

Operator Product Expansions and Composite Field Operators in the General Framework of Quantum Field Theory

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Received July 26, 1971

Abstract. The short distance behavior of field operator products is analyzed. It is shown that under certain conditions operator product expansions can be derived which give complete information on the short distance behavior and lead to the construction of composite field operators.

1. Introduction

A central problem in local quantum field theory is the definition of products of field operators at the same point. Important examples of such composite field operators are Lagrangian densities, energy-momentum tensors, interaction terms of local field equations and current operators associated with internal symmetries. Let A_1, \dots, A_n be field operators satisfying the usual postulates of local quantum field theory¹. The difficulty in constructing composite operators such as

$$A_1(x) \dots A_n(x) \tag{1.1}$$

originates in the singularity of the operator product

$$A_1(x_1) \dots A_n(x_n) \tag{1.2}$$

for coinciding arguments. Such singularities inevitably occur as a consequence of relativistic invariance and positive definite metric in Hilbert space [1].

In Ref. [2] an expansion of the operator product (1.2) was proposed which exhibits the singularities near $x_j = x$ and simultaneously allows

* Supported in parts by funds from the National Science Foundation Grant No. GP-25609.

¹ Each of the operators A_j may have several components transforming like a tensor under homogeneous Lorentz transformations.

the construction of composite operators. The hypothesis is that any operator product (1.2) may be expanded in the form²

$$A_1(x + \xi_1) \dots A_a(x + \xi_a) = \sum_{k=1}^n f_k(\xi_1, \dots, \xi_a) B_k(x) + R(x, \xi_1, \dots, \xi_a) \quad (1.3)$$

where the remainder R vanishes at $\xi_j = 0$ while the functions f_k become singular (or non-vanishing) in the limit $\xi_j \rightarrow 0$. The operators B_k are local and may be regarded as composite field operators associated with the formal product (1.1).

In Ref. [3] this hypothesis was generalized by assuming that any operator product (1.2) may be represented as a series

$$A_1(x + \xi_1) \dots A_a(x + \xi_a) = \sum_{k=1}^{\infty} f_k(\xi_1, \dots, \xi_a) B_k(x). \quad (1.4)$$

The series is asymptotic in the sense that the coefficients

$$f_k(\varrho \lambda_1, \dots, \varrho \lambda_a)$$

vanish stronger than ϱ^N for $k \geq k(N)$ provided $k(N)$ is chosen large enough. In addition dimensional rules were given in Ref. [2, 3] which provide detailed information on the operators B_k and the singularities of the coefficients f_k .

We summarize the results concerning operator product expansions and the properties of composite field operators which so far have been obtained in the perturbation theory of renormalizable interactions, and exactly for lower dimensional models. Local field equations with properly defined interaction terms were first introduced by Valatin [4] and have been verified in all orders of perturbation theory [5, 6]. Independent of perturbation theory Glimm and Jaffe have derived local field equations for the model of ϕ^4 -coupling in two dimensions [7]. Similar results have been obtained for other twodimensional models [8, 9]³. The principal parts (1.3) of operator products were derived by Brandt [5], moreover the asymptotic expansion (1.4) has been confirmed in perturbation theory [10, 11]. Operator product expansions and composite operators have been studied in detail for the Thirring model [8]. In perturbation theory a definition of composite operators can easily be given by applying Bogoliubov's renormalization technique to the Gell-Mann Low expansion of the relevant Green's functions. Under certain conditions it has been possible to define current operators which have the required properties, such as conservation laws, commutation relations, etc. [10, 12]. In

² The product $f_k B_k$ may include a sum over some Lorentz indices of f_k and B_k .

³ Earlier references can be found in Ref. [11].

particular it has been shown that in a renormalizable theory it is always possible to construct an energy-momentum tensor [10, 12].

The purpose of this paper is to show how an operator product expansion can be derived from general principles. We begin with a brief outline of the method, leaving questions of mathematical rigor for the detailed discussion of the following sections. Our aim is to

(i) define local operators associated with the formal product (1.1),

(ii) to completely analyze the asymptotic behaviour of (1.2) for $x_j \rightarrow x$.

We introduce fixed vectors and a scaling parameter ϱ by

$$x_j = x + \varrho \lambda_j(\eta), \quad \lambda_j(\eta) = \sum_{i=1}^{a-1} C_{ji} \eta_i \quad (1.5)$$

with

$$\frac{\partial(x_1 \dots x_a)}{\partial(x\eta_1 \dots \eta_{a-1})} \neq 0 \quad \text{for } \varrho \neq 0$$

and consider the operator product (1.2) as a function of x , η_j and ϱ

$$P(x, \eta, \varrho) = A(x_1) \dots A(x_a) \quad (1.6)$$

$$\eta = (\eta_1, \dots, \eta_{a-1}).$$

If P diverges for $\varrho \rightarrow 0$ we define an operator

$$C_1(x\eta) = \lim_{\varrho \rightarrow 0} \frac{P(x\eta\varrho)}{f_1(\varrho)} \quad (1.7)$$

dividing P by a suitable function f_1 . The singularity of f_1 is restricted by the condition that the result be finite and different from zero. A suitable f_1 can be found if we make the assumption that there are "most singular" matrix elements

$$(\tilde{\Phi}, P(\tilde{x}\tilde{\eta}\varrho) \tilde{\Psi}) \quad (1.8)$$

of P . A matrix element (1.8) is called most singular if it is as singular or more singular than any other matrix element of P near $\varrho = 0$. Choosing one of the matrix elements (1.8) as the function f_1 the operator (1.7) will be finite and not identically zero. Evidently C_1 is a local operator. In many cases C_1 turns out to be a multiple of the identity [13]. However, more composite operators can be found by further examining the behavior of $P(x\eta\varrho)$ for $\varrho \rightarrow 0$. Defining an operator P_2 by

$$P(x\eta\varrho) = f_1(\varrho) C_1(x\eta) + P_2(x\eta\varrho) \quad (1.9)$$

we have

$$\lim_{\varrho \rightarrow 0} \frac{P_2}{f_1} = 0$$

as follows from (1.7). If

$$\lim_{\varrho \rightarrow 0} P_2 = 0$$

(1.9) already gives complete information on the singularities of P near $\varrho = 0$. If, however, some of the matrix elements of P_2 diverge at $\varrho = 0$ the product P has additional singularities. In this case we repeat the procedure for P_2 and obtain

