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On the Thermodynamic Limit of the B.C.S. State

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Abstract. We consider vector states in the Fock representation of the C.A.R. algebra, representing condensed pair states. We prove that in the thermodynamic limit these states give rise to a direct integral of gauge-dependent B.C.S. states.

1. Introduction

The algebra of the anticommutation relations (C.A.R. algebra) is an essential tool in the description of an infinite non relativistic Fermi system. A state on the C.A.R. algebra can be defined by giving the set of its (n, m)-point correlation functions $W_{n,m}$ (expectation values of Wick ordered monomials of fields operators). A simple class of states that has been extensively studied [1] ⋯ [3] is formed by the "quasi-free" states or "generalized" free states; a quasi-free state has the property that its truncated (n, m) point functions $W^T_{n,m}$ vanish if $n+m>2$. One of the most important example of quasi-free state is provided by the B.C.S. state [4, 5].

The fact that for gauge-invariant quantities the state ϱ over the C.A.R. algebra arising from the "Schafroth-condensed pair wave function" and the gauge-dependent B.C.S. state become equivalent in the thermodynamic limit, is well known by physicists since long time [6]; however, to the best of our knowledge, nobody has produced a rigorous proof of the identification of ϱ with a direct integral of gauge dependent B.C.S. states. We shall produce here such a proof, at least for particular classes of "Cooper pairs".

However, the main reason for performing this work is to test some methods that might be useful when searching for physical states where the role of the "Cooper pairs" is played by "atoms" of more than two

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particles. We expect that some of these states remain non quasi-free even in the thermodynamic limit [1].

Following the procedure introduced for the free Bose gas by Araki and Woods [7], we look for states of a one-dimensional infinite system by

i) including the system in a finite interval of length L.

ii) considering vector states $\varrho_{L,N}$ in the Fock representation of the C.A.R. algebra representing $N/2$ condensed pair states.

iii) taking the limit, in the weak topology, of a sequence $\{\varrho_{L,N}\}$ of such states when $L \to \infty$, $N \to \infty$, and the density N/L is equal to a constant value d.

In order to prove the above result, we use methods adapted from statistical mechanics. We reduce our problem to that of proving the equivalence, in the thermodynamic limit, of the correlation functions in the canonical and grand-canonical ensemble for a free Fermi system with arbitrary single-particle energies. Our technique is similar to that used recently by Dobrushin and Minlos [8] for proving the continuity of pressure in classical statistical mechanics.

2. The Thermodynamic Limit

We consider for simplicity the case of one-dimensional fermions without spin (the generalization to the three-dimensional case with spin is straightforward). Let $\mathfrak{A}(L)$ be the Von Neumann algebra generated by the field operators

$$a_F(f) = \int_L a_F(x) f(x)\, dx; \qquad a_F(f)^+ = \int_L a_F(x)^+ \overline{f(x)}\, dx \tag{1}$$

belonging to the Fock representation π_F^L of the C.A.R. algebra constructed over the Hilbert space $H_L = \mathscr{L}^2(L)$ of square-integrable functions f having their support in the interval L of the real line $\boldsymbol{R}$ [2]. Let $|\psi_{0,F}^L\rangle$ be the Fock vacuum belonging to the Hilbert space H_F^L of the representation π_F^L, and finally let us put

$$a_{k_j} = a_F(f_j) \tag{2}$$

where f_j is the wave function

$$f_j(x) = \frac{1}{\sqrt{L}}\, e^{ik_j x}, \; x \in L, \; k_j = j\,\frac{2\pi}{L}; \qquad j = 0, \pm 1, \pm 2 \ldots \tag{3}$$

[1] G. Fano and G. Loupias: "Conjectures on a class of physical states of Fermi systems..." (Unpublished report).

[2] We indicate with the same symbol the set L and its linear measure.

The anticommutation relations of the a_{k_j} are

$$[a_{k_i}, a_{k_j}]_+ = [a^+_{k_i}, a^+_{k_j}]_+ = 0\,; \qquad [a^+_{k_i}, a_{k_j}] = \delta_{ij}\,. \tag{4}$$

Furthermore $a_{k_j}|\psi^L_{0,F}\rangle = 0$ for any k_j.

Consider now the operator

$$B^+_{2,L} = \frac{1}{2}\sum_{-\infty j}^{+\infty} c(k_j)\, a^+_{k_j} a^+_{-k_j} = \frac{1}{2}\sum_{-\infty j \neq 0}^{+\infty} c(k_j)\, a^+_{k_j} a^+_{-k_j} \tag{5}$$

where c is a real function defined on $\boldsymbol{R}$. The label 2 indicates that $B^+_{2,L}$ creates a pair of fermions.

For the moment we will assume only that c is decreasing, that it vanishes at infinity faster than k^{-1} and that it is square-integrable over $\boldsymbol{R}$. Because of (4), only the odd part of c contributes to the sum (5). Therefore we can restrict ourselves to functions c such that $c(k) = -c(-k)$ and write

$$B^+_{2,L} = \sum_{j=1}^{\infty} c(k_j)\, a^+_{k_j} a^+_{-k_j} \tag{6}$$

$B^+_{2,L}$ is a bounded operator; in fact, since $\|a^+_{k_j}\| = 1$ we have

$$\|B^+_{2,L}\| \leqq \sum_{j=1}^{\infty} |c(k_j)| < \infty\,. \tag{7}$$

Let us define now a sequence $\{v^{(n)}_{2,L}\}$ of vectors $v^{(n)}_{2,L} \in H^L_F$

$$|v^{(n)}_{2,L}\rangle = (B^+_{2,L})^n |\psi^L_{0,F}\rangle\,. \tag{8}$$

Each vector $|v^{(n)}_{2,L}\rangle$ is an eigenstate of the total particle number corresponding to the eigenvalue $N = 2n$ and defines a vector state $\varrho_{L,N}$ on the algebras $\mathfrak{A}(L')$, $L' \subseteqq L$ by

$$\varrho_{L,N}(A) = \frac{\langle v^{(n)}_{2,L}|\pi^L_F(A)|v^{(n)}_{2,L}\rangle}{\langle v^{(n)}_{2,L}|v^{(n)}_{2,L}\rangle}\,; \qquad A \in \mathfrak{A}(L')\,,\ N = 2n\,. \tag{9}$$

We want to study the limit state $\varrho_{B_2,d}$ defined by

$$\varrho_{B_2,d} = \operatorname*{weak\ limit}_{\substack{L\to\infty \\ N\to\infty}\big\} \frac{N}{L} = d} \varrho_{L,N}\,. \tag{10}$$

Of course $\varrho_{B_2,d}$ will no longer be a vector state nor a density matrix in the Fock representation over $H = L^2(R)$.

Let us compute the norm of $|v^{(n)}_{2,L}\rangle$:

$$\begin{aligned}\langle v^{(n)}_{2,L}, v^{(n)}_{2,L}\rangle &= \langle \psi^L_{0,F}|(B_{2,L})^n (B^+_{2,L})^n|\psi^L_{0,F}\rangle \\ &= \sum_{\substack{j_1, j_2 \dots j_n > 0 \\ j'_1, j'_2 \dots j'_n > 0}} c(k_{j_1})\, c(k_{j_2}) \dots c(k_{j_n}) c(k_{j'_1})\, c(k_{j'_2}) \dots c(k_{j'_n}) \\ &\quad \cdot \langle \psi^L_{0,F}|a_{-k_{j'_n}} a_{k_{j'_n}} \dots a_{-k_{j'_1}} a_{k_{j'_1}} a^+_{k_{j_1}} a^+_{-k_{j_1}} \dots a^+_{k_{j_n}} a^+_{-k_{j_n}}|\psi^L_{0,F}\rangle\end{aligned} \tag{11}$$

Because of the Fermi statistics, the only terms which survive are those such that $j_1 \neq j_2 \cdots \neq j_n; j'_1 \neq j'_2 \cdots \neq j'_n$. Furthermore, because of the condition $a_{k_j} | \psi^L_{0,F} \rangle = 0$, the set $\{j'_1, j'_2 \ldots j'_n\}$ can only be a permutation of the set $\{j_1, j_2 \ldots j_n\}$. Therefore

$$\begin{aligned} \|v^{(n)}_{2,L}\|^2 &= n! \sum_{j_1 \neq j_2 \cdots \neq j_n > 0} c(k_{j_1})^2 \ldots c(k_{j_n})^2 \\ &= (n!)^2 \sum_{0 < j_1 < j_2 \cdots < j_n} c(k_{j_1})^2 \ldots c(k_{j_n})^2 \end{aligned} \tag{12}$$

and we have the inequality

$$\|v^{(n)}_{2,L}\|^2 < n! \sum_{j_1, j_2 \ldots j_n > 0} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2 = n! \left[\sum_{j=1}^{\infty} c(k_j)^2 \right]^n < \infty . \tag{13}$$

