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World Lines of Dust in C-Field Cosmology*

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Abstract. Geodesic and non-geodesic world lines of dust are investigated in Pryce's C-field cosmology. A Raychaudhuri type equation is derived for the non-geodesic, rotational dust flow.

I. Introduction

In their investigation of the steady state theory, Raychaudhuri and Banerji [1] obtained a generalization of Raychaudhuri's equation [2]. A similar investigation of the C-field equations of Pryce [3] yielded inconclusive results; however, their conclusions were based on the assumption that dust always follows geodesics and also, that rotational motion must be excluded by these C-field equations. Nariai [4] pointed out that rotational motion is not excluded by the assumption of a geodesic dust flow and further, that non-geodesic world lines are permissible. Nariai's investigation of the non-geodesic flow led him to postulate an irrotational velocity field.

Here we investigate all possible world lines for dust in Pryce's theory, paying particular attention to the non-geodesic flow. In section III of this paper we show that Nariai's postulate is unnecessary and in section IV we derive a general version of Raychaudhuri's equation for non-geodesic, rotational motion of dust.

II. The Field Equations of Pryce

We are concerned with the following C-field equations which Pryce derived from an action principle:

$$R^\alpha_\beta - \tfrac{1}{2} g^\alpha_\beta R = -8\pi [T^\alpha_\beta - f(C^\alpha C_\beta - \tfrac{1}{2} g^\alpha_\beta C^\gamma C_\gamma)] , \tag{2.1}$$

$$f C^\alpha_{;\alpha} = J^\alpha_{;\alpha} \tag{2.2}$$

where C_α is the gradient of a scalar ($C_\alpha = C_{,\alpha}$) and f is a coupling constant. We adopt the usual notation and denote ordinary differentiation by a

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comma, covariant differentiation by a semi-colon, covariant differentiation along world lines by a dot and the lie derivative with respect to a vector field u^α by $\mathscr{L}_u$. So, for example, for an arbitrary vector A_α we have $\dot{A}_\alpha = A_{\alpha;\beta} u^\beta$ as the effective time derivative for an observer moving with the standard velocity and

$$\mathscr{L}_u A_\alpha = A_{\alpha;\beta} u^\beta + u_{\beta;\alpha} A^\beta \tag{2.3}$$

as the lie derivative defined by Schouten [5]. Greek indices $\alpha, \beta, \ldots = 1, 2, 3, 4$. We consider the simple case of a continuous medium where the energy momentum tensor T^α_β, and the mass current vector J^α are given by

$$T^\alpha_\beta = \varrho u^\alpha u_\beta \,, \tag{2.4}$$

$$J^\alpha = \varrho u^\alpha \,, \tag{2.5}$$

where ϱ is the density and u^α is the 4-velocity of the matter. We take this vector to be normalized so that

$$u^\alpha u_\alpha = -1 \tag{2.6}$$

and therefore

$$u^\alpha u_{\alpha;\beta} = 0 \,. \tag{2.7}$$

We refer to this continuous medium as "dust". It follows immediately from the above equations and assumptions that:

$$C_{\alpha;\beta} = C_{\beta;\alpha} \,, \tag{2.8}$$

$$T^{\alpha\beta}_{;\beta} = f C^\alpha C^\beta_{;\beta} \,, \tag{2.9}$$

$$\dot{u}_\alpha = \lambda (C_\alpha - u_\alpha) \tag{2.10}$$

where

$$\varrho\lambda \overset{\text{def}}{=} (\varrho u^\alpha)_{;\alpha} \,. \tag{2.11}$$

By contracting (2.10) with u^α, and using (2.7), we have (Nariai, 1964):

$$\lambda u_\alpha A^\alpha = 0 \tag{2.12}$$

where

$$A_\alpha \overset{\text{def}}{=} C_\alpha - u_\alpha \,. \tag{2.13}$$

Lemma. 1) $\mathscr{L}_u C_\alpha = 0 \Leftrightarrow$ 2) $\mathscr{L}_u A_\alpha = -\dot{u}_\alpha \Leftrightarrow$ 3) $C_\alpha u^\alpha = constant.$

Proof. 1)$\Leftrightarrow$2): $\mathscr{L}_u C_\alpha = 0 \Leftrightarrow \mathscr{L}_u A_\alpha + \mathscr{L}_u u_\alpha = 0$ by (2.13),
$\Leftrightarrow \mathscr{L}_u A_\alpha + \dot{u}_\alpha = 0$ by (2.3).