$$P_2(x\eta\varrho) = f_2(\varrho) C_2(x\eta) + P_3(x\eta\varrho) \quad (1.10)$$

where

$$C_2(x\eta) = \lim_{\varrho \rightarrow 0} \frac{P_2(x\eta\varrho)}{f_2(\varrho)} \quad (1.11)$$

is a local, non-vanishing field operator. The function f_2 is one of the most singular matrix elements of P_2 . It diverges in the limit

$$\lim_{\varrho \rightarrow 0} f_2(\varrho) = \infty,$$

but is less singular than f_1

$$\lim_{\varrho \rightarrow 0} \frac{f_2(\varrho)}{f_1(\varrho)} = 0.$$

Proceeding in this way we construct operators $P_k(x, \eta\varrho)$, $C_k(x, \eta)$ and functions $f_k(\varrho)$ by the recursion formulae

$$C_k(x\eta) = \lim_{\varrho \rightarrow 0} \frac{P_k(x\eta\varrho)}{f_k(\varrho)}, \quad (1.12)$$

$$P_k(x\eta\varrho) = f_k(\varrho) C_k(x\eta) + P_{k+1}(x\eta\varrho), \quad (1.13)$$

where $f_k(\varrho)$ is chosen to be one of the most singular matrix elements of P_k . We continue this procedure until we arrive at an operator P_{k+1} which vanishes in the limit $\varrho \rightarrow 0$. Then the operator product becomes

$$P(x\eta\varrho) = f_k(\varrho) C_1(x\eta) + \dots + f_n(\varrho) C_n(x\eta) + P_{n+1}(x\eta\varrho) \quad (1.14)$$

with

$$\lim_{\varrho \rightarrow 0} P_{n+1}(x\eta\varrho) = 0. \quad (1.15)$$

(1.14) represents the principal part of the expansion near $\varrho = 0$. The functions f_k carry the singularities of P and are ordered according to decreasing strength of singularity

$$\lim_{\varrho \rightarrow 0} \frac{f_{k+1}}{f_k} = 0, \quad \lim_{\varrho \rightarrow 0} f_k(\varrho) = \infty \quad \text{for } k = 1, \dots, n-1, \quad (1.16)$$

$$\lim_{\varrho \rightarrow 0} f_n(\varrho) = \infty \quad \text{or} \quad \lim_{\varrho \rightarrow 0} f_n(\varrho) \neq 0. \quad (1.17)$$

The field operators C_k are given by

$$C_k = \lim_{\varrho \rightarrow 0} \frac{P - \sum_{l=1}^k f_l C_l}{f_k} \quad (1.18)$$

and satisfy the causality conditions

$$[C_k(x, \eta), O(y)]_{\pm} = 0, \quad (1.19)$$

$$[C_k(x, \eta), C_l(y, \zeta)]_{\pm} = 0 \quad (x - y)^2 < 0 \quad (1.20)$$

with commutators or anticommutators taken appropriately. $O(y)$ denotes a field operator which (anti)commutes with A_1, \dots, A_n at spacelike distances.

In deriving the principal part (1.14–15) some assumptions have been used which seem to be plausible but at present cannot be inferred from Wightman's postulates. The precise formulation of the hypothesis needed will be given in Section 2 (Eq. (2.15) and Hypothesis 1). Eq. (2.15) excludes oscillations at $\varrho = 0$. For models which violate this hypothesis the following alternatives will be found in Section 5:

(a) $P(x, \eta\varrho)$ has no leading singularities as operator, i.e., to any matrix element of P another one can be found which is more singular at $\varrho = 0$.

(b) The remainder $P_{n+1}(x, \eta\varrho)$ has no leading singularity as an operator, for some value of n . Then (1.14) holds with

$$\lim_{\varrho \rightarrow 0} \frac{f_{k+1}}{f_k} = 0, \quad \lim_{\varrho \rightarrow 0} f_k(\varrho) = \infty \quad \text{for } k = 1, \dots, n, \quad (1.21)$$

$$\lim_{\varrho \rightarrow 0} \frac{P_{n+1}}{f_k} = 0.$$

In this case (1.14) gives the leading singularity for only some of the matrix elements of P .

(c) (1.14) and (1.21) hold for any n , but no P_n vanishes at $\varrho = 0$ with all matrix elements.

In order to obtain the asymptotic expansion (1.4) we construct an infinite sequence of operators C_k and functions f_k using the recursion formulae (1.12–13). If P_k vanishes at $\varrho = 0$ we take f_k as one of the matrix elements of P_k which vanish most weakly for $\varrho \rightarrow 0$. Under suitable assumptions on the matrix elements of P (Hypothesis 2 of Section 2) it will be shown in Section 4 that P_k and f_k vanish stronger than any power of ϱ provided k is sufficiently large. With this we have

$$P(x\eta\varrho) = f_1(\varrho) C_1(x\eta) + \dots + f_k(\varrho) C_k(x\eta) + P_{k+1}(x\eta\varrho) \quad (1.22)$$

where

$$\lim_{\varrho \rightarrow 0} \frac{P_{k+1}(x\eta\varrho)}{\varrho^N} = 0 \quad (1.23)$$

for any N provided $k > k(N)$. In addition the functions f_k satisfy

$$\lim_{\varrho \rightarrow 0} \frac{f_{k+1}(\varrho)}{f_k(\varrho)} = 0. \quad (1.24)$$

The local operators C_k are given by (1.18) and can be chosen to be linearly independent. The Eq. (1.22–23) represent by definition the asymptotic expansion

$$P(x\eta\varrho) = \sum_{k=1}^{\infty} f_k(\varrho) C_k(x\eta). \quad (1.25)$$

The following sections contain the rigorous derivation of these results. In Section 2 the assumptions are formulated from which the principal part (Section 3) and the asymptotic expansion (Section 4) are derived. Alternatives to the assumptions and modifications are discussed in Section 5.

2. General Assumptions

We consider field operators

$$O_1(x), \dots, O_c(x) \quad (2.1)$$

which satisfy Wightman's postulates of a local, relativistic quantum field theory [14]. Each of the operators O_j may have several components $O_{j\alpha}$ transforming under a representation of the homogeneous Lorentz group. With O_j also the hermitian conjugate O_j^* is listed in (2.1). D_0 denotes the joint domain of definition of the smeared operators

$$O_j(f) = \int dx f(x) O_j(x) \quad f \in \mathcal{S}(R_4), \quad (2.2)$$

which is obtained by applying a polynomial of the $O_j(f)$ to the vacuum state Ω . D_0 is dense in the Hilbert space \mathcal{H} .