Let us consider now the entire function f^L of the variable λ defined by the infinite product

$$f^L(\lambda) = \prod_{j=1}^{\infty} (1 + \lambda c(k_j)^2) \tag{14}$$

which is convergent according to the hypothesis on c. All the zeros of $f^L(\lambda)$ lie on the negative real axis and are given by

$$\lambda_j = -\frac{1}{c(k_j)^2}; \quad j = 1, 2, 3 \ldots . \tag{15}$$

The coefficients a^L_n of the power series expansion of f^L are simply connected with the $\|v^{(n)}_{2,L}\|^2$. In fact

$$\begin{aligned} f^L(\lambda) = \sum_{n=0}^{\infty} a^L_n \lambda^n &= 1 + \lambda \sum_{j=1}^{\infty} c(k_j)^2 + \lambda^2 \sum_{0 < j_1 < j_2} c(k_{j_1})^2 c(k_{j_2})^2 + \cdots \\ &+ \lambda^n \sum_{0 < j_1 < j_2 \cdots < j_n} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2 + \cdots . \end{aligned} \tag{16}$$

Therefore [3]

$$a^L_n = \frac{1}{n!} \left. \frac{d^{(n)} f^L}{d\lambda^n} \right|_{\lambda=0} = \sum_{0 < j_1 < j_2 \cdots < j_n} c(k_{j_1})^2 c(k_{j_2})^2 \ldots c(k_{j_n})^2 = \frac{1}{(n!)^2} \|v^{(n)}_{2,L}\|^2 . \tag{17}$$

Useful formulae for the computation of the a^L_n are the following: Defining

$$\sigma^L_p = \sum_{j=1}^{\infty} c(k_j)^{2p}; \quad p = 1, 2, 3 \ldots \tag{18}$$

[3] Notice that if the function c has compact support, then a^L_n vanishes for n sufficiently large, i.e. for $2n/L$ larger than a certain value $d_{\max}$, where $d_{\max}$ is simply related to the measure of the support of c.

it follows that (see Ref. [9], Theorem 3.7, and Ref. [10], Formula 38)

$$a_n^L = \sum_{j_1+2j_2+3j_3+\cdots+mj_m=n} (-1)^{j_2+j_4+j_6+\cdots} \frac{(\sigma_1^L)^{j_1}}{j_1!} \frac{(\sigma_2^L)^{j_2}}{2^{j_2} j_2!} \cdots \frac{(\sigma_m^L)^{j_m}}{m^{j_m} j_m!}$$

$$= \frac{1}{n!} \operatorname{Det} \begin{Vmatrix} \sigma_1^L & 1 & 0 & 0 & \dots & 0 \\ \sigma_2^L & \sigma_1^L & 2 & 0 & \dots & 0 \\ \sigma_3^L & \sigma_2^L & \sigma_1^L & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & & & & n-1 \\ \sigma_n^L & \sigma_{n-1}^L & \sigma_{n-2}^L & \sigma_{n-3}^L & \dots & \sigma_1^L \end{Vmatrix} \tag{19}$$

$$a_n^L = \frac{1}{n} \sum_{m=1}^{n} (-1)^{m+1} \sigma_m^L a_{n-m}^L . \tag{20}$$

We shall not make further use of formulae (19), (20); they are, however, interesting since they exploit the analogy between our case and Fredholm's theory of integral equations.

From (14) it follows that

$$f^L(\lambda) = \exp\left\{ \sum_{j=1}^{\infty} \log[1 + \lambda c(k_j)^2] \right\} . \tag{21}$$

Hence, defining

$$\frac{1}{L} \log f^L(\lambda) = p(L, \lambda) \tag{22}$$

we have

$$\frac{d}{d\lambda} p(L, \lambda) = \frac{1}{L} \sum_{j=1}^{\infty} \frac{c(k_j)^2}{1 + \lambda c(k_j)^2} . \tag{23}$$

We define now a family of vectors of the B.C.S. type depending on the positive number λ, on the lenght L and on the function c:

$$|\lambda, L, c\rangle = e^{\sqrt{\lambda} B_{2,L}^+} |\psi_{0,F}^L\rangle = e^{\sqrt{\lambda} \sum_{j=1}^{\infty} c(k_j) a_{k_j}^+ a_{-k_j}^+} |\psi_{0,F}^L\rangle . \tag{24}$$

The square of the norm of this vector coincides with $f^L(\lambda)$. In fact, since

$$\langle (B_{2,L}^+)^n \psi_{0,F}^L | (B_{2,L}^+)^m \psi_{0,F}^L \rangle = 0 \quad \text{for} \quad n \neq m \tag{25}$$

we have[4]

$$\langle \lambda, L, c | \lambda, L, c \rangle = \left\langle \sum_{n=0}^{\infty} \frac{\lambda^{\frac{n}{2}} (B_{2,L}^{+})^n}{n!} \psi_{0,F}^{L} \middle| \sum_{m=0}^{\infty} \frac{\lambda^{\frac{m}{2}} (B_{2,L}^{+})^m}{m!} \psi_{0,F}^{L} \right\rangle$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} \langle (B_{2,L}^{+})^n \psi_{0,F}^{L} | (B_{2,L}^{+})^n \psi_{0,F}^{L} \rangle \tag{26}$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} \| v_{2,L}^{(n)} \|^2 = \sum_{n=0}^{\infty} a_n^L \lambda^n = f^L(\lambda) .$$

We notice that $\|\lambda, L, c\| \neq 0$ since λ is positive. Let us now put, for any monomial $\pi_F(A)$ in the field operators (we assume that the test functions appearing in $\pi_F(A)$ vanish outside L, i.e. $\pi_F(A) \in \mathfrak{A}(L)$)

$$\varrho_{(\lambda,L)}(A) = \frac{\langle \lambda, L, c | \pi_F(A) | \lambda, L, c \rangle}{\langle \lambda, L, c | \lambda, L, c \rangle}$$
$$= \frac{1}{f^L(\lambda)} \left\langle \sum_{n=0}^{\infty} \frac{\lambda^{\frac{n}{2}} (B_{2,L}^{+})^n}{n!} \psi_{0,F}^{L} \middle| \pi_F(A) \middle| \sum_{m=0}^{\infty} \frac{\lambda^{\frac{m}{2}} (B_{2,L}^{+})^m}{m!} \psi_{0,F}^{L} \right\rangle \tag{27}$$

If A is gauge invariant (i.e. A contains an equal number of creation and annihilation operators), then

$$\langle (B_{2,L}^{+})^n \psi_{0,F}^{L} | \pi_F(A) | (B_{2,L}^{+})^m \psi_{0,F}^{L} \rangle = 0 \quad \text{for} \quad n \neq m \tag{28}$$

and therefore

$$\varrho_{(\lambda,L)}(A) = \frac{\sum_{n=0}^{\infty} A_n^L \lambda^n}{\sum_{n=0}^{\infty} a_n^L \lambda^n} \tag{29}$$

where

$$A_n^L = \frac{1}{(n!)^2} \langle v_{2,L}^{(n)} | \pi_F(A) | v_{2,L}^{(n)} \rangle . \tag{30}$$

We notice that since A is a bounded operator, the expansion $\sum_{n=0}^{\infty} A_n^L \lambda^n$ converges in the entire complex plane of the variable λ. The functional $\varrho_{(\lambda,L)}$ is a vector state for $\lambda \geqq 0$. Defining our states $\varrho_{L,2n}$ on $\mathfrak{A}(L)$ as in

[4] The series expansion of $e^{\sqrt{\lambda} B_{2,L}^{+}}$ in powers of $\sqrt{\lambda}$ converges in the uniform topology since $B_{2,L}^{+}$ is a bounded operator. Further, we note that the same results are obtained substituting $|\lambda, L, c\rangle$ by $\sum_{n=0}^{\infty} \frac{\lambda^{\frac{n}{2}}}{n!} (B_{2,L}^{+})^n e^{i\theta_n} |\psi_{0,F}^{L}\rangle$ where θ_n are arbitrary real numbers.

formula (9), the formulae (17), (30) imply that

$$\varrho_{L,2n}(A) = \frac{A_n^L}{a_n^L}. \tag{31}$$

Let us start computing the expectation value in the state $\varrho_{L,2n}$ of monomials of the type $a(f)^+ a(g)$, where the supports of f and g are contained in L[5]; from this and from the continuity of $\varrho_{L,2n}$, it follows that

$$\varrho_{L,2n}[a(f)^+ a(g)] = \frac{2\pi}{L} \sum_{j=-\infty}^{+\infty} \varrho_{L,2n}(a_{k_j}^+ a_{k_j}) \overline{\tilde{f}(k_j)}\, \tilde{g}(k_j) \tag{32}$$

where

$$\left.\begin{aligned} \tilde{f}(k_j) &= \frac{1}{\sqrt{2\pi}} \int_L f(x)\, e^{-ik_j x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ik_j x}\, dx \\ \tilde{g}(k_j) &= \frac{1}{\sqrt{2\pi}} \int_L g(x)\, e^{-ik_j x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x)\, e^{-ik_j x}\, dx. \end{aligned}\right\} \tag{33}$$

In order to compute the thermodynamic limit of (32), we will choose a particular sequence of lengths:

$$L_i = 2^i L_0 \quad \text{with} \quad n_0 = \tfrac{1}{2} d L_0 \quad \text{integer}. \tag{34}$$

With this choice the sets $S^{(i)} = \left\{x : x = j\frac{2\pi}{L_i};\ j = 1, 2, 3 \ldots\right\}$ are ordered by inclusion

$$S^{(1)} \subset S^{(2)} \subset S^{(3)} \ldots \subset S^{(i)} \ldots$$

so that putting $L = L_i$ and $2n = 2n_i = dL_i$ into Eq. (32), we can take the limit of $\varrho_{L_i,2n_i}(a^+_{k_j^{(i)}} a_{k_j^{(i)}})$ when $i \to \infty$, $j \to \infty$ and both $d = 2n_i/L_i$ and $k_j^{(i)} = j\, 2\pi/L_i \in S^{(i)}$ are kept fixed.