1)$\Leftrightarrow$3): $C_\beta u^\beta = \text{constant} \Leftrightarrow (C_\beta u^\beta)_{;\alpha} = 0$
$\Leftrightarrow C_{\beta;\alpha} u^\beta + C^\beta u_{\beta;\alpha} = 0$
$\Leftrightarrow \mathscr{L}_u C_\alpha - 2u^\beta C_{[\alpha;\beta]} = 0$
$\Leftrightarrow \mathscr{L}_u C_\alpha = 0$ by (2.8).

III. The World Lines of Dust

In general relativity the contracted Bianchi identities are

$$T^{\alpha\beta}_{;\alpha} = 0 \tag{3.1}$$

which for dust give

$$\dot{u}^{\alpha} = 0, \quad \dot{\varrho} + \varrho\theta = 0, \tag{3.2}$$

that is, the world lines are geodesic and the mass is conserved. Here $\dot{u}^{\alpha}$ is the acceleration vector and $\theta = u^{\alpha}_{;\alpha}$ is the volume expansion. However, in the C-field theory we must replace (3.1) by the Eq. (2.9) which for dust gives

$$\dot{u}^{\alpha} = \lambda A^{\alpha}, \quad \dot{\varrho} + \theta\varrho = \lambda\varrho, \tag{3.3}$$

in which case, unless $\lambda A^{\alpha} = 0$, the world lines are not geodesic. Furthermore, if $\lambda \neq 0$ then we have continuous creation of matter. In what follows we attempt to classify the various types of world lines that dust will follow in accordance with Eqs. (3.3), the classification is in two parts. First, however, we note that on combining the Eqs. (2.2), (2.5), and (2.11) we can write

$$f C^{\alpha}_{;\alpha} = \lambda\varrho \tag{3.4}$$

as the source equation for the C-field.

Type A: $A_{\alpha} = 0$

It follows immediately from Eqs. (3.3) and (2.13) that for $A_{\alpha} = 0$ we have

$$\dot{u}^{\alpha} = 0, \quad \dot{\varrho} + \varrho\theta = \lambda\varrho, \quad u_{\alpha} = C_{,\alpha} \tag{3.5}$$

that is, the world lines are geodesic, matter is continuously created (is conserved) if $\lambda \neq 0$ (if $\lambda = 0$) and the velocity field is irrotational, i.e. $u_{[\alpha;\beta]} = 0$ (where square brackets denote skew-symmetrized indices).

A(i) is given by $\lambda = 0$: we have $A_{\alpha} = 0 \Rightarrow C_{\alpha} = u_{\alpha}$ and therefore the expansion $\theta = C^{\alpha}_{;\alpha}$. It follows from (3.4) that $\theta = 0$.

A(ii) is given by $\lambda \neq 0$: again we have $\theta = C^{\alpha}_{;\alpha} = \lambda\varrho/f$ by (3.4). On substituting for λ from (2.11) we find that the expansion is given by

$$\theta = \dot{\varrho}(f - \varrho)^{-1}.$$

The A(ii) motion has been discussed by Raychaudhuri and Banerji in [1].

Type B: $A_{\alpha} \neq 0$

We divide the type B motion into two cases, B(i) given by $\lambda = 0$ and B(ii) given by $\lambda \neq 0$:

B(i) The Eqs. (3.3) together with the condition $\lambda = 0$ reproduce the Bianchi identities (3.2), and (3.4) gives the homogeneous source equation

$$C^{\alpha}_{;\alpha} = 0. \tag{3.6}$$

Now, the velocity field u^α is given by (2.13) as $u^\alpha = C^\alpha - A^\alpha$, and is therefore rotational (irrotational) if A_α is (is not) the gradient of a scalar. The B(i) motion has been investigated, in the irrotational case, by Hoyle and Narlikar [6].

