Commun. math. Phys. 22, 162—165 (1971)

On Stable Potentials

N. ANGELESCU, G. NENCIU, and V. PROTOPOPESCU
Institute for Atomic Physics, Bucharest, Romania

Received April 23, 1971

Abstract. Examples are given of one- and two-dimensional stable potentials which cannot be decomposed into the sum of a non-negative function and a continuous stable potential.

§ 1

In his book on Statistical Mechanics [1], D. Ruelle raised the question whether every measurable stable potential on R^ν can be decomposed into the sum of a continuous function of positive type and a non-negative function. In a recent paper [2], Lenard and Sherman studied a class of step potentials on R^1 and found inside this class an example of stable potential which cannot be decomposed in this manner. Moreover, they were able to change this potential into a continuous stable potential preserving however the indecomposability property.

This note is concerned with finding further examples of indecomposable stable potentials. The idea is that for a subclass of the step potentials considered in [2], even a weaker decomposability requirement cannot be satisfied. Namely, we are looking for potentials which cannot be written as the sum of a continuous stable potential and a non-negative function. This enables us to considerably simplify the indecomposability proof and, moreover, to find a two-dimensional example. Of course, our examples will consist of surely non-continuous potentials.

§ 2

Let us consider the two-parameter family of potentials $\varphi_{t,d}: R^\nu \to R$, $0 \leqq t \leqq 2$, $d \geqq 0$, defined through:

$$\varphi_{t,d}(x) = \begin{cases} d & \text{for } 0 \leqq |x| \leqq t \\ -1 & \text{for } t < |x| < 2 \\ 0 & \text{for } 2 \leqq |x| \end{cases} \tag{1}$$

$$\varphi_{2,d}(x) = \begin{cases} d & \text{for } 0 \leqq |x| \leqq 2 \\ 0 & \text{for } 2 < |x| \end{cases} \tag{1'}$$

For every $d \geqq 0$, there is one critical value of $t, t_c(d) > 0$, such that $\varphi_{t,d}$ is stable for $t_c(d) \leqq t \leqq 2$ and unstable for $0 \leqq t < t_c(d)$. Indeed, for every fixed configuration $(x_1, \ldots, x_n) = (x)_n$, the function

$$\Phi_{n,t,d}(x)_n = \sum_{i,j=1}^{n} \varphi_{t,d}(x_i - x_j)$$

is continuous to the right in t, therefore, if $\varphi_{t,d}$ is stable for all $t_0 < t \leqq 2$, then $\varphi_{t_0,d}$ is stable. As $\varphi_{0,d}$ is unstable, $t_c(d) > 0$.

Moreover, $\lim_{d \to \infty} t_c(d) = 0$. Indeed, for every $0 < t < 2$, define:

$$p(t) = \max \{ n \in \mathcal{N} \mid \exists (x)_n,\ t < |x_i| < 2 \text{ for } i = 1, \ldots, n \text{ and } \min_{\substack{i,j=1,\ldots,n \\ i \neq j}} |x_i - x_j| > t \} . \tag{2}$$

Applying an induction argument as in the proof of Theorem 1 in [2] (see also Example 2^0 below), one obtains at once that $\varphi_{t,2p(t)}$ is stable, therefore $t_c(2p(t)) < t$.

Proposition. Suppose $d > 0$ is given such that, for a $\delta > 0$, $t_c(d) = t_c(d + \delta) < 2$. Then, there is no continuous stable potential $\Psi(x) \leqq \varphi_{t_c(d),d}(x)$.

Proof. Supoose there is one such $\Psi(x)$. From continuity in the neighbourhood of $|x| = t_c(d)$, we obtain: For every $N > 0$, there is an $\eta_N > 0$ $(\eta_N < t_c(d)/2)$, such that $t_c(d) - \eta_N < |x| \leqq t_c(d)$ implies $\Psi(x) < -1 + 1/N$. Define then:

$$\varphi_N^*(x) = \begin{cases} d & \text{for } 0 \leqq |x| \leqq t_c(d) - \eta_N \\ -1 + 1/N & \text{for } t_c(d) - \eta_N < |x| < t_c(d) \\ -1 & \text{for } t_c(d) \leqq |x| < 2 \\ 0 & \text{for } 2 \leqq |x| . \end{cases} \tag{3}$$

If we prove that φ_N^* is unstable for at least one N, the proposition is proved, because φ_N^* majorizes Ψ, a contradiction. To this end, define for every N:

$$\varphi_N^{**}(x) = \begin{cases} \delta & \text{for } 0 \leqq |x| \leqq t_c(d) - \eta_N \\ -1/N & \text{for } t_c(d) - \eta_N < |x| < t_c(d) \\ 0 & \text{for } t_c(d) \leqq |x| \end{cases} \tag{4}$$

and choose N such that φ_N^{**} be stable. This is certainly possible, in view of the fact that $N\,\varphi_N^{**}(2x/t_c(d))$ belongs to the family (1) and, therefore, as already remarked, is stable for $N\delta \geqq 2p(1) \geqq 2p(2 - 2\eta_N/t_c(d))$. Note that:

$$\varphi_{t_c(d) - \eta_N, d + \delta}(x) = \varphi_N^*(x) + \varphi_N^{**}(x) .$$

By hypothesis, $\varphi_{t_c(d) - \eta_N, d+\delta}$ is unstable, which implies φ_N^* unstable.