In the work that follows a family

$$Q(g) = \int dz_1 \dots dz_n g(z_1 \dots z_n) Q(z_1 \dots z_n)$$

of operators in Hilbert space is called a distribution in $\mathcal{S}'_{z_1 \dots z_n}(\Delta)$

$$Q(z_1 \dots z_n) \in \mathcal{S}'_{z_1 \dots z_n}(\Delta)$$

if $Q(g)$ is defined for every $g \in \mathcal{S}_{z_1 \dots z_n}$ on the dense set Δ and if each matrix element

$$(\Phi, Q(g) \Psi) \quad \Phi, \Psi \in \Delta$$

is a distribution in $\mathcal{S}'_{z_1 \dots z_n}(\Delta)$. We further write

$$\lim_{\varrho \rightarrow 0} Q_\varrho(z_1 \dots z_n) = Q(z_1 \dots z_n) \quad \text{in } \mathcal{S}'_{z_1 \dots z_n}(\Delta) \quad (2.3)$$

if

$$Q \in \mathcal{S}'_{z_1 \dots z_n}(\Delta), \quad Q_\varrho \in \mathcal{S}'_{z_1 \dots z_n}(\Delta)$$

and

$$\lim_{\varrho \rightarrow 0} (\Phi, Q_\varrho(g) \Psi) = (\Phi, Q(g) \Psi) \quad \text{for } \Phi \in \mathcal{H}, \Psi \in \Delta \quad (2.4)$$

(2.4) implies

$$\lim_{\varrho \rightarrow 0} (\Phi, Q_\varrho(z_1 \dots z_n) \Psi) = (\Phi, Q(z_1 \dots z_n) \Psi) \quad \text{in } \mathcal{S}'_{z_1 \dots z_n} \quad \text{if } \Phi, \Psi \in \Delta.$$

Instead of the operator product (1.2) we introduce the smeared operator

$$P(t, \varrho) = \int dx d\eta t(x\eta) P(x\eta\varrho) \\ d\eta = d\eta_1 \dots d\eta_{a-1} \quad (2.5)$$

$$P(x\eta\varrho) = A_1(x_1) \dots A_a(x_a)$$

with the variables (1.5). A_1, \dots, A_n denote linear combinations of operators O_j or derivatives thereof. Our aim is to study the behavior of $P(t, \varrho)$ for $\varrho \rightarrow 0$. First we show that for every real $\varrho \neq 0$ $P(t, \varrho)$ can be defined as a distribution in $\mathcal{S}'_{x\eta}(D_0)$. For $\Phi, \Psi \in D_0$ the matrix elements

$$(\Phi, A_1(f_1) \dots A_a(f_a) \Psi) = \int dx_1 \dots dx_a f_1(x_1) \dots f_a(x_a) (\Phi, A_1(x_1) \dots A_a(x_a) \Psi) \quad (2.6)$$

may be extended to distributions in the variables x_1, \dots, x_a . We introduce new independent variables $x, \eta_1, \dots, \eta_{a-1}$ by (1.5). Then

$$(\Phi, P(t\varrho) \Psi) = \int dx d\eta_1 \dots d\eta_{a-1} t(x\eta_1 \dots \eta_{a-1}) (\Phi, P(x\eta\varrho) \Psi) \quad (2.7)$$

is a distribution in the variables $x, \eta_1, \dots, \eta_{a-1}$ for $\varrho \neq 0$. Likewise the vacuum expectation values

$$\langle Q_1 \dots Q_n \rangle \quad (2.8)$$

with either

$$Q_\kappa = O_{j\kappa}(x_\kappa)$$

or

$$Q_\kappa = P(x_\kappa \eta^{(\kappa)} \varrho) \quad (\varrho \neq 0)$$

may be interpreted as distributions in all the variables $x_\kappa, \eta_{\kappa_1}^{(\kappa)} \dots \eta_{\kappa_{a-1}}^{(\kappa)}$. From the vacuum expectation values (2.8) the operator (2.5) is constructed on the domain D_0 in the usual manner. The distributions (2.7) are the matrix elements of $P(t, \varrho)$ between vectors D_0 . Hence (2.5) is a distribution in $\mathcal{S}'_{x\eta}(D_0)$.

We state the following properties of P .

(i) *Lorentz Invariance*. Under an inhomogeneous Lorentz transformation

$$x' = \Lambda x + a, \quad \eta'_r = \Lambda \eta_r, \quad \Phi' = U(\Lambda, a) \Phi, \quad \Psi' = U(\Lambda, a) \Psi \\ t'(x', \eta'_1, \dots, \eta'_{a-1}) = t(x, \eta_1, \dots, \eta_{a-1}) \quad (2.9)$$

the components of P_α transform according to

$$(\Phi', P_{\alpha'}(t', \varrho) \Psi') = \sum_{\alpha} D_{\alpha'\alpha}(A) (\Phi, P_{\alpha}(t, \varrho) \Psi) \quad \Phi \in H, \quad \Psi \in D_0 \quad (2.10)$$

where $D(A)$ is a finite dimensional representation of the homogeneous Lorentz group.

(ii) *Causality*. Let $t \in \mathcal{D}_{x\eta}$, $f \in \mathcal{D}_\eta$ be test functions satisfying

$$t(x\eta_1 \dots \eta_{a-1}) f(\eta) = 0 \quad (2.11)$$

if $(x_r - y)^2 \geq 0$ for at least one $r = 1, \dots, n$. Then

$$[P(t, \varrho), O_j(f)]_{\pm} \Psi = 0 \quad \text{if } \Psi \in D_0. \quad (2.12)$$

We want to formulate a hypothesis concerning the relative growth of matrix elements of P for $\varrho \rightarrow 0$. To this end we consider the set \mathcal{A}_N of all functions

$$\phi(\varrho) = \sum_{\kappa=1}^A (\Phi_{\kappa}, P_{\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}), \quad \Psi_{\kappa} \in D_0 \quad (2.13)$$

which do not vanish stronger than ϱ^N in the limit $\varrho \rightarrow 0$, i.e. ⁴

$$\lim_{\varrho \rightarrow 0} \frac{\phi(\varrho)}{\varrho^N} \neq 0. \quad (2.14)$$

In (2.13) the Φ_{κ} are arbitrary vectors of Hilbert space, the Ψ_{κ} are elements of the domain D_0 . The t_{κ} denote test functions in $\mathcal{S}(R_{4a})$.

