leqq M(u)\,. \tag{6.35}$$

Then the family $D(\alpha;\eta)$ is uniformly bounded:

$$|\textstyle\int d\eta\, D(\alpha;\eta)\,u(\eta)|\leqq M\,\|u\|_{r,s} \tag{6.36}$$

for some norm $\|\ \|_{r,s}$, finite number M, and every testing function $u(\eta)$.

In order to use this theorem we construct Δ as in the last section and indicate the dependence on Φ, Ψ, x explicitly

$$\Delta_{\mathrm{I}}(x,\eta;\Phi,\Psi)=\lim_{\varepsilon\to+0}\frac{1}{(\sqrt{-\eta^2+i\varepsilon\eta_0})^c}(\Phi,(Q^{\text{even}}(x,\eta)-Q^{\text{even}}(x,-\eta))\,\Psi)$$
$$\Delta_{\mathrm{II}}(x,\eta;\Phi,\Psi)=\lim_{\varepsilon\to+0}\frac{1}{(\sqrt{-\eta^2+i\varepsilon\eta_0})^{c+1}}(\Phi,(Q^{\text{odd}}(x,\eta)+Q^{\text{odd}}(x,-\eta))\,\Psi) \tag{6.37}$$

consider $\Delta_{(\,)}(x,\eta;\Phi,\Psi)$ as a family of distributions parameterized by Φ for fixed Ψ, use the continuity in Φ (for fixed Ψ and testing function $u(x,\eta)\in\mathscr{S}_{x\eta}$) to bound the family for fixed u, then appeal to the uniform boundedness theorem to obtain

$$\frac{1}{1+\|\Phi\|}\int dx\,d\eta\,u(x,\eta)\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)\leqq M_{(\,)}(\Psi)\,\|u\|_{r(\Psi),s(\Psi)}\,. \tag{6.38}$$

By the lemma of Section 6a this implies (since $\Delta_{(\,)}=0$ for $\eta^2\neq 0$) that

$$(\eta^2)^{m_{(\,)}(\Psi)}\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)=0\,. \tag{6.39}$$

Finally, locality in x is used to remove the Ψ-dependence of $m_{(\,)}(\Psi)$ and so complete the task. The details are as follows.

For every $\Phi\in B$, and testing function $u(x,\eta)\in\mathscr{S}_{x\eta}$

$$\int d\eta\,dx\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)\,u(x,\eta)\qquad\Psi\in B$$

is an inner product $(\Phi,\tilde{\Psi})$ (because of Theorem 4) and is thus continuous (or equivalently, bounded) in Φ:

$$\left|\int d\eta\,dx\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)\,u(x,\eta)\right|<M(\Psi,u)\,\|\Phi\|\,; \tag{6.40}$$

thus by the uniform boundedness theorem (6.38) and (6.39) are established and it remains only to remove the Ψ-dependence of $m_{(\,)}(\Psi)$. To resolve the last problem we choose the vacuum Ω for the vector Ψ. Then

$$(-\eta^2)^{m_{(\,)}(\Omega)}\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)=0$$

implies

$$N_{(\,)}(x,\eta)\,\Omega=0 \tag{6.41}$$

where $N_{(\,)}$ denotes one of the operators

$$\left.\begin{aligned}N_{\rm I}(x,\eta)&=(-\eta^2)^{m_{\rm I}-\frac{c}{2}}\,(C^{\rm even}(x,\eta)-C^{\rm even}(x,-\eta))\\ N_{\rm II}(x,\eta)&=(-\eta^2)^{m_{\rm II}-\frac{c+1}{2}}\,(C^{\rm odd}(x,\eta)+C^{\rm odd}(x,-\eta))\end{aligned}\right\} \tag{6.42}$$

$$(\Phi,N_{(\,)}(x,\eta)\,\Psi)=(-\eta^2)^{m_{(\,)}(-\Omega)}\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)\,.$$

Since $N_{(\,)}$ is local in x relative to the basic fields O_j

$$[N_{(\,)}(x,\eta)\,O_j(y)]_{\pm}=0\quad\text{on}\quad D_0\quad\text{if}\quad (x-y)^2<0$$

the relation (6.41) necessarily implies [10]

$$N_{(\,)}(x,\eta)=0\,. \tag{6.43}$$

Hence

$$(-\eta^2)^{m_{(\,)}}\,\Delta_{(\,)}(x,\eta;\Phi,\Psi)=0\qquad\Phi,\Psi\in B \tag{6.44}$$

where $m_{(\ )}$ is independent of x, Φ and Ψ. Thus the degree of Π^{ab} in (6.6) is uniformly bounded for x and Φ, $\Psi \in B$

$$(\Phi, Q^{ab}(x,\eta)\,\Psi) = \lim_{\varepsilon\to+0} (\sqrt{-\eta^2+i\varepsilon\eta_0})^c\, \Pi^{ab}(x,\zeta,\Phi,\Psi)$$
$$\zeta = \frac{\eta}{\sqrt{-\eta^2+i\varepsilon\eta_0}}, \quad \Phi,\ \Psi\in B; \quad \operatorname{degr}\Pi^{ab} \leqq N. \tag{6.45}$$

Smearing out in x and η with a test function of $\mathscr{S}_{x\eta}$ and using continuity of the left hand side in Ψ for $\Phi\in B$ given we can extend (6.45) to vectors $\Phi\in B$, $\Psi\in D_0$. Keeping $\Psi\in D_0$ fixed and using continuity in Φ we can further extend (6.45) to vectors $\Phi\in H$, $\Psi\in D_0$. It is then not difficult to verify that the coefficients of H^{ab} represent matrix elements of field operators.

