Commun. math. Phys. 19, 31–64 (1970)

A Class of Homogeneous Cosmological Models

II. Observations

M. A. H. MacCallum and G. F. R. Ellis
Department of Applied Mathematics and Theoretical Physics, Cambridge, England

Received March 30, 1970

Abstract. This paper discusses the application of geometric optics to the study of observational properties of cosmological models examined in a previous paper. A number of results concerning these properties are derived, the most interesting of which is the invariance of observational relations under certain discrete isotropy groups. Closed form expressions are obtained in certain cases.

1. Introduction

This paper discusses the observational properties of a class of homogeneous cosmological models studied in previous papers [1–3]. These are spacetimes which satisfy Einstein's field equations for a perfect fluid and which admit a three-parameter group of motions simply-transitive on spacelike sections (surfaces of homogeneity) orthogonal to the fluid flow vector[1], u^i. They are therefore universes homogeneous in the rest-space of any fundamental observer.

In this paper we will quote freely from the results of the earlier work. The matter in these spaces has no rotation or acceleration. One can choose coordinates $\{t, x_\nu\}$ such that $\{x_\nu\}$ are comoving coordinates, $\{t = \text{constant}\}$ are the surfaces of homogeneity, and t is the proper time along the world-lines of the matter (Latin indices run from 0 to 3, Greek from 1 to 3; $a, b, c \dots \alpha, \beta \dots$ will be used for components referred to an orthonormal tetrad $\{e_a\}$ with $e_0 = u$; $i, j, k \dots$ will be used for coordinate components). $\{e_\kappa\}$ span the tangent plane to the surface of homogeneity at each point. The signature is $+2$ and u^a is normalised ($u^a u_a = -1$). The first derivatives of u_a are determined by the expansion tensor θ_{ab},

$$u_{a;b} = \theta_{ab}; \qquad \theta_{ab} = \theta_{(ab)}; \qquad \theta_{ab} u^b = 0. \tag{1.1}$$

[1] Spacetimes admitting a multiply-transitive group acting on such three-dimensional spacelike surfaces belong to the class of L.R.S. (locally rotationally symmetric) spaces [4,49]. The only such spacetimes not admitting a simply-transitive subgroup acting on these surfaces are those of Case I of Kantowski and Sachs [5].

We write $\theta_{ab}=\sigma_{ab}+1/3\,\theta\,h_{ab}$ where $h_{ab}=g_{ab}+u_a u_b$. σ_{ab} is the shear tensor and θ the expansion.

The commutator of two vectors $X=X^i\partial/\partial x^i$ and $Y=Y^j\partial/\partial x^j$ is defined by

$$[X, Y]f := X(Yf) - Y(Xf) \quad \text{for all functions } f. \tag{1.2}$$

One finds, on writing $[e_a, e_b] =: \gamma^c{}_{ab} e_c$, that in these models

$$\left.\begin{aligned} \gamma^0{}_{0\alpha} &= \gamma^0{}_{\beta\alpha} = 0, \\ \gamma^\beta{}_{0\alpha} &= -\theta_{\beta\alpha} + \varepsilon_{\beta\alpha\delta}\Omega^\delta, \\ \gamma^\alpha{}_{\beta\gamma} &= \varepsilon_{\beta\gamma\delta} n^{\alpha\delta} + \delta^\alpha{}_\gamma a_\beta - \delta^\alpha{}_\beta a_\gamma, \end{aligned}\right\} \tag{1.3}$$

where

$$\left.\begin{aligned} \Omega^a &:= \tfrac{1}{2}\eta^{abcd} u_b e_c \cdot \dot{e}_d, \\ n^{\alpha\delta} &:= \tfrac{1}{2}\gamma^{(\alpha}{}_{\gamma\sigma}\varepsilon^{\delta)\gamma\sigma}, \\ a_\beta &:= \tfrac{1}{2}\gamma^\alpha{}_{\beta\alpha}, \end{aligned}\right\}. \tag{1.4}$$

η^{abcd} is the skew pseudo-tensor with $\eta^{0123}=1$, and signifies covariant differentiation in the u^a direction. $\theta_{\alpha\beta}, n_{\alpha\beta}, a_\beta, \Omega_\beta$ depend only on t and behave as symmetric three-tensors and three-vectors respectively under proper orthogonal transformations of $\{e_\nu\}$ dependent only on t. In general, we choose $\{e_\nu\}$ such that $n_{\alpha\beta}=\operatorname{diag}(n_1, n_2, n_3)$ and $a^\beta=(a, 0, 0)$; then the Jacobi identities for $\{e_\nu\}$ are $n_1 a=0$. When $n^\alpha{}_\alpha=0$, one can choose an alternative basis such that $n_{23}=q$, and the remaining $n_{\alpha\beta}$ are zero.