Later we shall be able to prove that for any $k \in \bigcup_{i=1}^{\infty} S^{(i)}$

$$\lim_{i \to \infty,\, d = \frac{2n_i}{L_i}} \varrho_{L_i,2n_i}(a_k^+ a_k) = \frac{\lambda(d)\, c(k)^2}{\lambda(d)\, c(k)^2 + 1} \tag{35}$$

where $\lambda(d)$ is a positive number depending only on d. Therefore it will be sufficient to make very broad hypothesis of regularity on the functions

[5] This means that we limit ourselves to strictly local observables; the supports of f and g will not move when $L \to \infty$.

c, f and g in order to conclude from (32) that[6]

$$\lim_{i\to\infty, d=\frac{2n_i}{L_i}} \varrho_{L_i, 2n_i}[a(f)^+ a(g)] = \int_{-\infty}^{+\infty} \overline{\tilde{f}(k)}\, g(k) \frac{\lambda(d)\, c(k)^2}{\lambda(d)\, c(k)^2+1}\, dk\,. \quad (36)$$

A well known formula of the B.C.S. theory giving the expectation value of the particle number operator $a_{k_j}^+ a_{k_j}$ in the state $\varrho_{(\lambda,L)}$ is[7]:

$$\varrho_{(\lambda,L)}(a_{k_j}^+ a_{k_j}) = \frac{\lambda\, c(k_j)^2}{1+\lambda\, c(k_j)^2} \quad (37)$$

We have the expansion:

$$\varrho_{(\lambda,L)}(a_{k_j}^+ a_{k_j}) = \sum_{m=1}^{\infty} e_m(k_j)\lambda^m \quad (38)$$

where

$$e_m(k_j) = (-1)^{m+1} c(k_j)^{2m}; \qquad m \geqq 1\,. \quad (39)$$

We notice that, whereas the preceeding expansions of $f^L(\lambda)$ and $\langle \lambda, L, c | \pi_F(A) | \lambda. L, c\rangle$ in power series of λ have an infinite radius of convergence, the radius of convergence of the expansion (38) is:

$$0 < R_j(L) = \frac{1}{c(k_j)^2}\,. \quad (40)$$

Putting $A = a_{k_j}^+ a_{k_j}$ into Eqs. (29), (30) and taking into account the expansion (38) we get:

$$\sum_{n=0}^{\infty} A_n^L(k_j)\lambda^n = \left[\sum_{m=1}^{\infty} e_m(k_j)\lambda^m\right]\left[\sum_{p=0}^{\infty} a_p^L \lambda^p\right] \quad (41)$$

where

$$A_n^L(k_j) = \frac{1}{(n!)^2} \langle v_{2,L}^{(n)} | a_{k_j}^+ a_{k_j} | v_{2,L}^{(n)}\rangle\,; \qquad n = 0, 1, 2 \ldots. \quad (42)$$

[6] Defining the function (we write k_j in the place of $k_j^{(i)}$)

$$\varrho_i(k) = \varrho_{L_i, 2n_i}(a_{k_j}^+ a_{k_j})\, \overline{\tilde{f}(k_j)}\, \tilde{g}(k_j) \quad \text{for} \quad k_j \leqq k < k_{j+1}\,; \; k_j = j\,\frac{2\pi}{L_i}$$

we have (see Eq. (32))

$$\varrho_{L_i, 2n_i}[a(f)^+ a(g)] = \int_{-\infty}^{+\infty} \varrho_i(k)\, dk\,.$$

Since $|\varrho_i(k)| \leqq |\tilde{f}(k_j)\, \tilde{g}(k_j)|$ for $k_j \leqq k < k_{j+1}$, and since $\tilde{f}(k)\, \tilde{g}(k)$ is summable, we are allowed to carry out the limit for $i\to\infty$ inside the integral if $\varrho_i(k)$ tends to $\frac{\lambda c(k)^2}{\lambda c(k)^2+1}\, \overline{\tilde{f}(k)}\, g(k)$ almost everywhere. For instance this is true if c, f, g are continuous functions.

[7] This relation can be easily proved by writing the vector $|\lambda, L, c\rangle$ as:

$$|\lambda, L, c\rangle = \prod_{j=1}^{\infty} (1 + \sqrt{\lambda}\, c(k_j)\, a_{k_j}^+ a_{-k_j}^+)\, |\psi_{0,F}^L\rangle$$

Therefore

$$A_n^L(k_j) = \sum_{m=1}^{n} e_m(k_j)\, a_{n-m}^L\,, \qquad n \geqq 1\,; \qquad A_0^L(k_j) = 0\,. \tag{43}$$

Finally from (31), (43) we obtain

$$\varrho_{L,2n}(a_{k_j}^+ a_{k_j}) = \sum_{m=1}^{n} e_m(k_j)\,\frac{a_{n-m}^L}{a_n^L}\,. \tag{44}$$

So we are left with the problem of calculating the thermodynamic limit of the right hand side of (44).

We shall prove that the thermodynamic limit of $\dfrac{a_{n-m}^L}{a_n^L}$ (performed when $n \to \infty$, $L \to \infty$ and m is fixed) is simply $\lambda(d)^m$, where $\lambda(d)$ is a positive number depending only on the density d. Relation (35) will then follow easily. In order to do this, we shall need the following lemmas:

Lemma 1[8]. *If* $\dfrac{L_1}{L_1 + L_2}$ *is a rational number, the following inequality holds:*

$$a_n^{L_1+L_2} \geqq \sum_{n_1+n_2=n} a_{n_1}^{L_1}\, a_{n_2}^{L_2}\,. \tag{45}$$

Proof. For simplicity let us prove (45) in the particular case $L_1 = L_2 = L$. Let us put

$$S_L = \left\{k : k = j\,\frac{2\pi}{L}\,;\; j = 1, 2, 3 \ldots\right\}. \tag{46}$$

The lattice S_{2L} is the union of two disjoint lattices S_L, S'_L, where S'_L is obtained from S_L by shifting it of an amount π/L to the left, i.e.:

$$S'_L = \left\{k' : k' = \left(j - \frac{1}{2}\right)\frac{2\pi}{L}\,;\; j = 1, 2, 3 \ldots\right\}. \tag{47}$$

We have

$$\begin{aligned} a_n^{2L} &= \frac{1}{n!} \sum_{\substack{k_1 \neq k_2 \cdots \neq k_n \\ k_i \in S_{2L}}} c(k_1)^2\, c(k_2)^2 \ldots c(k_n)^2 \\ &= \frac{1}{n!} \sum_{n_1=0}^{n} \binom{n}{n_1} \sum_{\substack{k_1 \neq k_2 \cdots \neq k_{n_1} \\ k_i \in S_L}} c(k_1)^2 \ldots c(k_{n_1})^2 \sum_{\substack{k'_1 \neq k'_2 \cdots \neq k'_{n-n_1} \\ k'_i \in S'_L}} c(k'_1)^2 \ldots c(k'_{n-n_1})^2 \\ &= \sum_{n_1+n_2=n} \left(\frac{1}{n_1!} \sum_{\substack{k_1 \neq k_2 \cdots \neq k_{n_1} \\ k_i \in S_L}} c(k_1)^2 \ldots c(k_{n_1})^2\right)\left(\frac{1}{n_2!} \sum_{\substack{k'_1 \neq k'_2 \cdots \neq k'_{n_2} \\ k'_i \in S'_L}} c(k'_1)^2 \ldots c(k'_{n_2})^2\right) \end{aligned} \tag{48}$$

[8] This Lemma, Theorem 1 and the analogy with statistical mechanics are unpublished results due to G. Gallavotti.