B(ii) We have $\lambda \neq 0$, $A_\alpha \neq 0$ but $A^\alpha u_\alpha = 0$, that is $C_\alpha u^\alpha = -1$. A_α is therefore a non zero space-like vector orthogonal to the velocity field vector u_α. Since $C_\alpha u^\alpha$ is constant it follows from the lemma that

$$\dot{u}_\alpha = -\mathscr{L}_u A_\alpha . \tag{3.7}$$

On substituting for A_α from (3.3) in (3.7) we obtain

$$\dot{u}_\alpha = -\mathscr{L}_u (\dot{u}_\alpha/\lambda) \tag{3.8}$$

which, on using the definition (2.3), becomes

$$\ddot{u}_\alpha + \dot{u}^\beta u_{\beta;\alpha} = \dot{u}_\alpha(\dot{\lambda} - \lambda^2)/\lambda . \tag{3.9}$$

The Eq. (3.9) says that the lie derivative of the acceleration vector $\dot{u}_\alpha$ with respect to the velocity field u_α, is proportional to the acceleration vector. The congruence of world lines of type B(ii) is non geodesic (cf. (3.3)) and rotational (since if the flow were irrotational then $u_{[\alpha;\beta]} = 0$, which together with (2.7) would imply $\dot{u}_\alpha = 0$). On specifying a source equation for the C-field we may obtain a value for λ from (3.4). This value of λ in (3.9) then specifies the congruence of non geodesic, rotational dust world lines.

IV. General Relations for Type B(ii)

We recall that the congruence of world lines for dust type B(ii) is characterized by the relations

$$A_\alpha = C_\alpha - u_\alpha \neq 0, \qquad \lambda = (\varrho u^\alpha)_{;\alpha}/\varrho \neq 0, \qquad A^\alpha u_\alpha = 0 , \tag{4.1}$$

and the total energy E created by the occurrence of the C-field is given by $E^2 = C^\alpha C_\alpha$ where

$$\begin{aligned} C^\alpha C_\alpha &= (A^\alpha + u^\alpha)(A_\alpha + u_\alpha) \\ &= A^\alpha A_\alpha - 1 . \end{aligned}$$

The world lines for dust type B(ii) are not geodesic, we ask now what is the next simplest timelike world line? Following Synge [7], we define an orthonormal tetrad $(u^\alpha, A^\alpha, B^\alpha, D^\alpha)$ where

$$\dot{u}^\alpha = \lambda A^\alpha , \tag{4.2}$$

$$\dot{A}^\alpha = \lambda u^\alpha \tag{4.3}$$

and the vectors B^α and D^α are defined by

$$\dot{B}^\alpha = \dot{D}^\alpha = 0 \tag{4.4}$$

together with

$$A^\alpha A_\alpha = B^\alpha B_\alpha = D^\alpha D_\alpha = 1, \qquad u^\alpha u_\alpha = -1 \tag{4.5}$$

and

$$u^\alpha B_\alpha = u^\alpha D_\alpha = u^\alpha A_\alpha = 0\,. \tag{4.6}$$

On choosing $\lambda = \text{constant}$ we obtain the Frenet-Serret formula (Eqs. (4.2)–(4.6)) for a "hyperbola of constant curvature". However, in obtaining this particular world line we have assumed that $A^\alpha A_\alpha = 1$, or in terms of the scalar field C_α that

$$C^\alpha C_\alpha = 0\,, \tag{4.7}$$

which we interpret to mean the creation of photons. On eliminating A^α between (4.2) and (4.3) we find that the time-like world line for dust satisfying (4.7) has velocity field u^α where

$$\ddot{u}^\alpha = \lambda^2 u^\alpha\,. \tag{4.8}$$

We now derive two general equations for this congruence in terms of its expansion, shear and rotation. First we obtain the components of the Ricci tensor $R_{\alpha\beta}$ from the field Eqs. (2.1), (2.2), (2.4), and (2.5); we find

$$R_{\alpha\beta} = 8\pi f C_\alpha C_\beta - 8\pi\varrho(u_\alpha u_\beta + \tfrac{1}{2} g_{\alpha\beta})\,. \tag{4.9}$$