Three linearly independent spacelike Killing vectors $\{\xi_\mu\}$, which generate the simply-transitive group of motions, can be chosen so that at any one given point $\xi_\kappa \triangleq -e_\kappa$, $C^\mu{}_{\kappa\nu} \triangleq \gamma^\mu{}_{\kappa\nu}$, where $C^\mu{}_{\kappa\nu}$ are defined by

$$[\xi_\mu, \xi_\nu] = C^\kappa{}_{\mu\nu}\xi_\kappa. \tag{1.5}$$

For this choice of basis $N_{\kappa\mu}$, A^μ can be defined from $C^\kappa{}_{\mu\nu}$ by equations similar to (1.4). We may then set N_κ, A to ± 1 or 0 by rescaling the Killing vectors, unless $AN_2N_3 \neq 0$ (see [1]). By the definition of the vectors $\{e_a\}$, any Killing vectors commute with them;

$$[e_a, \xi_\mu] = 0. \tag{1.6}$$

The possible group types have previously been classified by Bianchi [6] and Behr [7]. We follow the modification of Behr's classification described in [1]. If $a=0$ the space is Class A and if $a \neq 0$, Class B. If $n_{\alpha\beta}=0$ the space is in subclasses Aa or Ba, and otherwise Ab, Bb. Case Bb is subdivided according as a^β is a shear eigenvector (Bbi) or not (Bbii). Case Bbii can only occur in a group of Bianchi type VI in which

$$n_2 n_3 + 9a^2 = 0 = N_2 N_3 + 9A^2. \tag{1.7}$$

In this paper we investigate the behaviour of null geodesics in these spacetimes. Section 2 introduces the required formulae from geometric optics and Section 3 discusses their use and evaluation in cosmology. These sections apply to any spacetime, while Sections 4–6 apply specifically to the class defined above. Section 4 studies the relation of homogeneity to discrete isotropy, Section 5 is concerned with closed form expressions, and Section 6 deals with the observational relations down the principal axes of shear.

Section 2, which is included for completeness, consists mostly of known results necessary for an understanding of the later work. However, it incorporates some previously unpublished derivations and some novelty of presentation which we hope will prove valuable. An amplified account will appear elsewhere [8]. In this and the remaining Sections results for which no reference is given are, as far as the authors are aware, new.

2. Geometric Optics

We suppose that spacetime is (pseudo-)Riemannian and that the electromagnetic tensor F_{ab} for the light emitted by a source obeys Maxwell's equations for a charge and current free region

$$F_{[ab;c]}=0\,, \tag{2.1 a}$$

$$F^{ab}{}_{;b}=0\,. \tag{2.1 b}$$

From (2.1 a), using freedom of gauge, one can choose a vector potential Φ^a such that

$$F_{ab}=2\Phi_{[a;b]}\,;\qquad \Phi^a{}_{;a}=0\,. \tag{2.2}$$

We assume that there are approximate solutions of (2.1) of the form $\Phi^a=A^a f(\phi)$, where f is an arbitrary function of ϕ and varies on a length scale much shorter than that on which A^a varies (cf. Trautmann [9] and Dehnen [10]). Defining $k_a:=\phi_{,a}$ so that $k_{[a;b]}=0$, and $A^2:=A^aA_a$, we find by substituting in (2.1), (2.2) and equating coefficients of f, $f':=df/d\phi$ and $f'':=d^2f/d\phi^2$ that:

$$k^ak_a=0 \tag{2.3 a}$$

implying

$$k_{a;b}k^b=k_{b;a}k^b=0\,; \tag{2.3 b}$$

$$A_bk^b=0\,; \tag{2.4}$$

$$2A^{a;b}k_b+A^ak^b{}_{;b}=0 \tag{2.5 a}$$

implying

$$(A^2)_{;a}k^a+A^2k^b{}_{;b}=0\,; \tag{2.5 b}$$

and

$$F_{ab}=f'(k_bA_a-k_aA_b)+2fA_{[a;b]}\,. \tag{2.6}$$

We now assume it is reasonable to ignore the last term in (2.6) (this is the geometric optics approximation [2]). The energy-momentum tensor of the electromagnetic field is then

$$S_{ab} = A^2 (f')^2 k_a k_b \,. \tag{2.7}$$

Eq. (2.3) shows that light travels on null geodesics ("rays") $x^a(\lambda)$ on which ϕ is constant. If two observers A and B measure the rate of change of f at points on the same ray, their results are in the ratio

$$\frac{(k^a u_a)_A}{(k^b u_b)_B} = 1 + z \tag{2.8}$$

where z is the redshift [3] observed by B in light emitted by A, since $f(\phi)_{;a} u^a = f'(k_a u^a)$.