For each point $k' \in S'_L$ there exists a point $k = k' + \frac{\pi}{L} \in S_L$ and the correspondence $k' \to k$ maps S'_L onto S_L; therefore

$$\begin{aligned}\sum_{\substack{k'_1 \neq k'_2 \cdots \neq k'_{n_2} \\ k'_i \in S'_L}} c(k'_1)^2 \ldots c(k'_{n_2})^2 &= \sum_{\substack{k_1 \neq k_2 \cdots \neq k_{n_2} \\ k_i \in S_L}} c\left(k_1 - \frac{\pi}{L}\right)^2 \ldots c\left(k_{n_2} - \frac{\pi}{L}\right)^2 \\ &\geqq \sum_{\substack{k_1 \neq k_2 \cdots \neq k_{n_2} \\ k_i \in S_L}} c(k_1)^2 \ldots c(k_{n_2})^2\end{aligned} \tag{49}$$

where in the last step we have used the fact that $c(k)^2$ is a decreasing function. Substituting (49) into (48) we obtain

$$a_n^{2L} \geqq \sum_{n_1 + n_2 = n} a_{n_1}^L a_{n_2}^L \tag{50}$$

which proves (45) for $L_1 = L_2$[9]. In particular for $n = 2m$, picking only the term $n_1 = n_2 = m$ in the R.H.S. of (50), we have

$$a_{2m}^{2L} > (a_m^L)^2 \,. \tag{51}$$

Lemma 2. *Let us put*

$$g(L, n) = \frac{1}{L} \log a_n^L \,. \tag{52}$$

The limit

$$\lim_{\substack{n \to \infty \\ L \to \infty}} \ _{\frac{2n}{L} = d} \ g(L, n) = g(d) \tag{53}$$

exists when L goes to infinity according to the sequence $L_i = 2^i L_0 (i = 1, 2 \ldots)$. This limit is a concave, continuous, left and right differentiable function g of the density d for all $0 < d < d_{\max}$, where $d_{\max} = \inf\{d : g(d) = -\infty\}$.

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[9] We sketch the proof for the general case: let's put $\frac{L_1}{L_1 + L_2} = \frac{m_1}{m_1 + m_2}$, where m_1 and m_2 are mutually prime. Consider then the following sets of positive numbers

$$S = \left\{x : x = j \frac{2\pi}{L_1 + L_2}; \quad j = 1, 2, 3, \ldots\right\},$$

$$S_1 = \left\{x : x = j_1 \frac{2\pi}{L_1}; \quad j_1 = 1, 2, 3, \ldots\right\},$$

$$S_2 = \left\{x : x = j_2 \frac{2\pi}{L_2}; \quad j_2 = 1, 2, 3, \ldots\right\},$$

and the mapping $\varphi : S \to (S_1 \cup S_2)$ defined for every $x \in S$ by

$$\varphi(x) = \inf\{y : y \in S_1 \cup S_2;\ y \geqq x\} \,.$$

The idea of the proof consists simply in "shifting" by means of φ the arguments of the functions c^2 from elements of S to elements of S_1 or S_2, and using the decreasing character of c^2. We leave the proof to the reader.

Proof. From inequality (51) we have $\log a_{2n}^{2L} > 2 \log a_n^L$, i.e.:

$$g(2L, 2n) > g(L, n). \tag{54}$$

Putting

$$n_i = \frac{dL_i}{2} = 2^{i-1} dL_0 \quad (i = 1, 2 \ldots), \tag{55}$$

$$\psi(i) = g(L_i, n_i) \qquad (i = 1, 2 \ldots) \tag{56}$$

we conclude from (54) that

$$\psi(i+1) > \psi(i) \quad (i = 1, 2 \ldots). \tag{57}$$

On the other hand ψ is a bounded function of i. In fact from the inequality (13) we have:

$$a_n^L < \frac{1}{n!} \left[\sum_{j=1}^{\infty} c(k_j)^2 \right]^n \leqq \frac{1}{n!} (DL)^n = \left(\frac{2D}{d} \right)^n \frac{n^n}{n!} \tag{58}$$

where

$$D = \frac{1}{2\pi} \int_0^\infty c(k)^2 \, dk < \infty. \tag{59}$$

Hence, using Stirling's theorem in the form $\ln n! \geqq n \ln n - n$, we have

$$g(L, n) < \frac{d}{2} \left(1 + \log \frac{2D}{d} \right) \quad \forall n; \quad \frac{2n}{L} = d. \tag{60}$$

Therefore, according to the sequence $L_i = 2^i L_0$, $g(L_i, n_i)$ tends to a function $g(d)$ such that

$$g(d) \leqq \frac{d}{2} \left(1 + \log \frac{2D}{d} \right). \tag{61}$$

On the other hand from Lemma 1 we have, putting $L = L_1 + L_2$, $n = n_1 + n_2$:

$$g(L, n) > \omega_1 \, g(L_1, n_1) + \omega_2 \, g(L_2, n_2); \quad \omega_1 = \frac{L_1}{L}, \ \omega_2 = \frac{L_2}{L} \tag{62}$$

where ω_1 is a rational number. In particular putting

$$L = 2^i L_0, \quad L_1 = L_2 = 2^{i-1} L_0, \quad n_1 = 2^{i-2} d_1 L_0, \quad n_2 = 2^{i-2} d_2 L_0$$

into the inequality (62), and taking the limit for $i \to \infty$, we get

$$g(d) > \tfrac{1}{2} [g(d_1) + g(d_2)] \tag{63}$$

where $d = \frac{1}{2}(d_1 + d_2)$, and d_1/d_2 is a rational number. This inequality can be easily extended to general values of the densities by defining $g(L, n)$ for all positive n by means of linear interpolation [11]. Therefore:

a) From (61), (63) and the lower bound $g(d) > g(L, n)$ we can conclude [12a] that g is a continuous function and has everywhere left- and right-hand derivatives $\frac{dg^-}{dd}$ and $\frac{dg^+}{dd}$, for all $0 < d < d_{\max}$.

b) From (63) and the continuity of g we obtain the general convexity relation [12b]:

$$g(d) \geqq \sum_i \omega_i g(d_i); \quad \sum_i \omega_i = 1; \quad d = \sum_i \omega_i d_i; \quad \omega_i \geqq 0. \tag{64}$$

It is possible to prove that the limit (53) exists and is equal to $g(d)$ for any sequence L_i tending to infinity (see for instance Ref. [11]), so that the choice (34) is not at all restrictive.

The analogy with the usual formalism of statistical mechanics is now clear: λ corresponds to the activity, $f^L(\lambda)$ corresponds to the grand canonical partition function, and $g(d)$ corresponds to the free energy per unit volume of an infinite system.

The following question now arises: which physical system has a grand canonical partition function equal to $f^L(\lambda)$? The answer is simply obtained[10] from the well-known formula

$$\operatorname{Tr} e^{-\beta \sum\limits_{j=1}^{\infty} [\varepsilon(k_j) - \mu] a_{k_j}^+ a_{k_j}} = \prod_{j=1}^{\infty} [1 + e^{-\beta[\varepsilon(k_j) - \mu]}]$$

(β, μ and ε are positive numbers with the usual physical meaning). This expression becomes equal to $f^L(\lambda)$ by putting $e^{\beta\mu} = \lambda$ and $e^{-\beta\varepsilon(k_j)} = c(k_j)^2$. Hence the desired physical system is simply a free Fermi gas with single particle energies $\varepsilon(k_j)$ equal to $-\frac{1}{\beta} \log c(k_j)^2$. We can exploit this analogy with statistical mechanics, by proving the existence of the "grand canonical pressure" $p(\lambda) = \lim\limits_{L\to\infty} p(L, \lambda)$, and showing that it satisfies the familiar relation $p(\lambda) = \left[g(d) - d\frac{dg}{dd}\right]_{d=d(\lambda)}$ where $d(\lambda)$ is the mean density. These results will be useful in solving our problem.

From our hypothesis on the function c it follows that the limit (see Eq. (21))

$$p(\lambda) = \lim_{L\to\infty} \frac{1}{L} \log f^L(\lambda) = \lim_{L\to\infty} \frac{1}{2\pi} \sum_{j=1}^{\infty} \log[1 + \lambda c(k_j)^2] \frac{2\pi}{L} \tag{65}$$

exists for any $\lambda > 0$, and is given by:

$$p(\lambda) = \frac{1}{2\pi} \int_0^{\infty} \log[1 + \lambda c(k)^2]\, dk. \tag{66}$$

[10] We thank Professor D. W. Robinson for pointing out to us the answer to this question.

