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On the Self-Adjointness of the $(g(x)\,\phi^4)_2$ Hamiltonian*

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Abstract. An alternate proof to that provided by Glimm and Jaffe of the essential self-adjointness of the Hamiltonian H for a relativistic scalar quantum field in two dimensional space-time with a "space cut-off" quartic interaction $H_I(g)$ is given. The proof depends mainly on the estimate $H_I^2(g) \leqq \text{const.}\,(N+I)^4$ and on the semiboundedness of $H = H_0 + H_I(g)$.

I. Introduction

We give an alternate proof of the essential self-adjointness of the total Hamiltonian $H = H_0 + H_I$ for a relativistic scalar quantum field in two-dimensional space-time with a "space cut-off" quartic interaction $H_I(g) = \int : \phi^4(x) : g(x)\,dx$. This result has previously been established by Glimm and Jaffe using their singular perturbation theory [1] and a number of inequalities relating H, H_0, H_I and the number operator N [2].

II. Proof

We need the following information in our proof:

(a) Any vector ψ in the Fock Hilbert space $\mathscr{F}$ may be written $\psi = \sum_{n=0}^{\infty} \psi_{(n)}$ where the vector $\psi_{(n)}$ corresponds to an "n-particle state" (we will use the bracketed subscript exclusively to denote such vectors).

(b) H_0 is defined on a certain linear domain $\mathscr{D}(H_0) \subsetneqq \mathscr{F}$. The domain of H_I contains the space $\mathscr{D}'$ of all finite linear combinations of vectors $\psi_{(n)} \in \mathscr{F}$. The domain $\mathscr{D} = \mathscr{D}' \cap \mathscr{D}(H_0)$ is dense in $\mathscr{F}$ and H_0, H_I and H are symmetric operators on $\mathscr{D}$.

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(c) If $\phi_{(m)}$, $\psi_{(n)} \in \mathscr{D}$ then $\langle \phi_{(m)} | H_0 \psi_{(n)} \rangle = 0$ unless $m = n$. If $\phi_{(m)}$, $\psi_{(n)} \in \mathscr{D}'$ then $\langle \phi_{(m)} | H_I \psi_{(n)} \rangle = 0$ unless $|m-n| = 0$, 2 or 4.

(d) H_0 is self-adjoint on $\mathscr{D}(H_0)$.

(e) If $\phi_{(m)}, \psi_{(n)} \in \mathscr{D}'$ then $|\langle \phi_{(m)} | H_I \psi_{(n)} \rangle| \leqq \text{const.}\,(n+1)^2 \, \|\phi_{(m)}\| \, \|\psi_{(n)}\|$.

(f) There exists a real constant B such that $\langle \phi | H \phi \rangle \geqq B \langle \phi | \phi \rangle$ for all $\phi \in \mathscr{D}$.

We refer to Glimm and Jaffe [2–5] for precise definitions of the Fock space $\mathscr{F}$ and the operators H_0 and H_I and for the derivation of (a)–(e). (f) was established by Glimm [6] using a functional integration technique invented by Nelson [7].

Theorem. *The total Hamiltonian H is essentially self-adjoint on $\mathscr{D}$.*

Proof. It is sufficient [8, p. 177] to show that if $\lambda < B$ then there is no non-zero vector $\psi \in \mathscr{F}$ such that:

$$\langle \psi | (H-\lambda)\chi \rangle = 0 \qquad (\text{for all} \quad \chi \in \mathscr{D}). \tag{1}$$

Assume that such a vector $\psi = \sum_{n=0}^{\infty} \psi_{(n)}$ exists. Let $\phi_n = \psi_{(4n)} + \psi_{(4n+1)} + \psi_{(4n+2)} + \psi_{(4n+3)}$. (For convenience of notation we take $\phi_n = \psi_{(n)} = 0$ when $n < 0$.) Then $\psi = \sum_{n=0}^{\infty} \phi_n$ and $\|\psi\|^2 = \sum_{n=0}^{\infty} \|\phi_n\|^2$.

We first show that $\phi_n \in \mathscr{D}$. To see this, use (c) and (1) to write $\langle \psi_{(n)} | H_0 \chi \rangle = \langle \psi | H_0 \chi_{(n)} \rangle = \lambda \langle \psi_{(n)} | \chi_{(n)} \rangle - \langle (\psi_{(n+4)} + \psi_{(n+2)} + \psi_{(n)} + \psi_{(n-2)} + \psi_{(n-4)}) | H_I \chi_{(n)} \rangle$. By (e) the last expression is bounded when $\|\chi_{(n)}\| \leqq \|\chi\| \leqq 1$ so that $\langle \psi_{(n)} | H_0 \chi \rangle$ is a bounded linear form for $\chi \in \mathscr{D}$. Thus $\psi_{(n)} \in \mathscr{D}(H_0^+)$. By (d) $\psi_{(n)} \in \mathscr{D}(H_0)$ and hence $\psi_{(n)} \in \mathscr{D}$ and $\phi_n \in \mathscr{D}$ for $n = 0, 1, \ldots$.