A displacement $k^a \delta\lambda$ at a point p along a null geodesic will be interpreted by an observer at p with velocity u^a as a time difference δt and a spatial distance δx where

$$\delta t = \delta x = (-k^a u_a)\,\delta\lambda \,. \tag{2.9}$$

The results of Jordan, Ehlers, and Sachs [13, 14] on null geodesic congruences show that the size and shape of a small cross-section of a given bundle of rays is independent of the observer's four-velocity and that its area dS is propagated according to

$$dS_{;a} k^a = dS(k^a{}_{;a}) \,. \tag{2.10}$$

From this, (2.3) and (2.5) one finds that

$$A^2 f'^2 dS \quad \text{is constant along a ray.} \tag{2.11}$$

If the observer A sees an object G with intrinsic area dS_G which subtends a solid angle $d\Omega_A$ at A, one can define an area distance r_A between A and G by

$$dS_G =: r_A{}^2 \, d\Omega_A \,,$$

while if an angle $d\Omega_G$ at G subtends an area dS_A at A one can define an area distance r_G between A and G by $r_G^2 \, d\Omega_G := dS_A$. The observer A sees a flux [4] $L_A = (S_{ab} u^a u^b)_A = A^2 f'^2 (k^a u_a)_A^2$ from the source at G, while an observer at unit distance from, and moving with, the source measures a flux $L_G = L/4\pi$ where L is the total output of the source. One can define

[2] For a discussion of the validity of the approximations see [11] (cf. [12]).

[3] There is no distinction here between "Doppler" and "gravitational" effects.

[4] The energy flux vector is $q_a = h_a{}^c S_{cb} u^b$. We can check from this that L_A is the rate of receipt of energy per unit area by a screen orthogonal to u^a and k^a at A.

a third distance between A and G, the luminosity distance D, by $D^2 := L_G/L_A$ [15]. From (2.11) and the definitions, $D^2 = r_G^2(1+z)^2$. These distances are related to r_A by

$$r_G^2 = r_A^2 \ (1+z)^2, \tag{2.12}$$

which is known as the reciprocity theorem. It was first proved by Etherington [16] and was recently rediscovered by Penrose [17] following a conjecture of Kristian and Sachs [18]. A simple proof suggested by Sachs is given in [8]; the essential step applies the known first integral of the second order geodesic deviation equation (the "Lagrange identity") to a pair of geodesic deviation vectors which are orthogonal at both A and G. (It is the different propagation of the magnitudes of these two vectors which gives rise to the distortion effect [18, 19].)

The area distance r_A depends on u_A^a but not on u_G^a. Since the fluxes L_G, L_A are related by

$$L_G = L_A {r_A}^2 (1+z)^4 = L_A {r_G}^2 (1+z)^2, \tag{2.13}$$

momentarily coincident observers of the same source see fluxes proportional to $(1+z)^{-2}$ (r_G being the same for both), while an observer of two equal momentarily coincident sources sees fluxes proportional to $(1+z)^{-4}$ (r_A being the same)[5]. Although r_A depends on the behaviour of a small bundle of rays it can be regarded as a function assigned only along the central ray of the bundle.

So far we have treated G as a point source. If the intensity of radiation is defined (in the terminology of Chandrasekhar and Ehlers, cf. [8]) by $I_A := L_A/d\Omega_A$ and $I_G := L_G/dS_G$ one finds[6]

$$I_G = I_A (1+z)^4. \tag{2.14}$$

Moreover we have so far considered monochromatic or bolometric fluxes, while in practice one observes over some frequency range $\Delta\omega_A = \Delta\omega_G/(1+z)$. Defining specific flux $F(\omega)$ and specific intensity $I(\omega)$ as the flux and intensity per unit frequency range at the frequency ω,

$$F_A(\omega_A)\,\Delta\omega_A = \frac{F_G(\omega_A(1+z))\,\Delta\omega_G}{{r_A}^2(1+z)^4} \Rightarrow F_A(\omega_A) = \frac{F_G(\omega_A(1+z))}{{r_A}^2(1+z)^3}, \tag{2.15}$$

[5] This leads to two definitions of corrected luminosity distance [18]; in fact, a number of "luminosity distances" appear in the literature. [18] and [20] use r_A, [21–23] use r_G and [15] uses D. Note that a beam may refocus so that r_G is the same at two points, although for fundamental observers the redshift factors would usually lead to different values of r_A at the two points.