On the other hand for any $\alpha > 0$ such that αL is integer, we can write:

$$f^L(\lambda) = \sum_{n=0}^{\alpha L} a_n^L \lambda^n + \sum_{n=\alpha L+1}^{\infty} a_n^L \lambda^n . \tag{67}$$

Since (see inequality (58))

$$\begin{aligned}\sum_{n=\alpha L+1}^{\infty} a_n^L \lambda^n &\leqq \sum_{n=\alpha L+1}^{\infty} \frac{1}{n!} (DL\lambda)^n \leqq \sum_{n=\alpha L+1}^{\infty} (DL\lambda)^n n^{-n} e^n \\ &= \sum_{k=1}^{\infty} \left(\frac{L}{\alpha L+k}\right)^{\alpha L+k} (D\lambda e)^{\alpha L+k} < \sum_{k=1}^{\infty} \left(\frac{D\lambda e}{\alpha}\right)^{\alpha L+k}\end{aligned} \tag{68}$$

it is possible to determine α in such a way that the last expression is less than ε, independently of L.

Therefore we can write, with a suitable choice of C:

$$f^L(\lambda) < \sum_{n=0}^{\alpha L} a_n^L \lambda^n + \varepsilon \leqq \alpha L \max_{n \leqq \alpha L} a_n^L \lambda^n + \varepsilon \leqq C \alpha L \max_{n \leqq \alpha L} a_n^L \lambda^n , \tag{69}$$

$$\begin{aligned}\frac{1}{L} \log f^L(\lambda) &\leqq \frac{1}{L} \log C\alpha L + \frac{1}{L} \log \max_{n \leqq \alpha L} a_n^L \lambda^n \\ &= \frac{1}{L} \log C\alpha L + \frac{1}{L} \max_{n \leqq \alpha L} \log (a_n^L \lambda^n) .\end{aligned} \tag{70}$$

On the other hand $\max\limits_{n \leqq \alpha L} (a_n^L \lambda^n) \leqq f^L(\lambda)$ for $\lambda > 0$; hence

$$\frac{1}{L} \log f^L(\lambda) \geqq \frac{1}{L} \max_{n \leqq \alpha L} \log (a_n^L \lambda^n) = \max_{n \leqq \alpha L} \left[\frac{1}{L} \log a_n^L + \frac{n}{L} \log \lambda\right] \tag{71}$$

Taking the limit for $L \to \infty$ into (70), (71), we have:

$$p(\lambda) = \max_{\substack{d \leqq 2\alpha \\ d < d_{\max}}} \left[g(d) + \frac{d}{2} \log \lambda\right]; \qquad \lambda > 0 \tag{72}$$

Since $\varepsilon \to 0$ when $\alpha \to \infty$ we can write:

$$p(\lambda) = \max_{0 < d < d_{\max}} \left[g(d) + \frac{d}{2} \log \lambda\right] . \tag{73}$$

Our next step shall be to prove that g is differentiable. Since g is a concave bounded function, it is differentiable except perhaps for a denumerable set of values of d. Let M denote the set where g is differentiable. We define

$$\lambda(d) = e^{-2 \frac{dg}{dd}} \quad \text{for} \quad d \in M . \tag{74}$$

Since $\frac{dg}{dd}$ is a decreasing function, $\lambda(d)$ is increasing. Hence $\lambda(d)$ is an invertible function. Let us call $d(\lambda)$ the inverse function, defined on the set $\lambda(M)$. $d(\lambda)$ is increasing, and hence it is differentiable except perhaps for a denumerable set of values of λ. Let Λ denote the set where $d(\lambda)$ exists and is differentiable.

Let us take a point $\lambda \in \Lambda$. Then $d(\lambda) \in M$, and $g(d)$ is differentiable at $d = d(\lambda)$. By differentiating $g(d) + \frac{d}{2}\log\lambda$ with respect to d, we obtain the expression $\frac{dg}{dd} + \frac{1}{2}\log\lambda$ that vanishes for $d = d(\lambda)$ (see Eq. (74)). Hence the function

$$P(\lambda) = \max_{d \in M}\left[g(d) + \frac{d}{2}\log\lambda\right] \tag{75}$$

coincides with $g[d(\lambda)] + \frac{d(\lambda)}{2}\log\lambda$ for $\lambda \in \Lambda$. It follows that $P(\lambda)$ coincides with $p(\lambda)$ (see Eq. (73)), since a continuous concave function can have at most one maximum value. Therefore

$$p(\lambda) = g[d(\lambda)] + \frac{d(\lambda)}{2}\log\lambda \quad \text{for} \quad \lambda \in \Lambda\,. \tag{76}$$

Differentiating this equation we have

$$\frac{dp}{d\lambda} = \frac{dd}{d\lambda}\left[\frac{\log\lambda}{2} + \frac{dg}{dd}\right]_{d=d(\lambda)} + \frac{d(\lambda)}{2\lambda} = \frac{d(\lambda)}{2\lambda} \quad \text{for} \quad \lambda \in \Lambda\,. \tag{77}$$

Due to our hypothesis on the function c, we can differentiate under the integral sign in Eq. (66), and we obtain

$$2\lambda\frac{dp}{d\lambda} = \frac{1}{\pi}\int_0^\infty \frac{\lambda\, c(k)^2}{1 + \lambda\, c(k)^2}\,dk \quad \text{for all} \quad \lambda > 0 \tag{78}$$

and by (77) this expression coincides with $d(\lambda)$ for $\lambda \in \Lambda$. From (78) we see that $2\lambda\frac{dp}{d\lambda}$ is an increasing positive analytic function of λ for $\lambda > 0$. Hence the inverse function $\bar{\lambda}(d)$ exists; $\bar{\lambda}(d)$ is analytic for $0 < d < d_{\max}$ and coincides with $\lambda(d)$ on a dense set. Therefore the function $\frac{dg}{dd} = -\frac{1}{2}\log\lambda(d)$ coincides with the analytic function $-\frac{1}{2}\log\bar{\lambda}(d)$ on a dense set. Since g is concave, $\frac{dg^+}{dd}$ is continuous from the right, and

$\frac{dg^-}{dd}$ is continuous from the left [13]. Hence for all d

$$\frac{dg^+}{dd}(d) = \lim_{\varepsilon \to 0} \frac{dg^+}{dd}(d+\varepsilon) = -\frac{1}{2}\log \bar{\lambda}(d)$$

$$\frac{dg^-}{dd}(d) = \lim_{\varepsilon \to 0} \frac{dg^-}{dd}(d-\varepsilon) = -\frac{1}{2}\log \bar{\lambda}(d)$$

so that g is everywhere differentiable. Furthermore

$$d(\lambda) = \frac{1}{\pi}\int_0^\infty \frac{\lambda\, c(k)^2}{1+\lambda\, c(k)^2}\,dk \quad \text{for all} \quad \lambda > 0\,. \tag{79}$$

Lemma 3. *The following inequality holds*[11]*:*

$$\frac{a_n^L}{a_{n-1}^L} > \frac{a_{n+1}^L}{a_n^L} \qquad n = 1, 2 \ldots . \tag{80}$$

Proof. Let us put

$$Q_n^L = n!\, a_n^L = \sum_{0 < j_1 \neq j_2 \cdots \neq j_n} c(k_{j_1})^2\, c(k_{j_2})^2 \ldots c(k_{j_n})^2 \tag{81}$$

Consider now the expressions

$$(Q_n^L)^2 = \sum_{\substack{j_1 \neq j_2 \cdots \neq j_n \\ j_1' \neq j_2' \cdots \neq j_n'}} c(k_{j_1})^2\, c(k_{j_2})^2 \ldots c(k_{j_n})^2\, c(k_{j_1'})^2\, c(k_{j_2'})^2 \ldots c(k_{j_n'})^2\,, \tag{82}$$

$$Q_{n-1}^L\, Q_{n+1}^L = \sum_{\substack{j_1 \neq j_2 \cdots \neq j_{n-1} \\ j_1' \neq j_2' \cdots \neq j_{n+1}'}} c(k_{j_1})^2\, c(k_{j_2})^2 \ldots c(k_{j_{n-1}})^2\, c(k_{j_1'})^2\, c(k_{j_2'})^2 \ldots c(k_{j_{n+1}'})^2\,. \tag{83}$$

We arrange the contributions to the sum appearing in (82), according to the number v of coincidences between the indices $j_1, j_2 \ldots j_n$ and $j_1', j_2' \ldots j_n'$ (respectively we arrange the contributions to (83) according to the number v of coincidences between $j_1, j_2 \ldots j_{n-1}$ and $j_1', j_2' \ldots j_{n+1}'$). More precisely, we define for $2v + \tau \geqq 2$:

$$A_{v,\tau}^L = \sum_{l_1 \neq l_2 \cdots \neq l_v \neq m_1 \neq m_2 \neq m_\tau} c(k_{l_1})^4\, c(k_{l_2})^4 \ldots c(k_{l_v})^4\, c(k_{m_1})^2\, c(k_{m_2})^2 \ldots c(k_{m_\tau})^2 \tag{84}$$

and we notice that there are $\binom{n}{v}^2$ ways of choosing v indices $l_1, l_2 \ldots l_v$ among $j_1, j_2 \ldots j_n$ and v indices $l_1', l_2' \ldots l_v'$ among $j_1', j_2' \ldots j_n'$. Furthermore there are $v!$ different ways of identifying the indices $l_1, l_2 \ldots l_v$ and $l_1', l_2' \ldots l_v'$, and all these ways give rise to the same contribution $A_{v,2(n-v)}^L$.