We first assume that the ratio of any given two functions of \mathcal{A}_N has only one accumulation point, i.e.,

$$\lim_{\varrho \rightarrow 0} \frac{\phi_1(\varrho)}{\phi_2(\varrho)} \text{ exists or } \lim_{\varrho \rightarrow 0} \frac{\phi_1(\varrho)}{\phi_2(\varrho)} = \infty. \quad (2.15)$$

We then define an equivalence relation in \mathcal{A}_N . ϕ_1, ϕ_2 are said to have the same behavior near $\varrho = 0$

$$\phi_1 \sim \phi_2$$

if their ratio is finite in the limit $\varrho \rightarrow 0$

$$\lim_{\varrho \rightarrow 0} \frac{\phi_1}{\phi_2} \neq 0, \infty.$$

Otherwise we say that ϕ_1 and ϕ_2 behave differently near $\varrho = 0$ and use the notation

$$\phi_1 > \phi_2 \quad \text{if } \lim_{\varrho \rightarrow 0} \frac{\phi_1}{\phi_2} = \infty$$

⁴ This includes the case that $\lim_{\varrho \rightarrow 0} \frac{\phi(\varrho)}{\varrho^N} = \infty$.

and

$$\phi_1 < \phi_2 \quad \text{if} \quad \lim \frac{\phi_1}{\phi_2} = 0.$$

With respect to the relation \sim we introduce equivalence classes of \mathcal{A}_N . For two equivalence classes $\mathcal{C}, \mathcal{C}'$ we write

$$\mathcal{C} > \mathcal{C}'$$

if

$$\lim_{\varrho \rightarrow 0} \frac{\phi}{\phi'} = \infty \quad \text{for} \quad \phi \in \mathcal{C}, \quad \phi' \in \mathcal{C}'.$$

We formulate the following hypotheses:

Hypothesis 1. There are at most n functions (2.13) which do not vanish at $\varrho = 0$ and behave differently near $\varrho = 0$. In other words, the number of equivalence classes of \mathcal{A}_0 is finite.

Hypothesis 2. For every N there are at most $n(N)$ functions (2.13) which do not vanish stronger than ϱ^N and behave differently near $\varrho = 0$. In other words, the number of equivalence classes of \mathcal{A}_N is finite.

Hypothesis 3. \mathcal{A} denotes the set of all functions (2.13) without further restrictions. (2.15) is again assumed for any two functions of \mathcal{A} .

Let ϕ be a function of \mathcal{A} and \mathcal{A}_ϕ denote the set of all $\chi \in \mathcal{A}$ with $\chi < \phi$. Then we assume that \mathcal{A} and any \mathcal{A}_ϕ contain maximal elements.

An element ψ of a subset $\mathcal{B} \subseteq \mathcal{A}$ is called maximal if

$$\psi > \chi \quad \text{or} \quad \psi \sim \chi$$

for any other $\chi \in \mathcal{B}$.

3. Principal Part

The operator $P_\alpha(t, \varrho)$ was defined by Eq. (2.5) as a distribution in $\mathcal{S}'_{x\eta}(D_0)$. We prove for P_α the following

Theorem 1. *Hypothesis 1 is necessary and sufficient for the existence of the principal part*

$$P_\alpha(t, \varrho) = \sum_{k=1}^n f_k(\varrho) C_{k\alpha}(t) + R_\alpha(t, \varrho) \tag{3.1}$$

with the properties

(a) *The remainder R is a distribution in $\mathcal{S}'_{x\eta}(D_0)$ and vanishes weakly for $\varrho \rightarrow 0$*

$$\lim_{\varrho \rightarrow 0} (\Phi, R(t, \varrho) \Psi) = 0 \quad \text{for} \quad \Phi \in \mathcal{H}, \quad \Psi \in D_0. \tag{3.2}$$

(b) Near $q=0$ the functions $f_k(q)$ are non-vanishing and ordered according to decreasing singularity

$$\lim_{q \rightarrow 0} \frac{f_{k+1}(q)}{f_k(q)} = 0. \quad (3.3)$$

$$\lim_{q \rightarrow 0} f_n(q) \neq 0. \quad (3.4)$$

(c) The $C_k(t)$ are linear operators on D_0 and distributions in $\mathcal{S}'_{x\eta}(D_0)$. The operators $C_k(t)$ can be chosen linearly independent. The Lorentz invariance of the theory implies

Lorentz Invariance of $C_k(t)$. Under a homogeneous Lorentz transformation (2.9) the components $C_{k\alpha}$ transform according to

$$(\Phi', C_{k\alpha'}(t') \Psi') = \sum_{\alpha} D_{\alpha'\alpha}(\Lambda) (\Phi, C_{k\alpha}(t) \Psi) \quad \Phi \in \mathcal{H}, \quad \Psi \in D_0. \quad (3.5)$$

The causality postulate implies

Locality of $C_k(t)$. The operators $C_k(t)$ and $O_j(f)$ (anti-) commute

$$[C_k(t), O_j(f)]_{\pm} \Psi = 0 \quad \text{for } \Psi \in D_0 \quad (3.6)$$

if the test functions $t \in \mathcal{D}_{x\eta}$ and $f \in \mathcal{D}_y$ satisfy

$$t(x\eta_1 \dots \eta_{a-1}) f(y) = 0 \quad \text{for } (x-y)^2 \geq 0. \quad (3.7)$$

The operators $C_k(t)$ and $C_l(s)$ (anti-) commute

$$(C_k^*(t) \Phi, C_l(s) \Psi) \pm (C_l^*(s) \Phi, C_k(t) \Psi) = 0 \quad \text{for } \Phi, \Psi \in D_0 \quad (3.8)$$

if the test functions satisfy

$$t(x\eta_1 \dots \eta_{a-1}) s(y\zeta_1 \dots \zeta_{a-1}) = 0 \quad \text{for } (x-y)^2 \geq 0. \quad (3.9)$$

Proof. We first show that Hypothesis 1 follows from (3.1) and (a)–(c). Let $\Phi \in \mathcal{H}$, $\Psi \in D_0$, then each matrix element of P_{α} is represented by

$$\begin{aligned} (\Phi, P_{\alpha}(t, \varrho) \Psi) &= \sum_{k=1}^n f_k(\varrho) (\Phi, C_{k\alpha}(t) \Psi) + (\Phi, R_{\alpha}(t, \varrho) \Psi) \\ \lim_{\varrho \rightarrow 0} (\Phi, R_{\alpha}(t, \varrho) \Psi) &= 0 \end{aligned}$$

which shows that \mathcal{A}_0 has only a finite number of equivalence classes.