[6] An alternative derivation of this result and those that follow has been given by Sachs [24] using a Boltzmann equation treatment for photons.

and similarly

$$I_A(\omega_A) = \frac{I_G(\omega_A(1+z))}{(1+z)^3}. \tag{2.16}$$

In particular for black body radiation at emitted temperature T_G and redshift z

$$I_G(\omega_G) = \frac{K\omega_G^3}{\exp(h\omega_G/kT_G)-1} \Rightarrow I_A(\omega_A) = \frac{K\omega_A^3}{\exp(h\omega_A(1+z)/kT_G)-1}$$

(K, k, h are constants), so that the observed radiation is black-body radiation [7] at a temperature

$$T_A = T_G/(1+z). \tag{2.17}$$

Finally if the spacetime contains matter with emissivity $j(\omega)$ per unit volume and absorption coefficient $K(\omega)$ (including stimulated emission), then [8]

$$\frac{dI(\omega)}{d\lambda} = \frac{3I(\omega)}{(1+z)}\frac{dz}{d\lambda} + (j(\omega) - K(\omega)\,I(\omega))\,(u^a k_a)$$

implying

$$\left[\frac{I(\omega_A(1+z))}{(1+z)^3}\exp - \tau(\lambda)\right]_0^{\lambda'} = -\int_0^{\lambda'} \frac{j(\omega_A(1+z))}{(1+z)^3}(\exp - \tau(\lambda))\,(-u_a k^a)\,d\lambda \tag{2.18}$$

where $\tau(\lambda)$, the optical depth, is defined by

$$\tau(\lambda) = \int_0^{\lambda} K(\omega_A(1+z))\,(-k^a u_a)\,d\lambda.$$

If the congruence ends at a source one can set $\lambda' = \lambda_G$ in (2.18); then the observed specific intensity contains a term due to the source and a term due to the integrated effect of other matter along the line of sight. Eq. (2.18) can be used to investigate the effects of specified absorption or emission processes, and to evaluate the intensity of light from a discrete source or a background flux. The right-hand side tells us that Olbers' paradox is resolved if $j(\omega)$ undergoes a suitable cutoff or if the redshift factors sufficiently attenuate the emission.

No particular cosmological models or gravitational field equations are involved in the above equations, nor any relation between r_A and z. A and G need not move as fundamental observers but (2.18) assumes a

---

[7] In an anisotropic universe replacing the assumed instantaneous decoupling of the black body radiation by more realistic scattering processes yields a distorted spectrum and (2.17) then requires modification [25].

[8] λ is any parameter along the geodesic curves. It could in particular be an affine parameter v; if it is not, k^a in (2.8–3.1) should, strictly, be replaced by $\tilde{k}^a$ (see (3.2) below).

unique velocity for the matter at each point. Therefore Eqs. (2.14–18) allow one to compare intrinsic properties of the sources without reference to a particular cosmological model, provided one can evaluate or ignore the effect of intervening matter (cf. [22, 26]).

3. Observations and Cosmological Models

We do not a priori know the intrinsic surface brightness I_G, cross-sectional area dS_G or luminosity L for a source or the emissivity $j(\omega)$ or absorption $K(\omega)$ of matter intervening between the source and observer. Thus we must proceed by evaluating the formulae of Section 2 for particular assumed matter evolution in a particular cosmological model and then comparing the results with observations. Some relations, as just remarked, can be used without specialising the cosmological model, but for others one needs the relationship of the three fundamental quantities r_A (the area distance), λ (the geodesic parameter) and z (the redshift). This relationship is usually calculated assuming that sources and observers move as fundamental observers; peculiar random motions, gravitational redshifts, and focussing by massive bodies being treated only as second approximations. When the function $z(\lambda)$ is known Eq. (2.18) can be used to determine the spectrum of background radiation, and (2.17) the temperature of primeval black-body radiation.

Eq. (2.15) is the basis of the comparison with individual sources. The specific flux at the galaxy F_G is deduced from the properties of nearby sources similar to those under consideration. The observed specific flux F_A is usually measured only out to a certain isophote (i.e. contour of observed specific intensity I_A). It is then corrected a) for the effect of the change with z of the relation of this contour to the contour of a fixed I_G (the so-called aperture correction) and b) to turn (2.15) into (2.13) by reference to a standard spectrum for the class of sources considered (the K-correction [27]). Correction a) requires knowledge of the brightness distribution in the object; the relation between the required correction and the angular diameter of the specified contour of I_A is cosmology dependent [28].

In principle one can measure r_A independently of F_A simply by measuring the solid angle $d\Omega_A$ subtended by the source, provided one knows dS_G. This is probably impractical due to the night-sky background which makes it difficult to decide where an extended source ends [28]. Thus in practice the measurements are expressed in terms of a corrected source magnitude m, which represents the total flux F_A received from the source. Once $r_A(\lambda)$ is known, one can combine it with $z(\lambda)$ to obtain the $m-z$ relation.