[11] Inequality (80) can also be derived using classical results in the theory of entire functions. See for instance R. P. Boas: Entire functions, Academic Press 1954, p. 24, Theorem 2.8.2.

Therefore we can write the expression (82) as

$$(Q_n^L)^2 = \sum_{\nu=0}^{n} \nu! \binom{n}{\nu}^2 A^L_{\nu,2(n-\nu)} \,. \tag{85}$$

Reasoning in the same way, we can write (83) as:

$$Q^L_{n-1} Q^L_{n+1} = \sum_{\nu=0}^{n-1} \nu! \binom{n-1}{\nu} \binom{n+1}{\nu} A^L_{\nu,2(n-\nu)} \,. \tag{86}$$

It follows that

$$\begin{aligned}(n+1) Q_n^2 - n Q_{n-1} Q_{n+1} = \sum_{\nu=0}^{n-1} \binom{n}{\nu} \frac{(n+1)!}{(n-\nu-1)!} \left[\frac{1}{n-\nu} - \frac{1}{n-\nu+1}\right] \\ \cdot A^L_{\nu,2(n-\nu)} + (n+1) n! A^L_{n,0}\end{aligned} \tag{87}$$

and the right-hand side of (87) is positive since all $A^L_{\nu,\tau}$ are positive. The inequality (80) then follows.

We can now prove our main theorem concerning the termodynamic limit of $\frac{a^L_{n+1}}{a^L_n}$. Let us first give an heuristic argument: from (52) it follows that

$$\frac{a^L_{n+1}}{a^L_n} = e^{\frac{g(L,n+1)-g(L,n)}{\frac{1}{L}}} = e^{\frac{g(L,n+1)-g(L,n)}{\frac{n+1}{L}-\frac{n}{L}}} \,. \tag{88}$$

Therefore, since the thermodynamic limit of $g(L,n)$ with $\frac{n}{L} = \frac{1}{2} d$ is the function $g(d)$, it seems reasonable[12] to conclude from (88) that the thermodynamic limit of a^L_{n+1}/a^L_n is $e^{2\frac{dg}{dd}}$. This is indeed the case and we have the theorem:

Theorem 1. *Let c be a real function defined over $\boldsymbol{R}^+ = \{x : x \geqq 0\}$ verifying the following conditions:*

i) *c is decreasing.*

ii) *c vanishes at infinity faster than $k^{-\frac{1}{2}}$.*

iii) *c is square-integrable over $\boldsymbol{R}^+$.*

Then for $0 < d < d_{\max}$ we have:

$$\lim_{\substack{n\to\infty, L\to\infty \\ 2n/L = d}} \frac{a^L_{n+1}}{a^L_n} = e^{2\frac{dg}{dd}} = \lambda(d)^{-1} \,. \tag{89}$$

[12] From the physical point of view, it is evident that there are no terms in (88) depending on the boundary of our system.

Proof. By repeated application of the inequality (80) we find

$$\frac{a_{n+\alpha}^L}{a_n^L}=\frac{a_{n+\alpha}^L}{a_{n+\alpha-1}^L}\,\frac{a_{n+\alpha-1}^L}{a_{n+\alpha-2}^L}\cdots\frac{a_{n+1}^L}{a_n^L}<\left(\frac{a_{n+1}^L}{a_n^L}\right)^{\alpha}<\left(\frac{a_n^L}{a_{n-1}^L}\right)^{\alpha} \tag{90}$$

where α is any positive integer. Therefore

$$\frac{a_n^L}{a_{n-1}^L}>\left(\frac{a_{n+\alpha}^L}{a_n^L}\right)^{\frac{1}{\alpha}}=e^{\frac{L}{\alpha}\frac{1}{L}(\log a_{n+\alpha}^L-\log a_n^L)}. \tag{91}$$

Letting $L\to\infty$ with $2n=Ld$ and $2\alpha=\varepsilon L$, we find

$$\liminf_{\substack{L\to\infty\\2n/L=d}}\frac{a_n^L}{a_{n-1}^L}\geqq e^{2\frac{g(d+\varepsilon)-g(d)}{\varepsilon}}. \tag{92}$$

Similarly

$$\frac{a_n^L}{a_{n-\alpha}^L}=\frac{a_n^L}{a_{n-1}^L}\,\frac{a_{n-1}^L}{a_{n-2}^L}\cdots\frac{a_{n-\alpha+1}^L}{a_{n-\alpha}^L}\geqq\left(\frac{a_n^L}{a_{n-1}^L}\right)^{\alpha} \tag{93}$$

$$\limsup_{\substack{L\to\infty\\2n/L=d}}\frac{a_n^L}{a_{n-1}^L}\leqq e^{2\frac{g(d)-g(d-\varepsilon)}{\varepsilon}}. \tag{94}$$

Letting now $\varepsilon\to 0$ in (92), (94) and using the differentiability of $g(d)$ we obtain the desired result (89).

For a given density d, let $L_i=2^iL_0$ $(i=1,2,3\ldots)$ be a sequence of lengths such that the numbers $n_i=\frac{1}{2}dL_i$ are integers. We recall that $S^{(i)}=\left\{x:x=j\frac{2\pi}{L_i};\ j=1,2,3\ldots\right\}$, and we put $S=\bigcup_{i=1}^{\infty}S^{(i)}$ (notice that S is dense in $\boldsymbol{R}^+$). The following theorem holds:

Theorem 2. *Let us assume that c satisfies the conditions* i) ... iii) *of the preceeding theorem. Then for* $k\in S$ *and* $0<d<d_{\max}$, *the thermodynamic limit of the sequence* $\varrho_{L_i,2n_i}(a_k^+a_k)$ *is given by:*

$$\varrho_{B_2,d}(a_k^+a_k)=\lim_{i\to\infty}\varrho_{L_i,2n_i}(a_k^+a_k)=\frac{\lambda(d)\,c(k)^2}{\lambda(d)\,c(k)^2+1} \tag{95}$$

where $\lambda(d)=e^{-2\frac{dg}{dd}}>0$.

Proof. We treat separately the two cases $c(k)^2<\lambda(d)^{-1}$, $c(k)^2>\lambda(d)^{-1}$ (the case $c(k)^2=\lambda(d)^{-1}$ then follows by continuity):

1. $c(k)^2<\lambda(d)^{-1}$. We can write (see Eqs. (39), (44)):

$$\varrho_{L_i,2n_i}(a_k^+a_k)=\sum_{m=1}^{\infty}f_m^{(i)} \tag{96}$$

where

$$\left.\begin{aligned} &f_m^{(i)} = e_m(k)\frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}} = (-1)^{m+1}\, c(k)^{2m}\frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}} \quad \text{for} \quad 1 \leqq m \leqq n_i\,, \\ &f_m^{(i)} = 0 \quad \text{for} \quad m > n_i\,. \end{aligned}\right\} \tag{97}$$

We define, for any $\varepsilon > 0$

$$g_{m,\varepsilon} = [c(k)^2(\lambda(d)+\varepsilon)]^m \quad m = 1, 2, 3 \ldots\,. \tag{98}$$

We claim that, for i sufficiently large:

$$|f_m^{(i)}| < g_{m,\varepsilon} \quad m = 1, 2, 3 \ldots\,. \tag{99}$$

In fact from Lemma 3 it follows that

$$\frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}} = \frac{a_{n_i-m}^{L_i}}{a_{n_i-m+1}^{L_i}}\,\frac{a_{n_i-m+1}^{L_i}}{a_{n_i-m+2}^{L_i}} \cdots \frac{a_{n_i-1}^{L_i}}{a_{n_i}^{L_i}} < \left(\frac{a_{n_i-1}^{L_i}}{a_{n_i}^{L_i}}\right)^m. \tag{100}$$

By Theorem 1 it follows that for any $\varepsilon > 0$, it is possible to choose i sufficiently large in such a way that $|a_{n_i-1}^{L_i}/a_{n_i}^{L_i} - \lambda(d)| < \varepsilon$; therefore

$$\frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}} \leqq [\lambda(d)+\varepsilon]^m \tag{101}$$

and the inequality (99) is proved.