We next prove that Hypothesis 1 is sufficient. If \mathcal{A}_0 is empty

$$\lim_{\varrho \rightarrow 0} (\Phi, P_{\alpha}(t, \varrho) \Psi) = 0, \quad \Psi \in D_0$$

and we may set

$$P_{\alpha}(t, \varrho) = R_{\alpha}(t, \varrho).$$

If \mathcal{A}_0 is not empty we denote its equivalence classes by $\mathcal{C}_1, \dots, \mathcal{C}_n$ and number them according to decreasing strength of singularities

$$\mathcal{C}_1 > \dots > \mathcal{C}_n.$$

Let

$$\phi(\varrho) = \Sigma (\Phi_\kappa, P_{\alpha_\kappa}(t_\kappa, \varrho) \Psi_\kappa) \in \mathcal{C}_1 \quad (3.10)$$

be an element of \mathcal{C}_1 . Since either

$$\lim_{\varrho \rightarrow 0} (\Phi_\kappa, P_{\alpha_\kappa}(t_\kappa, \varrho) \Psi_\kappa) = 0$$

or

$$(\Phi_\kappa, P_{\alpha_\kappa}(t_\kappa, \varrho) \Psi_\kappa) \in \mathcal{C}_{j(\kappa)} \leq \mathcal{C}_1$$

there must be a matrix element

$$f_1(\varrho) = (\tilde{\Phi}, P_{\tilde{\alpha}}(\tilde{t}, \varrho) \tilde{\Psi}) \in \mathcal{C}_1 \quad \tilde{\Phi} = \Phi_\kappa, \quad \tilde{\Psi} = \Psi_\kappa, \quad t = \tilde{t}_\kappa, \quad \tilde{\alpha} = \alpha_\kappa \quad (3.11)$$

belonging to \mathcal{C}_1 for at least one value of κ . We will show that the limit

$$C_1(t) = \lim_{\varrho \rightarrow 0} \frac{P(t, \varrho)}{f_1(\varrho)} \quad (3.12)$$

exists in the weak sense and defines a non-vanishing field operator. First we note that the limit

$$\lim_{\varrho \rightarrow 0} \frac{(\Phi, P(t, \varrho) \Psi)}{f_1(\varrho)} \quad (3.13)$$

exists for all $\Phi \in \mathcal{H}$, $\Psi \in D_0$. It does not vanish identically since

$$\lim_{\varrho \rightarrow 0} \frac{(\tilde{\Phi}, P_{\tilde{\alpha}}(\tilde{t}, \varrho) \tilde{\Psi})}{f_1(\varrho)} = 1$$

(3.13) implies the existence of a vector $X(t)$ with

$$\lim_{\varrho \rightarrow 0} \left(\Phi, \frac{P(t, \varrho)}{f_1(\varrho)} \Psi \right) = (\Phi, X(t)) \quad (3.14)$$

for every $\Phi \in \mathcal{H}$, $\Psi \in D_0$ ⁵.

$$X = C_1(t) \Psi \quad (3.15)$$

defines a linear operator $C_1(t)$ on the domain D_0 . This yields (3.12) in the sense of weak convergence.

Since

$$(\Phi, C_1(t) \Psi) = \lim_{\varrho \rightarrow 0} \frac{(\Phi, P(t, \varrho) \Psi)}{f_1(\varrho)}$$

⁵ This follows from the weak completeness of the Hilbert space.

the matrix elements $(\Phi, C_1(t) \Psi)$ are distributions in $\mathcal{S}'_{x\eta}$ for every $\Phi, \Psi \in D_0$. Hence

$$C_1(x\eta) = \lim_{\varrho \rightarrow 0} \frac{P(t\eta\varrho)}{f_1(\varrho)} \quad (3.16)$$

exists as a distribution in $\mathcal{S}'_{x\eta}(D_0)$.

Starting from

$$P_1(t, \varrho) = P(t, \varrho), \quad C_1(t) \quad \text{and} \quad f_1(\varrho)$$

we will construct a sequence of operators $P_k(t, \varrho)$, $C_k(t)$ and functions $f_k(\varrho)$ which are related by the recursion formulae

$$C_k(t) = \lim_{\varrho \rightarrow 0} \frac{P_k(t, \varrho)}{f_k(\varrho)} \neq 0, \quad (3.17)$$

$$P_k(t, \varrho) = f_k(\varrho) C_k(t) + P_{k+1}(t, \varrho). \quad (3.18)$$

We use the following hypothesis of induction:

(i) Distribution properties. For $\varrho \neq 0$ the operator $P_k(t, \varrho)$ is a distribution in $\mathcal{S}'_{x\eta}(D_0)$.

(ii) Growth of matrix elements for $\varrho \rightarrow 0$. Let $\mathcal{A}_0(P_k)$ denote the set of all functions

$$\phi(\varrho) = \Sigma (\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}), \quad \Phi_{\kappa} \in \mathcal{H}, \quad \Psi_{\kappa} \in D_0, \quad t_{\kappa} \in \mathcal{S}(R_{4a}) \quad (3.19)$$

with the property

$$\lim_{\varrho \rightarrow 0} \phi(\varrho) \neq 0. \quad (3.20)$$

Then

$$\mathcal{A}_0(P_k) = \sum_{j=k}^n \mathcal{C}_j. \quad (3.21)$$

Under this hypothesis we will prove that (3.17) exists and that the operator P_{k+1} defined by (3.18) again satisfies the conditions of the hypothesis.

First we choose the function $f_k(\varrho)$ which will be used for defining C_k as a non-vanishing operator. According to statement (ii) of the hypothesis any function $\phi \in \mathcal{C}_k$ is also an element of $\mathcal{A}_0(P_k)$ and can therefore be written as

$$\phi(\varrho) = \Sigma (\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}), \quad \Psi_{\kappa} \in D_0.$$

Since either

$$\lim_{\varrho \rightarrow 0} (\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) = 0$$

or

$$(\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) \in \mathcal{C}_{j(\kappa)} \leq \mathcal{C}_k$$

it follows

$$(\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) \in \mathcal{C}_k \quad (3.22)$$

for at least one value of κ . We choose (3.22) as the function $f_k(\varrho)$. As in the case of C_1 it follows that the weak limit (3.17) defines a non-vanishing distribution in $\mathcal{S}'_{x\eta}(D_0)$. By (3.18) the distribution properties (i) follow for P_{k+1} .