The most important other direct test is the number-flux density relation for radio sources (in our terminology the number-specific flux relation). Consider a small parameter displacement $\delta\lambda$ on a null geodesic, and a small bundle of rays about this geodesic with cross-section dS. The volume element thus specified contains

$$n\, dS\, \delta x = n\, {r_A}^2 d\Omega_A (-k^a u_a)_G \delta\lambda$$

sources, where n is the number density of sources per unit proper volume, and we have used (2.9). So if N is the number of sources per unit solid angle at parameter distances less than λ down a certain ray bundle

$$\frac{dN}{d\lambda} = n\, r_A^2 (1+z)\,(-k^a u_a)_A\,. \tag{3.1}$$

Thus if in a cosmological model one knows $z(\lambda)$ and $r_A(\lambda)$ for a particular ray one can find the relation of N and F_A along that ray for any class of sources, with F_G as a parameter (or find dN/dF_A which may be more useful [29], cf. [30]).

The remaining problem in evaluating the theoretical predictions is to relate r_A, λ and z along any ray.

To relate λ and z for a given observer A and galaxy G one has to solve the geodesic equation (2.3b) for the null geodesic $x^j(v;\mu^1,\mu^2)$ joining a point $y^i = x^i(0;\mu^1,\mu^2)$ on the observer's world line to the galaxy's world line[9]. Here v is an affine parameter along the geodesic with tangent vector k^i, so $k^j(v;\mu^1,\mu^2) = \dfrac{\partial x^j(v;\mu^1,\mu^2)}{\partial v}$ and μ^1,μ^2 are constants specifying the initial direction at the point y^i. Substituting in (2.8) gives the observed redshift of the source at affine parameter distance v, i.e. determines the function $z(v)$ for the ray in direction (μ^1,μ^2) at the observer.

We shall wish to have the freedom to use non-affine parameters along the geodesic, i.e. to choose some other parameter $\lambda = \lambda(v)$. Re-expressing the equation in terms of the parameter λ, the geodesic $x^i(\lambda;\mu^1,\mu^2)$ has a tangent vector with coordinate components

$$\begin{aligned} \tilde{k}^i(\lambda;\mu^1,\mu^2) &= \frac{\partial x^i(\lambda;\mu^1,\mu^2)}{\partial\lambda} \\ &= \frac{dv}{d\lambda} k^i \end{aligned} \tag{3.2}$$

and the redshift will be known as a function $z(\lambda)$.

[9] We assume this geodesic is unique.

To determine r_A, we note that (3.2) implies

$$\frac{\partial \tilde{k}^i}{\partial \mu^M} = \frac{\partial^2 x^i}{\partial \mu^M \partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{\partial x^i}{\partial \mu^M}\right) \quad (M=1,2). \tag{3.3a}$$

This equation can be integrated along the ray to obtain the quantities

$$p^i_M := \left.\frac{\partial x^i}{\partial \mu^M}\right|_G = \int_A^G \frac{\partial \tilde{k}^i}{\partial \mu^M}\, d\lambda \tag{3.3b}$$

where the initial condition is taken to be $p^i_M|_A = 0$. We have in fact solved the first order geodesic deviation equation for null geodesics diverging from A; if one makes a small variation $\delta\mu^M$ of angular parameters at A, the resulting geodesic deviation vector at G is $p^i_M \delta\mu^M$, since the geodesic deviation equation is linear.

For an observer with four-velocity u^a at A

$$d\Omega_A = \lim_{\delta\lambda\to 0} dS/(u^a \tilde{k}_a)^2_A (\delta\lambda)^2$$

where dS is the cross-section of the ray bundle at $\delta\lambda$ from A. Using (2.9) and the definitions one finds

$$r_A{}^2 = \frac{|p_1{}^{[i} p_2{}^{j]}|_G}{\left|\frac{\partial p_1{}^{[i}}{\partial t} \frac{\partial p_2{}^{j]}}{\partial t}\right|_A},$$

where

$$|f^{[i} g^{j]}|^2 = f^{[i} g^{j]} f_{[i} g_{j]} = \begin{vmatrix} f^i f_i & f^j g_j \\ f^k g_k & g^m g_m \end{vmatrix}. \tag{3.4}$$

Clearly the arbitrariness in choice of (λ, μ^M) does not affect (3.4).

The method of finding r_A outlined here appears to be of wide applicability as well as being conceptually simple. One might be able to proceed in various other ways, e.g. one might be able to find $\{x^i(\lambda, \mu^1, \mu^2)\}$ explicitly and then differentiate to get (3.3b)[10], or one might integrate the second order geodesic deviation equation directly (cf. [18]) or indirectly [21, 31]. The method we use in this paper has the advantage that it could be easily adapted to numerical calculation (cf. [32]).