Choosing $\varepsilon < c(k)^{-2} - \lambda(d)$, the geometrical series $\sum\limits_{m=1}^{\infty} g_{m,\varepsilon}$ converges. Therefore the series (96) is dominated by a convergent series, and we can interchange the limit for $i \to \infty$ and the sum over m. Applying Theorem 1, we obtain:

$$\begin{aligned} \lim_{i\to\infty} \varrho_{L_i,2n_i}(a_k^+ a_k) &= \lim_{i\to\infty} \sum_{m=1}^{n_i} (-1)^{m+1}\, c(k)^{2m}\frac{a_{n_i-m}^{L_i}}{a_{n_i-m+1}^{L_i}}\,\frac{a_{n_i-m+1}^{L_i}}{a_{n_i-m+2}^{L_i}} \cdots \frac{a_{n_i-1}^{L_i}}{a_{n_i}^{L_i}} \\ &= \lim_{i\to\infty} \sum_{m=1}^{n_i} (-1)^{m+1}\, c(k)^{2m}\lambda(d)^m = \frac{\lambda(d)\, c(k)^2}{\lambda(d)\, c(k)^2+1}\,. \end{aligned} \tag{102}$$

2. $c(k)^2 > \lambda(d)^{-1}$. Let us put $L = L_i$, $n = n_i$ into Eq. (44), and let us perform the change of variable $m = n_i - m'$ into the sum. We obtain:

$$\begin{aligned} \varrho_{L_i,2n_i}(a_k^+ a_k) &= \sum_{m'=0}^{n_i-1} (-1)^{n_i-m'+1}\, c(k)^{2(n_i-m')}\frac{a_{m'}^{L_i}}{a_{n_i}^{L_i}} \\ &= (-1)^{n_i+1}\, c(k)^{2n_i}\left\{\sum_{m'=0}^{\infty}\left[-\frac{1}{c(k)^2}\right]^{m'}\frac{a_{m'}^{L_i}}{a_{n_i}^{L_i}} - \sum_{m'=n_i}^{\infty}\left[-\frac{1}{c(k)^2}\right]^{m'}\frac{a_{m'}^{L_i}}{a_{n_i}^{L_i}}\right\} \\ &= (-1)^{n_i+1}\, c(k)^{2n_i}\left\{\frac{1}{a_{n_i}^{L_i}}\, f^{L_i}\left(-\frac{1}{c(k)^2}\right) - \sum_{m=n_i}^{\infty}\left[-\frac{1}{c(k)^2}\right]^{m}\frac{a_{m}^{L_i}}{a_{n_i}^{L_i}}\right\}. \end{aligned} \tag{103}$$

Since $k \in S$, there exists an index i^* such that for $i > i^*$, $k \in S_i$. Therefore for i sufficiently large, $f^{L_i}\left(-\dfrac{1}{c(k)^2}\right) = 0$ (see Eq. (15)) and we get:

$$\begin{aligned}\lim_{i\to\infty} \varrho_{L_i,2n_i}(a_k^+ a_k) &= \lim_{i\to\infty} \sum_{m=n_i}^{\infty} \left[-\frac{1}{c(k)^2}\right]^{m-n_i} \frac{a_m^{L_i}}{a_{n_i}^{L_i}} \\ &= \lim_{i\to\infty} \sum_{m=0}^{\infty} \left[-\frac{1}{c(k)^2}\right]^{m} \frac{a_{n_i+m}^{L_i}}{a_{n_i}^{L_i}}.\end{aligned} \tag{104}$$

From Lemma 3 it follows that

$$\frac{a_{n_i+m}^{L_i}}{a_{n_i}^{L_i}} = \frac{a_{n_i+m}^{L_i}}{a_{n_i+m-1}^{L_i}} \frac{a_{n_i+m-1}^{L_i}}{a_{n_i+m-2}^{L_i}} \cdots \frac{a_{n_i+1}^{L_i}}{a_{n_i}^{L_i}} < \left(\frac{a_{n_i+1}^{L_i}}{a_{n_i}^{L_i}}\right)^m. \tag{105}$$

Therefore for any $\varepsilon > 0$, it follows by Theorem 1 that for i sufficiently large we can write

$$\frac{a_{n_i+m}^{L_i}}{a_{n_i}^{L_i}} \leqq (\lambda(d)^{-1} + \varepsilon)^m. \tag{106}$$

Therefore choosing $\varepsilon < c(k)^2 - \lambda(d)^{-1}$ the series (104) is dominated by a convergent geometrical series, and we can interchange the limit with the sum. Applying again Theorem 1 we obtain:

$$\begin{aligned}\lim_{i\to\infty} \varrho_{L_i,2n_i}(a_k^+ a_k) &= \lim_{i\to\infty} \sum_{m=0}^{\infty} \left[-\frac{1}{c(k)^2}\right]^{m} \frac{a_{n_i+m}^{L_i}}{a_{n_i+m-1}^{L_i}} \frac{a_{n_i+m-1}^{L_i}}{a_{n_i+m-2}^{L_i}} \cdots \frac{a_{n_i+1}^{L_i}}{a_{n_i}^{L_i}} \\ &= \sum_{m=0}^{\infty} [-\lambda(d)\, c(k)^2]^{-m} = \frac{\lambda(d)\, c(k)^2}{\lambda(d)\, c(k)^2 + 1}.\end{aligned} \tag{107}$$

This theorem assures that (35), (36) hold. Comparing these formulae with (37), we have finally

$$\varrho_{B_2,d}[a(f)^+ a(g)] = \varrho_{[\lambda(d),R]}[a(f)^+ a(g)] \tag{108}$$

where $\varrho_{[\lambda(d),R]}$ is the (weak) limit of the gauge dependent B.C.S. vector state $\varrho_{(\lambda,L)}$ when $\lambda = \lambda(d)$ and $L \to \infty$, and $\varrho_{B_2,d}$ is given by (10).

The physical meaning of Eq. (79) is now clear: it establishes the equality between the particle density in the state $\varrho_{B_2,d}$ and the mean particle density in the gauge-dependent B.C.S. state $\varrho_{[\lambda(d),R]}$.

Example. Let $c(k)^2$ be given by the function [13]

$$c(k)^2 = e^{-tk}; \quad t > 0 \tag{109}$$

that satisfies conditions i) … iii) of Theorem 1.

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[13] We are indebted to Dr. G. Pizzichini for showing to us that this example can be worked out explicitly.

We have, putting $\Delta = \frac{2\pi}{L}$ (see formula (17)):

$$\begin{aligned} a_n^L &= \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \dots \sum_{j_n=1}^{\infty} c(j_1 \Delta)^2 c[(j_1+j_2)\Delta]^2 \dots c[(j_1+j_2+\dots+j_n)\Delta]^2 \\ &= \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \dots \sum_{j_n=1}^{\infty} e^{-t\Delta j_1} e^{-t\Delta(j_1+j_2)} \dots e^{-t\Delta(j_1+j_2+\dots+j_n)} \\ &= \sum_{j_1=1}^{\infty} e^{-nt\Delta j_1} \sum_{j_2=1}^{\infty} e^{-(n-1)t\Delta j_2} \sum_{j_3=1}^{\infty} e^{-(n-2)t\Delta j_3} \dots \sum_{j_n=1}^{\infty} e^{-t\Delta j_n} \\ &= \frac{1}{e^{nt\Delta}-1} \frac{1}{e^{(n-1)t\Delta}-1} \dots \frac{1}{e^{t\Delta}-1}. \end{aligned} \tag{110}$$

Therefore

$$\lambda(d) = e^{-2\frac{dg}{dd}} = \lim_{\substack{n\to\infty \\ n\Delta=\pi d}} \frac{a_n^L}{a_{n+1}^L} = \lim_{\substack{n\to\infty \\ n\Delta=\pi d}} (e^{(n+1)t\Delta}-1) = e^{\pi dt}-1. \tag{111}$$

This value of $\lambda(d)$ satisfies the relation

$$\frac{1}{\pi} \int_0^{\infty} \frac{\lambda(d)\, e^{-tk}}{\lambda(d)\, e^{-tk}+1} dk = d \tag{112}$$

in agreement with (79).