In order to prove (3.21) for P_{k+1} we first show

$$\mathcal{A}_0(P_{k+1}) \subseteq \mathcal{A}_0(P_k). \quad (3.23)$$

Indeed we have

$$\Sigma(\Phi_{\kappa}, P_{k+1, \alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) = \Sigma(\hat{\Phi}_{\kappa}, P_{k\hat{\alpha}_{\kappa}}(\hat{t}_{\kappa}, \varrho) \hat{\Psi}_{\kappa}) \in \mathcal{A}_0(P_k)$$

since

$$\begin{aligned} (\Phi, P_{k+1, \alpha}(t, \varrho) \Psi) &= (\Phi, P_{k, \alpha}(t, \varrho) \Psi) - (\tilde{\Phi}, P_{k, \tilde{\alpha}}(\tilde{t}, \varrho) X) \\ X &= (\Phi, C_{k, \alpha}(t) \Psi) \tilde{\Psi}, \quad f_k(\varrho) = (\tilde{\Phi}, P_{k, \tilde{\alpha}}(\tilde{t}, \varrho) \tilde{\Psi}). \end{aligned}$$

Next we show that the type \mathcal{C}_k is missing in $\mathcal{A}_0(P_{k+1})$. Since

$$\lim_{\varrho \rightarrow 0} \frac{(\Phi, P_{k+1}(t, \varrho) \Psi)}{f_k(\varrho)} = \lim_{\varrho \rightarrow 0} \frac{(\Phi, C_k(t, \varrho) \Psi)}{f_k(\varrho)} - (\Phi, C_k(t) \Psi) = 0$$

holds for any matrix element of P_{k+1} between $\Phi \in \mathcal{H}$, $\Psi \in D_0$ it follows

$$\lim_{\varrho \rightarrow 0} \frac{\phi(\varrho)}{f_k(\varrho)} = 0$$

for $\phi(\varrho) \in \mathcal{A}_0(P_{k+1})$. Hence $\phi(\varrho)$ belongs to a type \mathcal{C} with $\mathcal{C} < \mathcal{C}_k$

$$\mathcal{C}_k \cap \mathcal{A}_0(P_{k+1}) = \emptyset. \quad (3.24)$$

Finally we check that any $\mathcal{C}_l (l > j)$ is contained in $\mathcal{A}_0(P_{k+1})$. Let $\phi \in \mathcal{C}_l (l > k)$, then (statement (ii) for P_k)

$$\begin{aligned} \phi(\varrho) &= \Sigma(\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) = f_k(\varrho) \Sigma(\Phi_{\kappa}, C_{k\alpha_{\kappa}}(t_{\kappa}) \Psi_{\kappa}) \\ &\quad + \Sigma(\Phi_{\kappa}, P_{k+1, \alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}). \end{aligned}$$

From $f_k \in \mathcal{C}_k$ and

$$(\Phi_{\kappa}, P_{k+1, \alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) < f_k \quad (3.25)$$

it follows that

$$\Sigma(\Phi_{\kappa}, C_{k\alpha_{\kappa}}(t_{\kappa}) \Psi_{\kappa}) = 0$$

and

$$\phi(\varrho) = \Sigma(\Phi_{\kappa}, P_{k+1, \alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}).$$

Hence

$$\mathcal{C}_l \subseteq \mathcal{A}_0(P_{k+1}) \quad \text{for } l > k.$$

(3.23–25) imply the desired relation

$$\mathcal{A}_0(P_{k+1}) = \sum_{\alpha=k+1}^n \mathcal{C}_k.$$

This completes the proof of the hypothesis of induction.

Iterating the recursion formula (3.18) we obtain (3.1) with

$$R(t, \varrho) = P_{n+1}(t, \varrho).$$

Since $A_0(P_{n+1}) = \emptyset$ we have (3.2). $f_k(\varrho) \in \mathcal{C}_k$ implies (3.3–4). Hence all conditions (a)–(c) are fulfilled.

In order to check the transformation law (3.5) we assume that for some value of k the relation

$$(\Phi', P_{k\alpha'}(t', \varrho) \Psi') = \sum_{\alpha} D_{\alpha'\alpha}(A) (\Phi, P_{k\alpha}(t, \varrho) \Psi) \quad \Phi \in \mathcal{H}, \quad \Psi \in D_0 \quad (3.26)$$

has been shown for a Lorentz transformation (2.9). Dividing (3.26) by $f_k(\varrho)$ and taking the limit $\varrho \rightarrow 0$ we obtain

$$(\Phi', C_{k\alpha'}(t') \Psi') = \sum_{\alpha} D_{\alpha'\alpha}(A) (\Phi, C_{k\alpha}(t) \Psi) \quad \Phi \in \mathcal{H}, \quad \Psi \in D_0 \quad (3.27)$$

as transformation law for C_k . By the recursion formula (3.18) relation (3.26) is then also true for P_{k+1} .

For the proof of the local properties we assume as hypothesis of induction that

(i) the (anti)commutation relation

$$[P_k(t, \varrho), O_j(f)]_{\pm} \Psi = 0, \quad \Psi \in D_0 \quad (3.28)$$

holds for test function $t \in \mathcal{S}_{x\eta}$, $f \in \mathcal{S}_y$ which satisfy

$$t(x, \eta_1 \dots \eta_{a-1}) f(y) = 0 \quad (3.29)$$

unless

$$(x-y)^2 < 0 \quad \text{and} \quad (x_r - y)^2 < 0, \quad r = 1, \dots, a, \quad x_r = x + \varrho \lambda_r(\eta). \quad (3.30)$$

(ii) The (anti)commutation relation

$$(P_k(t, \varrho)^* \Phi, C_j(s) \Psi) \pm (C_j^*(s) \Phi, P_k(t, \varrho) \Psi) = 0 \quad (3.31)$$

for $\Phi, \Psi \in D_0, \quad j = 1, \dots, k-1$

holds for test functions $t \in \mathcal{D}_{x\eta}$, $s \in \mathcal{D}_{y\zeta}$ which satisfy

$$t(x\eta_1 \dots \eta_{a-1}) s(y\zeta_1 \dots \zeta_{a-1}) = 0 \quad (3.32)$$

unless (3.30) is valid.