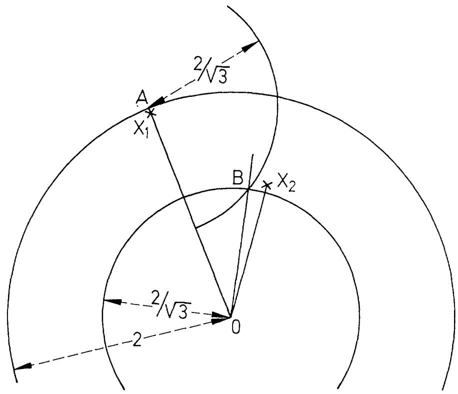

Fig. 1. x_1 must be at nonzero distance from A, therefore $\widehat{x_1 0 x_2} > \widehat{A0B} = \pi/6$

This proposition furnishes indecomposable potentials as soon as $t_c(d)$ is shown to be constant on an open interval. We shall consider the following two examples:

1°. For $\nu = 1$, it has been shown in [2] that $t_c(d)$ is the following right-continuous stop function:

$$t_c(d) = 2/k \quad \text{for} \quad 2(k-1) \leqq d < 2k\,, \qquad k = 1, 2, \ldots \tag{5}$$

Therefore, all potentials $\varphi_{t_c(d),d}$ with $d \geqq 2$ are indecomposable.

2°. For $\nu = 2$, it can be easily shown that, for $d < 12$ and $t < 2/\sqrt{3}$, $\varphi_{t,d}$ is unstable. For, suppose the points are arranged in a hexagonal lattice of constant a, $t < a < 2/\sqrt{3}$; then, for every i, $\varphi_{t,d}(x_i - x_j) = -1$ whenever x_j is a nearest- or next-nearest-neighbour of x_i, and:

$$\varphi_{n,t,d}(x)_n \leqq nd - 6n - 6n + 0(\sqrt{n}) < 0$$

for sufficiently large n. $0(\sqrt{n})$ is due to the fact that peripheral points have a smaller number of neighbours.

On the other hand, for $t = 2/\sqrt{3}$, $d \geqq 23/2$, $\varphi_{t,d}$ is stable. To prove this, the induction argument [2] refered to above can be applied. Let us first note that $p(2/\sqrt{3}) = 11$, as it can be seen on Fig. 1. Suppose $\Phi_{n-1,t,d}(x)_{n-1} \geqq 0$ for all configurations $(x)_{n-1}$, but there is one configuration $(x)_n$ such that $\Phi_{n,t,d}(x)_n < 0$. For every x_i in this configuration, denote n_i the number of those integers $r \neq i$, $r = 1, \ldots, n$, such that $|x_r - x_i| \leqq t$ and let $\bar{n} = \max_{i=1,\ldots,n} n_i$. Similarly, denote f_i the number of those integers $r = 1, \ldots, n$ such that $t < |x_r - x_i| < 2$. Clearly, $\bar{n} \geqq 1$, because otherwise:

$$\Phi_{n,t,d}(x)_n = 23n/2 - \sum_{i=1}^{n} f_i \geqq 23n/2 - 11n > 0\,.$$

Choose i_0 such that:

a) $n_{i_0} = \bar{n}$.

b) The x-coordinate of x_{i_0} is equal to the infimum of the x-coordinates of all the x_i satisfying a).

$$\begin{aligned}&\Phi_{n,t,d}(x)_n - \Phi_{n-1,t,d}(x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n) \\ &= \varphi_{t,d}(0) + 2 \sum_{i \neq i_0} \varphi_{t,d}(x_i - x_{i_0}) = (1 + 2\bar{n})\, 23/2 - 2 f_{i_0} < 0\,,\end{aligned} \tag{6}$$

because otherwise $\Phi_{n,t,d}(x)_n \geqq 0$ by the induction hypothesis. In view of $p(2/\sqrt{3}) = 11$, the set $\Delta_{i_0} = \{x_i \in (x)_n \mid t < |x_i - x_{i_0}| < 2\}$ can be covered with at most 11 circles of radius t, each of them centered in some $x_i \in \Delta_{i_0}$ and such that their centers be at a distance strictly greater than t. No such circle contains more than $\bar{n} + 1$ of the x_i in view of a), and at most 6 of them can contain exactly $\bar{n} + 1$ of the x_i in view of b). Thus:

$$2 f_{i_0} \leqq 2[5\bar{n} + 6(\bar{n} + 1)] = 11(2\bar{n} + 1) + 1\,, \tag{7}$$

which contradicts (6) for all $\bar{n} \geqq 1$.

Therefore, $t_c(d) = 2/\sqrt{3}$ for all $23/2 \leqq d < 12$, and, according to the proposition, $\varphi_{2/\sqrt{3},d}$ is indecomposable for $23/2 \leqq d < 12$.

For $v = 3$, such an argument is not sufficient to prove that, on some open interval, $t_c(d) < 2$ is constant. It seems however a natural conjecture that $t_c(d)$ is in fact a step function for all v.

Acknowledgements. We are grateful to Professor D. Ruelle for having pointed out to us that the argument we gave as a proof of this last conjecture was mistaken.

References

1. Ruelle,,D.: Statistical Mechanics, Rigorous Results. New York: W.A. Benjamin, Inc. 1969.
2. Lenard, A., Sherman, S.: Commun. math. Phys. **17**, 91 (1970).

N. Angelescu
G. Nenciu
V. Protopopescu
Institute for Atomic Physics
Laboratory of Theoretical Physics
P.O. Box 35
Bucharest, Romania