For the time-like congruence of world lines type B(ii) we have $C_\alpha u^\alpha = -1$, and therefore we obtain the components of $R_{\alpha\beta}u^\beta$ along the congruence and orthogonal to it, to be

$$R_{\alpha\beta}u^\alpha u^\beta = 8\pi f - 4\pi\varrho\,, \tag{4.10}$$

$$h^{\alpha\beta}R_{\beta\gamma}u^\gamma = 8\pi f(u^\alpha - C^\alpha)\,, \tag{4.11}$$

where, using the notation and results of Ehlers [8], we have

$$R_{\alpha\beta}u^\alpha u^\beta = \dot{\theta} + \tfrac{1}{3}\theta^2 - \dot{u}^\alpha_{;\alpha} + 2(\sigma^2 - \omega^2)\,, \tag{4.12}$$

$$h^{\alpha\beta}R_{\beta\gamma}u^\gamma = h^\alpha_\beta(\omega^{\beta\gamma}_{;\gamma} - \sigma^{\beta\gamma}_{;\gamma} + \tfrac{2}{3}\theta^{,\beta}) + (\omega^\alpha_\beta + \sigma^\alpha_\beta)\dot{u}^\beta\,; \tag{4.13}$$

$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the projection operator into the hyperplane orthogonal to u_α, $\sigma_{\alpha\beta} = u_{(\gamma;\delta)}h^\gamma_\alpha h^\delta_\beta - \frac{1}{3}\theta h_{\alpha\beta}$ is the shear tensor and $\omega_{\alpha\beta} = u_{[\gamma;\delta]}h^\gamma_\alpha h^\delta_\beta$ is the vorticity tensor. The magnitude of these tensors gives the shear $\sigma = (\frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta})^{\frac{1}{2}}$ and the vorticity $\omega = (\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})^{\frac{1}{2}}$. Round brackets denote symmetrized indices.

In Eq. (4.12) the component $\dot{u}^\alpha_{;\alpha}$ is given by (2.10) and (3.4) as $\dot{u}^\alpha_{;\alpha} = \lambda^2\varrho/f - \lambda\theta$, for $\lambda = \text{constant}$, which on combining with Eqs. (4.10) to (4.13) yields:

$$\dot{\theta}+\tfrac{1}{3}\theta^2+\lambda\theta+2(\sigma^2-\omega^2)=8\pi f-4\pi\varrho(1-\lambda^2/4\pi f) \tag{4.14}$$

and

$$h^\alpha_\beta(\omega^{\beta\gamma}_{;\gamma}-\sigma^{\beta\gamma}_{;\gamma}+\tfrac{2}{3}\theta^{,\beta})+(\omega^\alpha_\beta+\sigma^\alpha_\beta)\dot{u}^\beta=8\pi f(u^\alpha-C^\alpha), \tag{4.15}$$

for the B(ii) motion with constant λ.

To conclude, we list all equations relevant to each type of motion, and for comparison we include the corresponding results for the Einstein equations for dust, namely

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R-\Lambda g_{\alpha\beta}=-8\pi\varrho u_\alpha u_\beta, \tag{4.16}$$

given by Ellis [9]:

$$\dot{\theta}+\tfrac{1}{3}\theta^2+2(\sigma^2-\omega^2)=\Lambda-4\pi\varrho, \tag{4.17a}$$

$$h^\alpha_\beta(\tfrac{2}{3}\theta^{,\beta}-\sigma^{\beta\gamma}_{;\gamma}+\omega^{\beta\gamma}_{;\gamma})=0, \tag{4.17b}$$