From the solution (3.3b), one can also find the distortion of optical images due to the curvature of space-time. To do so, choose two varia-

[10] Various authors [32–36] have introduced (v, μ^1, μ^2) and a parameter τ defined along the world line of the observer as coordinates. As only v varies along the geodesics, these coordinates, with $x^B = v$, satisfy

$$k^a{}_{,a} = k^B{}_{,a} k^a / k^B \quad \text{(no sum over } B)$$

so that (2.10) can be integrated to obtain $dS = C\sqrt{-g}\, k^B$ where the constant C can be found in terms of $d\Omega_A$. τ may then be eliminated and the result re-expressed in terms of any other coordinate system. The practical difficulty lies in evaluating the coordinate transformations, which is in fact just equivalent to calculating (3.3b). (See e.g. [37].)

tions $\delta\mu_1^M$, $\delta\mu_2^M$ such that they represent orthogonal displacements of equal magnitude on the unit sphere representing the sky at A (e.g. if μ^1, μ^2 are polar coords Θ, Ψ one can choose variations $\delta\mu_1^M = \Delta\delta_\Theta^M$, $\delta\mu_2^M = \sin\Theta\,\Delta\delta_\Psi^M$) and denote the corresponding deviation vectors at G by p, q (so $p^i = p^i_M \delta\mu_1^M$, $q^i = p^i_M \delta\mu_2^M$). The magnitude of the distortion may be represented by the quantity d where

$$d^2 = \frac{p^2 + q^2 - ((p^2 - q^2)^2 + 4(p \cdot q)^2)^{\frac{1}{2}}}{p^2 + q^2 + ((p^2 - q^2)^2 + 4(p \cdot q)^2)^{\frac{1}{2}}}$$

and $p^2 = p^a p_a$, $q^2 = q^a q_a$, $p \cdot q = p^a q_a$.

This quantity has the following significance: a galaxy which appears to A to be spherical (i.e. of Hubble type E_0) would appear to an observer near the galaxy in the same direction as A to be an elliptical galaxy of type E_n, where $n = 10(1-d)$. This effect offers in principle a further test of cosmological models [18, 19].

An alternative to exact evaluation of the observational formulae is to obtain a power series solution. This method was used by Kristian and Sachs [18], who found power series in v and eliminated to get power series in r_A. dS_G was found by use of Taylor's theorem on the second order geodesic deviation equation. The main results of their paper, using our conventions, are

$$\begin{aligned} 1+z = 1 &+ (u_{a;b}K^aK^b)_A r_A + \tfrac{1}{2}(u_{a;bc}K^aK^bK^c)_A {r_A}^2 \\ &+ \frac{{r_A}^3}{6}\{(u_{a;bcd}K^aK^bK^cK^d)_A + \tfrac{1}{2}(R_{cd}K^cK^d)_A(u_{a;b}K^aK^b)_A\} \dots, \end{aligned} \tag{3.5}$$

$$dN = (1+z)\,{r_A}^2 dr_A (n_A + (n_{,a}K^a)_A r_A + \tfrac{1}{2}{r_A}^2((n_{;ab} + \tfrac{1}{2}nR_{ab})K^aK^b)_A \dots) \tag{3.6}$$

where K^a is defined by $K^a := k^a/(u_b k^b)_A$, and is a past-pointing null vector. To obtain power series in directly measurable quantities from these results, we invert Eq. (3.5), thus finding the series

$$\begin{aligned} {r_A}^{-2} = z^{-2}(u_{a;b}K^aK^b)_A^2 &\left\{1 + \frac{(u_{a;bc}K^aK^bK^c)_A z}{(u_{a;b}K^aK^b)_A^2}\right. \\ &- \left.\left[\frac{(u_{a;bc}K^aK^bK^c)^2}{4(u_{a;b}K^aK^b)^4} - \frac{(u_{a;bcd}K^aK^bK^cK^d)}{3(u_{a;b}K^aK^b)^3} - \frac{(R_{ab}K^aK^b)}{6(u_{a;b}K^aK^b)^3}\right]_A z^2 \dots\right\} \end{aligned} \tag{3.7}$$