We summarize the results of this Section in

Theorem 6. *The operator Q_{jl}^{ab} has the general form*

$$Q_{jl}^{ab}(x,\eta) = \lim_{\varepsilon\to+0} (\sqrt{-\eta^2+i\varepsilon\eta_0})^c \sum_{n=1}^{N} \sum_{\mu_1\ldots\mu_n} \zeta_{\mu_1}\ldots\zeta_{\mu_n} Q_{jl}^{ab\mu_1\ldots\mu_n}(x)$$
$$\zeta = \frac{\eta}{\sqrt{-\eta^2+i\varepsilon\eta_0}}. \tag{6.46}$$

The coefficient $Q_{ab}^{\mu_1\ldots\mu_n}(x)$ is an operator in $\mathscr{S}_x'(D_0)$. Under inhomogeneous Lorentz transformations $Q_j^{ab\mu_1\ldots\mu_n}$ follows the usual transformation law of a tensor of rank n. The operators $Q_j^{ab\mu_1\ldots\mu_n}$ are local relative to each other and relative to the basic fields.

Combining this theorem with Eq. (5.23) we have as final result

Theorem 7. *For $\eta^2 \neq 0$ given, the composite operators $C_j(x,\eta)$ appearing in the Wilson expansion*

$$A_a(x+\varrho\eta)\,A_b(x-\varrho\eta) = \sum_{j=1}^{h} f_j(\varrho)\, C_j^{ab}(x\eta) + P_{k+1}(x,\eta,\varrho) \tag{6.47}$$

are polynomials in the components of $\eta/\sqrt{-\eta^2}$. By a suitable equivalence transformation the Wilson expansion (6.47) *can be brought into the form*

$$A_a(x+\varrho\eta)\,A_b(x-\varrho\eta) = \sum_{j=1}^{n} \sum_{l=1}^{N_j} f_{jl}(\varrho)\, C_{jl}^{ab}(x\eta) + R(x\eta\varrho) \tag{6.48}$$

where the composite operators may be written as

$$C_{(j)}^{ab}(x,\eta) = \lim_{\varepsilon\to+0} (\sqrt{-\eta^2+i\varepsilon\eta_0})^{-c_j}\, s_j(\sqrt{-\eta^2+i\varepsilon\eta_0})\, Q_{(j)}^{ab}(x,\eta).$$

The $N_j\times N_j$ matrix s_j is given by (5.12). *The general form of the operators Q_{jl}^{ab} was stated in Theorem 6.*

Appendix

I: *Proof* of the Lemma, Section 6a.

Statement. *Let $D(\eta)$ be a distribution in $\mathcal{S}'(R^4)$, with $D(\eta)=0$ for $\eta^2 \neq 0$, and with $\int d\eta\, D(\eta)\, u(\eta) < M \|u\|_{rs}$, for each testing function $u(\eta) \in \mathcal{S}(R^4)$. Then for each integer $k > |s|$, $(\eta^2)^k D(\eta) = 0$.*

Proof. $\| \quad \|_{rs}$ denotes the norm $\|u\|_{rs} = \sum_{\substack{|\alpha|<r \\ |\beta|<s}} \sup_\eta |\eta^\alpha d_\eta^\beta u|$. Let $v \in \mathcal{S}(R)$ be such that $v(y)=1$ for $|y|<\frac{1}{2}$ and $v(y)=0$ for $|y|>1$; since D vanishes for $\eta^2 \neq 0$, one has for real $b>0$ and each testing function u that

$$\int d\eta\, D(\eta)\, (\eta^2)^k\, u(\eta) = \int d\eta\, D(\eta)\, (\eta^2)^k\, u(\eta)\, v(\eta^2/b)\,. \tag{A.1}$$

However, when $k > s$, we claim that given a real $\varepsilon > 0$ one may pick $b > 0$ such that $\|(\eta^2)^k\, v(\eta^2/b)\, u(\eta)\|_{rs} < \varepsilon/M$; thus the right hand side of (A.1) vanishes and the statement is proved. The claim may be verified as follows: $\|(\eta^2)^k\, u(\eta)\, v(\eta^2/b)\|_{rs}$ is a sum of terms of the form

$$\begin{aligned} &\sup_\eta |(\eta^l\, d_\eta^m\, u(\eta))\, ((\eta^2)^{k-k'}\, g^{(k'')}\, (\eta^2/b))|\, b^{-k''} \\ &\leqq \left(\sup_\eta |(\eta^l\, d_\eta^m\, u(\eta))|\right) (\sup |(\eta^2)^{k-k'}\, g^{(k'')}\, (\eta^2/b)|)\, b^{-k''} \qquad k'+k'' \leqq s \end{aligned}$$

each of which may be made arbitrarily small

$$\sup_\eta b^{-k''}\, |(\eta^2)^{k-k'}\, g^{(k'')}\, (\eta^2/b)| = b^{k-(k'+k'')} \sup_t |(t)^{k-k'}\, g^{(k'')}\, (t)|\,.$$

Corollary. *Since every distribution is bounded in some norm $\| \quad \|_{rs}$ it follows that for every $D(\eta) \in \mathcal{S}'(R^4)$ vanishing for $\eta^2 \neq 0$, there exists an integer k with $(\eta^2)^k D(\eta) = 0$.*

This corollary is sufficient for the purpose of Section 6a; Section 6b strictly speaking requires the stronger result (which may be proved analogously):

Corollary. *Let $D(x;\eta)$ be a distribution in $\mathcal{S}'(R_8)$ bounded in some norm $\| \quad \|_{rs}$ and vanishing for $\eta^2 \neq 0$; then $(\eta^2)^k D(x,\eta) = 0$ for $k > s$.*

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