$$\dot{\varrho}+\varrho\theta=0, \qquad \dot{u}_\alpha=0. \tag{4.17c}$$

For type A(i):

$$\begin{gathered}\sigma^2=2\pi(2f-\varrho), \qquad \omega=0,\\ \theta=0, \qquad \dot{\varrho}=0, \qquad \dot{u}_\alpha=0, \qquad C^\alpha_{;\alpha}=0;\end{gathered} \tag{4.18}$$

for type A(ii):

$$\begin{gathered}\dot{\theta}+\tfrac{1}{3}\theta^2+2\sigma^2=8\pi f-4\pi\varrho, \qquad \omega=0, \qquad \dot{u}_\alpha=0,\\ h^\alpha_\beta(\tfrac{2}{3}\theta^{,\beta}-\sigma^{\beta\gamma}_{;\gamma})=0, \qquad \theta=\dot{\varrho}(f-\varrho)^{-1}=C^\alpha_{;\alpha};\end{gathered} \tag{4.19}$$

for type B(i):

$$\begin{gathered}\dot{\theta}+\tfrac{1}{3}\theta^2+2(\sigma^2-\omega^2)=8\pi f-4\pi\varrho,\\ h^\alpha_\beta(\tfrac{2}{3}\theta^{,\beta}-\sigma^{\beta\gamma}_{;\gamma}+\omega^{\beta\gamma}_{;\gamma})=8\pi f(u^\alpha-C^\alpha),\\ \dot{\varrho}+\varrho\theta=0, \qquad \dot{u}_\alpha=0, \qquad C^\alpha_{;\alpha}=0;\end{gathered} \tag{4.20}$$

for type B(ii):

$$\begin{gathered}\dot{\theta}+\tfrac{1}{3}\theta^2+\lambda\theta+2(\sigma^2-\omega^2)=8\pi f-4\pi\varrho(1-\lambda^2/4\pi f),\\ h^\alpha_\beta(\omega^{\beta\gamma}_{;\gamma}-\sigma^{\beta\gamma}_{;\gamma}+\tfrac{2}{3}\theta^{,\beta})+(\omega^\alpha_\beta+\sigma^\alpha_\beta)\dot{u}^\beta=8\pi f(u^\alpha-C^\alpha),\\ \dot{u}_\alpha=(C_\alpha-u_\alpha)\lambda, \qquad \dot{\varrho}+\varrho\theta=\lambda\varrho, \qquad fC^\alpha_{;\alpha}=\lambda\varrho.\end{gathered} \tag{4.21}$$

The Eq. (4.17a) is referred to as Raychaudhuri's equation. We note that on identifying the cosmological constant Λ with the coupling constant $8\pi f$ then the Raychaudhuri equation includes the geodesic C-field motions A(i), A(ii) and B(i) as immediate consequences. That is, the C-field does not affect the components of $R_{\alpha\beta}u^\beta$ along the geodesic dust world lines. However, the presence of the C-field does affect a) the components of $R_{\alpha\beta}u^\beta$ in the hyperplane orthogonal to u_α in the B(i) motion and b) both components of $R_{\alpha\beta}u^\beta$ along and orthogonal to the congruence in the B(ii) motion.

A referee has drawn my attention to the fact that the Lagrangian for a scalar field theory of the type considered in this paper (where the "coupling constant" is a constant) can be transformed, by a conformal transformation of the metric tensor, into the Lagrangian for a scalar field theory of the Brans-Dicke type (where the "coupling constant" is a function of the scalar field). A question then arises as to whether the results presented here can be transformed and applied to the Brans-Dicke theory. Unfortunately the answer is negative since not all of the geometric properties dealt with are conformally invariant. For example, the condition $\dot{\bar{u}}_\alpha = 0$ for the time-like congruence of world lines to be geodesic in the one space-time does not transform under the conformal transformation of the metric tensor into the same condition $\dot{u}_\alpha = 0$ in the conformal space-time. In this case it appears to be necessary to repeat this work starting with the Brans-Dicke equations in order to get the equivalent results for that theory.

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