(which in the Robertson-Walker case reduces to that given by Bertotti [21]), and substitute in (2.13), with $m_{bol} = -2.5 \log_{10} L_A$ and $M = -2.5 \log_{10} L_G$. Thus

$$\begin{aligned} m_{bol} = M - 5\log_{10}(u_{a;b}K^aK^b)_A + 5\log_{10} z + \frac{5}{2}(\log_{10} e)&\left\{z\left(4 - \frac{(u_{a;bc}K^aK^bK^c)}{(u_{a;b}K^aK^b)^2}\right)\right. \\ + z^2\left(\frac{3(u_{a;bc}K^aK^bK^c)^2}{4(u_{a;b}K^aK^b)^4} - \frac{(u_{a;bcd}K^aK^bK^cK^d)}{3(u_{a;b}K^aK^b)^3} - \frac{R_{ab}K^aK^b}{6(u_{a;b}K^aK^b)^3} - 2\right)&\left.\dots\right\}_A . \end{aligned} \tag{3.8}$$

In any particular model, one may substitute for some of these terms from the field equations. In the Robertson-Walker case with vanishing pressure one obtains

$$\begin{aligned} m_{bol} &= M - 5\log_{10} H_0 + 5\log_{10} z \\ &+ (2.5\log_{10} e)\left((1-q_0)\,z + \frac{z^2}{4}(3q_0+1)(q_0-1) - \frac{2\Lambda z^2}{3H_0{}^2}\cdots\right) \end{aligned} \tag{3.9}$$

(in which H_0, q_0 and Λ have their usual meanings) as given by Solheim [38], who corrected Mattig's result [39]. In the L.R.S. spaces of Bianchi type I and of Kantowski and Sachs [5], which include Bianchi type III ($n^\alpha{}_\alpha = 0$), (3.8) reduces to the form given by Tomita [40].

Similarly one can find the number-flux relation. Assuming the emitted spectrum is $F_G \propto \omega_G{}^{-x}$ (x constant) the relation of N to F_A is

$$\begin{aligned} &\frac{d(\log N)}{d(\log F_A(\omega_A))} \\ &= -\frac{3}{2} - \frac{3}{8}\varepsilon\left[\frac{n_{,a}K^a}{n} - (5+2x)(u_{a;b}K^aK^b)\right]_A - 3\varepsilon^2\left[(u_{a;b}K^aK^b)^2\left(\frac{7+2x}{8} - \frac{3}{32}\right)\right. \\ &\quad - \frac{(13+5x)}{20}(u_{a;bc}K^aK^bK^c) - \frac{(29+10x)(n_{,a}K^a)(u_{a;b}K^aK^b)}{80n} \\ &\quad \left.- \frac{3}{32}\frac{(n_{,a}K^a)^2}{n^2} + \frac{n_{;ab}K^aK^b}{10n} + \frac{R_{ab}K^aK^b}{20}\right]_A \cdots \end{aligned} \tag{3.10}$$

where $\varepsilon = \sqrt{F_G(\omega_A)/F_A(\omega_A)}$. For $x = 1$, (3.10) becomes the $N - L_A$ relation if one replaces F_A by L_A and F_G by L_G. If one assumes there is no evolution of the comoving coordinate volume density of sources, one can recover from (3.10), on using the field equations, the results a) for Robertson-Walker spaces of Mattig [41], Bondi [42], and McVittie ([15], Eq. 9.306) and b) for L.R.S. Bianchi I and Kantowski-Sachs spaces of Tomita [40]. In the Robertson-Walker case one gets

$$N = \frac{n_A(DH_0)^3}{3}\left(1 - 3DH_0 + 3(DH_0)^2\left[\frac{5-q_0}{2} + \frac{K}{10H_0^2R_0^2}\right]\cdots\right) \tag{3.11}$$

where the symbols have their usual meanings.

We note that if there is no evolution either in luminosity or comoving coordinate density the slope of the source counts for bright (nearby) sources will be $-3/2$ as is well-known. It is clear from (3.10) that the deviation from this rule would initially be towards a flatter slope, unless $x < -2.5$ or we observe in particular directions in a highly anisotropic universe. Thus one is justified in regarding the observed numbers of sources [43] as evidence for intrinsic evolution of luminosity or density even if the universe is not exactly homogeneous and isotropic.

The power series method has two drawbacks. First the region of validity of the power series may not be sufficiently large for practical use – it certainly does not extend to the last scattering of the microwave background radiation. Secondly, in practical applications one usually compares only the first few terms, i.e. a truncated series, with observations. However, Solheim has shown, by comparing (3.9) truncated at the second order with the exact relations for Robertson-Walker models, that such a method will give rather inaccurate results [38]. (For $z<0.5$, the two formulae differ by more than $0^{\mathrm{m}}.1$ unless q_0 is small.)

We need further calculation to obtain the apparent proper motion of sources. The first-order effect ([8, 18]) is determined simply by θ_{ab} and ω_{ab}; Kristian and Sachs [18] give power series expressions for this effect and the distortion effect.