Writing (1) with $\chi = \phi_n$ and using (c) we get:

$$\langle \phi_{n-1} | H_I \phi_n \rangle + \langle \phi_n | H \phi_n \rangle + \langle \phi_{n+1} | H_I \phi_n \rangle = \lambda \langle \phi_n | \phi_n \rangle, \quad n = 0, 1, \ldots. \tag{2}$$

Now let M be the least integer such that $\|\phi_M\| \neq 0$. Using (f) and (2) one may calculate:

$$0 < \left\langle \left[\sum_{j=M}^{n} \phi_j \right] \middle| (H-\lambda) \left[\sum_{k=M}^{n} \phi_k \right] \right\rangle = -\langle \phi_{n+1} | H_I \phi_n \rangle. \tag{3}$$

This shows that $\langle \phi_{n+1} | H_I \phi_n \rangle = \langle \phi_n | H_I \phi_{n+1} \rangle = -|\langle \phi_{n+1} | H_I \phi_n \rangle|$ and that $\|\phi_n\| > 0$ for $n \geqq M$. Let $\lambda < \mu < B$ and $p_M = 1$ and define the real numbers $\{p_n | n = M+1, M+2, \ldots\}$ by the equations:

$$\begin{gathered}
p_M \frac{\langle \phi_M | H \phi_M \rangle}{\|\phi_M\|^2} + p_{M+1} \frac{\langle \phi_{M+1} | H_I \phi_M \rangle}{\|\phi_{M+1}\| \, \|\phi_M\|} = \mu p_M \,, \\
p_{n-1} \frac{\langle \phi_{n-1} | H_I \phi_n \rangle}{\|\phi_{n-1}\| \, \|\phi_n\|} + p_n \frac{\langle \phi_n | H \phi_n \rangle}{\|\phi_n\|^2} + p_{n+1} \frac{\langle \phi_{n+1} | H_I \phi_n \rangle}{\|\phi_{n+1}\| \, \|\phi_n\|} = \mu p_n \,, \\
n = M+1, M+2, \ldots .
\end{gathered} \tag{4}$$

Note that if we multiply Eq. (4) by p_n we get Eq. (2) with ϕ_n, λ replaced by $(p_n \phi_n / \|\phi_n\|), \mu$. Calculating as in (3) we find that

$$0 < -\left\langle \left(\frac{p_{n+1} \phi_{n+1}}{\|\phi_{n+1}\|} \right) \middle| H_I \left(\frac{p_n \phi_n}{\|\phi_n\|} \right) \right\rangle = p_{n+1} p_n \frac{|\langle \phi_{n+1} | H_I \phi_n \rangle|}{\|\phi_{n+1}\| \, \|\phi_n\|} ,$$

so that $p_n > 0$ for $n \geqq M$.

If we now multiply (2) by $p_n / \|\phi_n\|$ and (4) by $\|\phi_n\|$ and subtract we get:

$$|\langle \phi_{M+1} | H_I \phi_M \rangle| \left[\frac{p_M}{\|\phi_M\|} - \frac{p_{M+1}}{\|\phi_{M+1}\|} \right] = (\mu - \lambda) \, p_M \|\phi_M\| \,,$$

$$\begin{gathered}
|\langle \phi_{n+1} | H_I \phi_n \rangle| \left[\frac{p_n}{\|\phi_n\|} - \frac{p_{n+1}}{\|\phi_{n+1}\|} \right] - |\langle \phi_n | H_I \phi_{n-1} \rangle| \left[\frac{p_{n-1}}{\|\phi_{n-1}\|} - \frac{p_n}{\|\phi_n\|} \right] \\
= (\mu - \lambda) \, p_n \|\phi_n\| \,, \qquad n = M+1, M+2, \ldots .
\end{gathered}$$

Since $(\mu - \lambda) \, p_n \|\phi_n\| > 0$ we see that:

$$\begin{gathered}
0 < (\mu - \lambda) \, \|\phi_M\| = |\langle \phi_{M+1} | H_I \phi_M \rangle| \left[\frac{p_M}{\|\phi_M\|} - \frac{p_{M+1}}{\|\phi_{M+1}\|} \right] < \cdots \\
< |\langle \phi_{n+1} | H_I \phi_n \rangle| \left[\frac{p_n}{\|\phi_n\|} - \frac{p_{n+1}}{\|\phi_{n+1}\|} \right] < \cdots .
\end{gathered}$$

Dividing by $|\langle \phi_{n+1} | H_I \phi_n \rangle|$ we find that:

$$0 < \frac{(\mu - \lambda) \, \|\phi_M\|}{|\langle \phi_{n+1} | H_I \phi_n \rangle|} < \frac{p_n}{\|\phi_n\|} - \frac{p_{n+1}}{\|\phi_{n+1}\|} .$$

From this one can see that the series $\sum_{n=M}^{\infty} |\langle \phi_{n+1} | H_I \phi_n \rangle|^{-1}$ converges to some positive constant C. We may then use the Cauchy-Schwarz inequality for sequences to write:

$$\begin{aligned}
\sum_{n=M}^{\infty} \left[\frac{\|\phi_n\| \, \|\phi_{n+1}\|}{|\langle \phi_{n+1} | H_I \phi_n \rangle|} \right]^{\frac{1}{2}} &\leqq \left[\sum_{n=M}^{\infty} \frac{1}{|\langle \phi_{n+1} | H_I \phi_n \rangle|} \right]^{\frac{1}{2}} \left[\sum_{n=M}^{\infty} \|\phi_n\| \, \|\phi_{n+1}\| \right]^{\frac{1}{2}} \\
&\leqq C^{\frac{1}{2}} \left[\sum_{n=M}^{\infty} \|\phi_n\|^2 \right]^{\frac{1}{2}} = C^{\frac{1}{2}} \, \|\psi\| \,.
\end{aligned} \tag{5}$$

But by (e) we have $1/(n+1) \leqq \text{const.}\,[\|\phi_n\|\,\|\phi_{n+1}\|/|\langle\phi_{n+1}\,|\,H_I\,\phi_n\rangle|]^{\frac{1}{2}}$ so the first series in (5) diverges. Thus no non-zero vector ψ satisfying (1) exists and the proof is complete.

The reader familiar with the theory of Jacobi matrices will recognize Eq. (4) as defining a J-matrix. The proof that H is essentially self-adjoint reduces to a proof that this J-matrix is of type D [9, p. 25].

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