(iii) The (anti)commutation relation

$$(P_k(t, \varrho)^* \Phi, P_k(s, \sigma) \Psi) \pm (P_k(s, \sigma)^* \Phi, P_k(t, \varrho) \Psi) = 0 \quad (3.33)$$

for $\Phi, \Psi \in D_0$

holds for test functions $t \in \mathcal{D}_{x\eta_1}, s \in \mathcal{D}_{y\zeta}$ which satisfy (3.32) unless

$$(x-y)^2 < 0 \quad \text{and} \quad (x_r - y_n)^2 < 0 \quad (3.34)$$

$$x_r = x + \varrho \lambda_r(\eta), \quad y_n = y + \sigma \lambda_r(\zeta).$$

Under this hypothesis we will prove that the operator P_{k+1} again satisfies the conditions (i)–(iii).

Suppose the test function $t \in \mathcal{D}_{x\eta}$ and $f \in \mathcal{D}_y$ satisfy

$$t(x\eta_1 \dots \eta_{a-1})f(y) = 0 \quad \text{if} \quad (x-y)^2 \geq 0. \quad (3.35)$$

Then (3.28) holds provided ϱ is small enough that all $x + \varrho\eta_r$ in the support of t are sufficiently close to x such that (3.29–30) is satisfied. For matrix elements between vectors $\Phi, \Psi \in D_0$ one obtains

$$0 = (\Phi, [P(t, \varrho), O_j(t)]_{\pm} \Psi) = (\Phi, P(t, \varrho) \Psi') \pm (\Phi', P(t, \varrho) \Psi)$$

$$\Psi' = O_j(f) \Psi, \quad \Phi' = O_j^*(f) \Phi.$$

Dividing by $f_k(\varrho)$ and taking the limit $\varrho \rightarrow 0$

$$0 = (\Phi, C_k(t) \Psi') \pm (\Phi', C_k(t) \Psi) = (\Phi, [C_k(t), O_j(f)]_{\pm} \Psi)$$

follows. Since this relation holds for any $\Phi \in D_0$ we have

$$[C_k(t), O_j(f)]_{\pm} \Psi = 0 \quad \Psi \in D_0 \quad (3.36)$$

for test functions satisfying (3.35). (3.28) and (3.36) imply

$$[P_{k+1}(t, \varrho), O_j(f)]_{\pm} \Psi = 0 \quad \Psi \in D_0 \quad (3.37)$$

for test functions which satisfy (3.29) unless (3.30) is valid.

Dividing (3.33) by $f_k(\sigma)$ and taking the limit $\sigma \rightarrow 0$ we find that (3.31) is also valid for $j = k$ ⁶. Dividing (3.31) by $f_k(\varrho)$ and taking the limit $\varrho \rightarrow 0$ we obtain

$$(C_k(t)^* \Phi, C_j(s) \Psi) \pm (C_j^*(s) \Phi, C_k(t) \Psi) = 0 \quad (3.38)$$

for $\Phi, \Psi \in D_0, \quad j = 1, \dots, k$

⁶ It is used here that the adjoint operators $C_k^*(t), P_k^*(t\varrho)$ are defined on D_0 and satisfy

$$C_k^*(t) = \lim_{\varrho \rightarrow 0} \frac{P_k^*(t\varrho)}{F_k(\varrho)}$$

weakly on D_0 . A proof of this statement will be given in a separate paper by comparison of the expansions of $P_1(t\varrho)$ and $P_1^*(t\varrho)$.

for test functions satisfying

$$t(x\eta_1 \dots \eta_{a-1}) s(y\zeta_1 \dots \zeta_{a-1}) = 0 \quad \text{if } (x-y)^2 \geq 0. \quad (3.39)$$

From (3.31) and (3.38)

$$\begin{aligned} (P_{k+1}(t)^* \Phi, C_j(s) \Psi) \pm (C_j(s)^* \Phi, P_{k+1}(t) \Psi) = 0, \\ \Phi, \Psi \in D_0, \quad j = 1, \dots, k, \end{aligned} \quad (3.40)$$

follows for test functions which satisfy (3.29) unless (3.30) holds.

For test functions t, s with (3.32) unless (3.34) the relations

$$\begin{aligned} (P_{k+1}(t, \varrho)^* \Phi, P_{k+1}(s, \sigma) \Psi) \pm (P_{k+1}(s, \sigma)^* \Phi, P_{k+1}(t, \varrho) \Psi) = 0 \\ \text{for } \Phi, \Psi \in D_0 \end{aligned} \quad (3.41)$$

follow from (3.33), (3.31) (with $j = k$) and (3.38).

The relations (3.37), (3.40–41) confirm the local properties (i)–(iii) for the operator P_{k+1} . (3.36) and (3.38) represent the local properties (3.6–9) stated in Theorem 1.

We finally prove that the C_k can be chosen to be linearly independent without changing the properties (3.1–9). Suppose there is a linear relation

$$\sum_{\kappa=1}^n a_{\kappa} C_{k_{\kappa}} = 0$$

with all $a_{\kappa} \neq 0$. We express the $C_{k_{\kappa}}$ with the highest subscript $b = k_{\kappa}$ by C_1, \dots, C_{b-1}

$$C_b = - \sum_{k=1}^{b-1} \beta_k C_k.$$

Then

$$P = \sum_{k=1}^{b-1} (f_k + \beta_k f_b) C_k + \sum_{j=1}^n f_j C_k + P_{n+1}.$$

The new coefficients again satisfy the conditions (3.3–4). By induction all linear relations among the C_k can be eliminated until an expansion (3.1) with linearly independent C_k is obtained. This completes the proof of Theorem 1.