We notice that conditions i) ... iii) of Theorem 1 on the function $c(k)$ are far from necessary for our results to hold. Consider the simple example

$$c(k) = \frac{1}{k} \tag{113}$$

that does not satisfy condition iii) of Theorem 1. In this case the infinite product (14) is simply given by[14]:

$$f^L(\lambda) = \prod_{j=1}^{\infty} \left[1+\lambda\left(\frac{L}{2\pi}\right)^2 \frac{1}{j^2}\right] = \frac{\sin \mathrm{i}\sqrt{\lambda}\,\frac{L}{2}}{i\sqrt{\lambda}\,\frac{L}{2}} = \frac{\sinh\sqrt{\lambda}\,\frac{L}{2}}{\sqrt{\lambda}\,\frac{L}{2}}. \tag{114}$$

Therefore in this case

$$a_n^L = \left(\frac{L}{2}\right)^{2n} \frac{1}{(2n+1)!}, \tag{115}$$

$$g(d) = \lim_{\substack{n\to\infty \\ L=2n/d}} \frac{1}{L} \log a_n^L = d \log \frac{e}{2d}; \qquad \frac{dg}{dd} = -\log 2d, \tag{116}$$

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[14] We make use of the well known formula $\prod_{j=1}^{\infty}\left(1-\frac{z^2}{j^2}\right) = \frac{\sin \pi z}{\pi z}$.

$$\lim_{\substack{n\to\infty\\ L=2n/d}} \frac{a_{n+1}^L}{a_n^L} = \lim_{\substack{n\to\infty\\ L=2n/d}} \left(\frac{L}{2}\right)^2 \frac{1}{(2n+2)(2n+3)} = \frac{1}{4d^2} = e^{2\frac{dg}{dd}} = \lambda(d)^{-1}, \tag{117}$$

$$p(\lambda) = \lim_{L\to\infty} \frac{1}{L} \log f^L(\lambda) = \frac{1}{2}\sqrt{\lambda}, \tag{118}$$

$$\frac{1}{\pi}\int_0^\infty \frac{k^{-2}}{k^{-2}+\lambda(d)^{-1}}\,dk = \frac{1}{\pi}\int_0^\infty \frac{k^{-2}}{k^{-2}+\frac{1}{4d^2}}\,dk = d \tag{119}$$

etc. Therefore all our formulae hold also in this case. A more general class of functions c, which includes the case

$$c(k) = \frac{1}{k^m}, \qquad m \text{ real} > \frac{1}{2}$$

shall be studied in a subsequent paper.

Let now $A = a_{r_1}^+ a_{r_2}^+ \dots a_{r_p}^+ a_{s_1} a_{s_2} \dots a_{s_p}$ be a general gauge invariant monomial in the field operators, with $r_m \in S$, $s_n \in S$ $(m, n = 1, 2 \dots p)$. Since A contains an equal number of creation and annihilation operators, well known formulae [14] tell us that $\varrho_{(\lambda,L)}(A)$ can be written as a linear combination $\sum_{q=1}^{M(p)} h_q(\lambda)$ of terms of the type:

$$h_q(\lambda) = \prod_{j=1}^{\beta} \varrho_{(\lambda,L)}(a_{l_j}^+ a_{l_j}) \prod_{i=1}^{\alpha} \varrho_{(\lambda,L)}(a_{k_i}^+ a_{-k_i}^+)\, \varrho_{(\lambda,L)}(a_{-h_i} a_{h_i}) \tag{120}$$

where $2\alpha + \beta = p$ and the different choices of the indices l_j, k_i, h_i among $r_1 \dots r_p s_1 \dots s_p$ are labelled by the integer q. Since

$$\varrho_{(\lambda,L)}(a_{l_j}^+ a_{l_j}) = \frac{\lambda\, c(l_j)^2}{1+\lambda\, c(l_j)^2}, \tag{121}$$

$$\varrho_{(\lambda,L)}(a_{k_i}^+ a_{-k_i}^+)\, \varrho_{(\lambda,L)}(a_{-h_i} a_{h_i}) = \frac{\lambda\, c(k_i)\, c(h_i)}{[1+\lambda\, c(k_i)^2]\,[1+\lambda\, c(h_i)^2]} \tag{122}$$

we can expand $\varrho_{(\lambda,L)}(A)$ in power series of λ, obtaining a generalization of formula (38):

$$\varrho_{(\lambda,L)}(A) = \sum_{m=1}^{\infty} e_m(A)\,\lambda^m. \tag{123}$$

Also formula (44) can be immediately generalized; we obtain

$$\varrho_{L,2n}(A) = \sum_{m=1}^{n} e_m(A) \frac{a_{n-m}^L}{a_n^L}. \tag{124}$$

Taking into account (123), we can write

$$\begin{aligned}\varrho_{L,2n}(A) &= \sum_{m=1}^{n} \frac{1}{m!} \frac{d^m}{d\lambda^m} [\varrho_{(\lambda,L)}(A)]_{\lambda=0} \frac{a_{n-m}^L}{a_n^L} \\ &= \sum_{q=1}^{M(p)} \sum_{m=1}^{n} \frac{1}{m!} \left.\frac{d^m h_q(\lambda)}{d\lambda^m}\right|_{\lambda=0} \frac{a_{n-m}^L}{a_n^L}.\end{aligned} \tag{125}$$

From (120), (121), (122) we have

$$\begin{aligned}&\frac{1}{m!} \left.\frac{d^m h_q(\lambda)}{d\lambda^m}\right|_{\lambda=0} \\ &= \frac{1}{m!} \frac{d^m}{d\lambda^m} \left\{ \frac{\lambda^{\alpha+\beta} \prod_{j=1}^{\beta} c(l_j)^2 \prod_{i=1}^{\alpha} c(k_i)\, c(h_i)}{\prod_{j=1}^{\beta} [1+\lambda c(l_j)^2] \prod_{i=1}^{\alpha} [1+\lambda c(k_i)^2]\,[1+\lambda c(h_i)^2]} \right\}_{\lambda=0} \\ &= (-1)^{m-\alpha-\beta} \prod_{j=1}^{\beta} c(l_j)^2 \prod_{i=1}^{\alpha} c(k_i)\, c(h_i) \\ &\qquad \cdot \left[\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} c(l_j)^2 + c(k_i)^2 + c(h_i)^2 \right]^{\langle m-\alpha-\beta \rangle}\end{aligned} \tag{126}$$

where we have introduced the notation

$$(a_1 + a_2 + \cdots + a_n)^{\langle m \rangle} = \sum_{m_1+m_2+\cdots+m_n=m} a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}. \tag{127}$$

Since

$$(a_1 + a_2)^{\langle m \rangle} = a_1^m + a_1^{m-1} a_2 + \cdots + a_1 a_2^{m-1} + a_2^m = \frac{a_1^{m+1} - a_2^{m+1}}{a_1 - a_2} \tag{128}$$

$$\begin{aligned}&(a_1 + a_2 + a_3)^{\langle m \rangle} \\ &= \frac{a_1^{m+2}}{(a_1 - a_2)(a_1 - a_3)} + \frac{a_2^{m+2}}{(a_2 - a_1)(a_2 - a_3)} + \frac{a_3^{m+2}}{(a_3 - a_1)(a_3 - a_2)}\end{aligned} \tag{129}$$

etc., it is possible to write $e_m(A)$ as a linear combination of terms of the type $(-1)^{m+1} c(k)^{2m}$, the coefficients of the linear combination being independent of m. We are then led back to the case considered in Theorem 2; for each term we can perform the thermodynamic limit inside the sum, i.e.:

$$\lim_{i\to\infty} \sum_{m=1}^{n_i} (-1)^{m+1} c(k)^{2m} \frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}} = \sum_{m=1}^{\infty} (-1)^{m+1} c(k)^{2m} \lambda(d)^m. \tag{130}$$

So, finally:

$$\varrho_{B_2,d}(A) = \lim_{i\to\infty} \varrho_{L_i, 2n_i}(A) = \lim_{i\to\infty} \sum_{m=1}^{n_i} e_m(A) \frac{a_{n_i-m}^{L_i}}{a_{n_i}^{L_i}}$$
$$= \sum_{m=1}^{\infty} e_m(A)\, \lambda(d)^m = \varrho_{[\lambda(d),R]}(A)\,. \tag{131}$$

As in the previous case, it is easy to extend this result to any local observable $A = a(f_1)^+ \ldots a(f_p)^+ a(g_1) \ldots a(g_p)$. Since $\varrho_{B_2,d}$ and $\varrho_{[\lambda(d),R]}$ coincide on any local observable we can deduce, by continuity, that they coincide on any quasi local observable. Of course $\varrho_{B_2,d}$ vanishes on the non gauge invariant monomials A of the algebra, while in general $\varrho_{[\lambda(d),R]}$ does not vanish on such elements. It is well known that $\varrho_{[\lambda(d),R]}$ presents a broken symmetry; defining

$$\varrho_{(\lambda,R,\theta)} = \varrho_{[\lambda,R]} \circ \alpha_\theta\,; \qquad 0 \leqq \theta \leqq 2\pi \tag{132}$$

where α_θ is the automorphism of the algebra generated by the gauge transformation $a(f) \to a(f)\, e^{i\theta}$, it is easy to verify that

$$\varrho_{B_2,d} = \int \frac{d\theta}{2\pi}\, \varrho_{(\lambda(d),R,\theta)}\,. \tag{133}$$

In fact the state on the R.H.S. of (133) vanishes on non gauge invariant monomials of the algebra and coincides with $\varrho_{[\lambda(d),R]}$ on the gauge invariant elements.

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