4. Asymptotic Expansion

Concerning the asymptotic expansion of $P(t, \varrho)$ near $\varrho = 0$ we state the following

Theorem 2. *Hypothesis 2 is necessary and sufficient for the asymptotic expansion*

$$P(t, \varrho) = f_1(\varrho) C_1(t) + \dots + f_k(\varrho) (k(t) + P_{n+1}(t, \varrho)) \quad (4.1)$$

with the properties

(a) the remainder P_{k+1} is a distribution in $\mathcal{S}'_{x\eta}(D_0)$ and satisfies

$$\lim_{\varrho \rightarrow 0} \frac{(\Phi, P_{k+1}(t, \varrho) \Psi)}{\varrho^N} = 0 \quad \text{if } k > k(N), \tag{4.2}$$

$$\Phi \in H, \quad \Psi \in D_0,$$

$$\lim_{\varrho \rightarrow 0} \frac{(\Phi, P_{k+1}(t, \varrho) \Psi)}{f_k(\varrho)} = 0. \tag{4.3}$$

(b) the functions f_k satisfy

$$\lim_{\varrho \rightarrow 0} \frac{f_{k+1}(\varrho)}{f_k(\varrho)} = 0. \tag{4.4}$$

(c) The $C_k(t)$ are linear operators on D_0 and distributions in $\mathcal{S}'_{x\eta}(D_0)$. It can be arranged that a finite number of $C_k(t)$ are linearly independent. Moreover, the operators $C_k(t)$ have the invariance and locality properties stated in Theorem 1.

Proof. Let \mathcal{B} denote the set of all functions (2.13) which satisfy (2.14) for some value of N . We have

$$\mathcal{B} = \sum_{N=1}^{\infty} \mathcal{A}_N.$$

We denote the equivalence classes of \mathcal{B} by

$$\mathcal{C}_1 > \dots > \mathcal{C}_k > \dots$$

where

$$\mathcal{C}_1 > \dots > \mathcal{C}_{n(N)}$$

are the equivalence classes of \mathcal{A}_N . The induction procedure of the last section is then easily extended to any value of k . To this end statement (ii) (below Eq. (3.18)) is generalized to

(ii) Let $\mathcal{A}_N(P_k)$ denote the set of all functions

$$\begin{aligned} \phi(\varrho) &= \Sigma (\Phi_{\kappa}, P_{k\alpha_{\kappa}}(t_{\kappa}, \varrho) \Psi_{\kappa}) \\ \Phi_{\kappa} &\in \mathcal{H}, \quad \Psi_{\kappa} \in D_0, \quad t_{\kappa} \in \mathcal{S}'_{x\eta} \end{aligned}$$

with the property

$$\lim_{\varrho \rightarrow 0} \frac{\phi(\varrho)}{\varrho^N} \neq 0$$

then

$$\mathcal{A}_N(P_k) = \sum_{j=k}^{n(N)} \mathcal{C}_j.$$

Eq. (4.2) then follows from the fact that the set $\mathcal{A}_N(P_{k+1})$ is empty for $k > n(N)$.

We finally have

Theorem 3. *Hypothesis 3 is necessary and sufficient for the expansion (4.1) with properties (a), (b) and (c) except for Eq. (4.2).*

For the proof we construct an infinite sequence of equivalence classes

$$\mathcal{C}_1 > \mathcal{C}_2 > \dots \quad (4.5)$$

of \mathcal{A} in the following way. \mathcal{C}_1 is the set of all maximal elements of \mathcal{A} . Let ϕ be an element of \mathcal{C}_k . Then \mathcal{C}_{k+1} is defined as the set of all maximal elements of \mathcal{A}_ϕ . The induction procedure of the last section can then be applied to the sequence (4.5).

5. Miscellaneous Remarks

Throughout this work Eq. (2.15) was assumed which excludes oscillations at $q=0$. If (2.15) is violated operator product expansions may still be set up by considering special sequences ϱ_n with $\lim_{\varrho \rightarrow \infty} \varrho_n = 0$.

The Hypotheses 1–3 can be relaxed in many ways by restricting the test functions and state vectors in the definition of the functions (2.31). We list the following possibilities:

- (1) The test functions t_κ are different from zero for space-like η only

$$t_\kappa(x, \eta) = 0 \quad \text{if} \quad \eta_j^2 \geq 0.$$

- (2) The test functions t_κ are different from zero for time-like η only

$$t_\kappa(x, \eta) = 0 \quad \text{if} \quad \eta_j^2 \leq 0.$$

- (3) The Fourier transforms of t_κ have compact support.

- (4) The vectors Φ_κ, Ψ_κ represent states of bounded energy-momentum.

The definition of classes \mathcal{A}_N is then modified accordingly. If Hypotheses 1 or 2 are used with (4) the expressions

$$(\Phi, C_k(t) \Psi) \quad (5.1)$$

might not define linear operators $C_k(t)$ in Hilbert space. But (5.1) may still be interpreted as a bilinear form in Φ and Ψ .

We next discuss the alternatives which occur in case Hypothesis 1 is not valid in any acceptable form. Let \mathcal{A}_0 be the class of functions (2.13) which do not vanish for $q \rightarrow 0$ and are defined with appropriate restrictions on the test functions and state vectors. If Hypothesis 1 does not hold there are an infinite number of equivalence classes in \mathcal{A}_0 . They are totally ordered by the relation $>$. We then have the following cases

- (a) There exists no maximal element among the equivalence classes of \mathcal{A}_0 . Then to any function (2.31) which does not vanish at $q=0$ another

function ϕ' in \mathcal{A}_0 can be found such that

$$\lim_{\varrho \rightarrow 0} \frac{\phi'}{\phi} = \infty .$$

This means that $P(t, \varrho)$, considered as an operator, does not have a leading singularity.

(b) There exists a maximal element \mathcal{C}_1 among the equivalence classes of \mathcal{A}_0 . Then a sequence of classes

$$\mathcal{C}_1 > \mathcal{C}_2 > \dots > \mathcal{C}_j > \dots \quad (5.2)$$

can be constructed such that Γ_j is the maximal element of the difference set

$$\Delta_j = \mathcal{A}_0 - \bigcup_{\alpha=1}^{j-1} \mathcal{C}_\alpha .$$

We then have the two possibilities

(b₁) The sequence (5.2) terminates for $j=m$ because Δ_j has no maximal element. Then (3.1) holds for $n=m$, but some matrix elements of R diverge for $\varrho \rightarrow 0$ and the operator R does not have a leading singularity. Eq. (3.1) gives the leading singularity for at least some matrix elements of P .

(b₂) The sequence (5.2) does not terminate. Then (3.1) holds for any n , but some matrix elements of R diverge for $\varrho \rightarrow 0$.

The form of the principal part and the asymptotic series is of course not unique. New operators C'_k and functions f'_k may be introduced by certain triangular transformations without changing the conditions (3.2–6) and (4.2–4). These problems will be discussed in a separate paper.

One of us (W.Z.) thanks Dr. G. Dell'Antonio for many helpful discussions. We are grateful to the members of the Aspen Center for Physics for their hospitality during our stay at Aspen where this work was begun.

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