Further possible observational tests include, for instance, the use of morphological effects [44] and any type of change of observations with time (cf. [45]).

4. Homogeneity and Isotropy

Using tetrad components k^a, $\boldsymbol{k}$ is given by $\boldsymbol{k}=k^a\boldsymbol{e}_a$. One can find $k^0|_G$, and so $(1+z)$, from the components $k^\beta|_G$, since (2.3) and (2.8) show

$$1+z=k^0=((k^1)^2+(k^2)^2+(k^3)^2)^{\frac{1}{2}} \tag{4.1}$$

where (to simplify the formulae) we set $k^0=1$ at A. The geodesic Eq. (2.3 b) is

$$\frac{dk_a}{dv}=-\Gamma_{abc}k^b k^c=\gamma_{bca}k^b k^c\,. \tag{4.2}$$

Because of (4.1), one need only solve for the components k^β of $\boldsymbol{k}$; substituting from (1.3) the equations for these components are

$$\frac{dk_\alpha}{dv}=-\theta_{\alpha\beta}k^\beta k^0+\varepsilon_{\alpha\beta\gamma}(\Omega^\beta+n^{\beta\delta}k_\delta)\,k^\gamma+k_\alpha(k^\beta a_\beta)-a_\alpha(k^\beta k_\beta)\,. \tag{4.3}$$

We wish now to consider discrete symmetries defined with respect to the canonically-defined tetrad. Let the subspace of the tangent space T_p at a point p which is tangent to the surface $\{t=\text{constant}\}$ through p be denoted by H_p. We use the following notation for operators in H_p: $\mathscr{I}$ denotes the identity, $\mathscr{S}_\alpha$ denotes reflection in the α-axis, $\mathscr{R}_\alpha$ denotes reflection in the plane perpendicular to the α-axis, and $\mathscr{T}$ denotes total reflection. We can, with the obvious multiplication, generate finite groups from these operators. The groups G, H, K, L under which Eq. (4.3) is invariant in Classes Aa, Ab, Ba, Bbi respectively are shown in Table 1.

Table 1. *The discrete isotropies occuring in the spaces of types Aa, Ab, Ba, Bbi, Bbii*

	$A(a=0)$	$B(a \neq 0)$
$a(n_{\alpha\beta}=0)$	$G=\{\mathscr{I}, \mathscr{R}_\alpha, \mathscr{S}_\alpha, \mathscr{T}\}$	$K=\{\mathscr{I}, \mathscr{R}_2, \mathscr{R}_3, \mathscr{S}_1\}$
$b(n_{\alpha\beta} \neq 0)$	$H=\{\mathscr{I}, \mathscr{S}_\alpha\}$	i) $L=\{\mathscr{I}, \mathscr{S}_1\}$ ii) none

In case Bbii, there is no non-trivial subgroup of G under which (4.3) is invariant.

These groups are not necessarily the maximal isotropy groups of (4.3), since if the space-time is L.R.S. the continuous isotropy group will leave (4.3) invariant. An examination of the cases which can occur shows that the group G will then be a discrete isotropy group of (4.3). When no continuous isotropy group exists (i.e. when the space is not L.R.S.), the finite groups mentioned above are the maximal isotropy groups.

In fact, these groups are not merely invariance groups of (4.3) but are generated by isometries of the spacetime and are automorphisms of the Lie algebra of the reciprocal group, leaving invariant the structure constants with respect to the basis $\{e_\alpha\}$. They correspond in a natural way to isomorphisms of the underlying group of motions; their existence has been discussed from this point of view by Schmidt [2].

The full group of isometries of a particular three-surface of homogeneity is in general larger than that generated by the three-parameter simply-transitive isometry group and the appropriate discrete isotropy group. (For example, in Bianchi type I the three-spaces are three-spaces of constant curvature and so are invariant under a six-parameter group of motions.) To correspond to isometries of the whole spacetime, the isometries of the three-surface must leave the second fundamental form of the surface, i.e. the expansion tensor θ_{ab}, invariant, so that the initial data on a Cauchy surface is invariant [2]. Thus although the group in case Bbii has the same invariance properties as the same group (VI_h with $h=-1/9$) has in case Bbi, the isotropy groups for the spacetimes are not the same, for in Bbii the shear tensor isotropies no longer coincide with the isotropies of the three-space sections.

Schmidt [2] has proved a partial converse of the above results: he has shown that the invariance under H of u^a, R_{abcd} and its first three derivatives implies that the spacetime belongs to Class A.

We now return to the context in which the isotropies were initially noted, the invariance of (4.3). Since the invariance applies to every geodesic it applies to bundles of geodesics and therefore to all types of cosmological

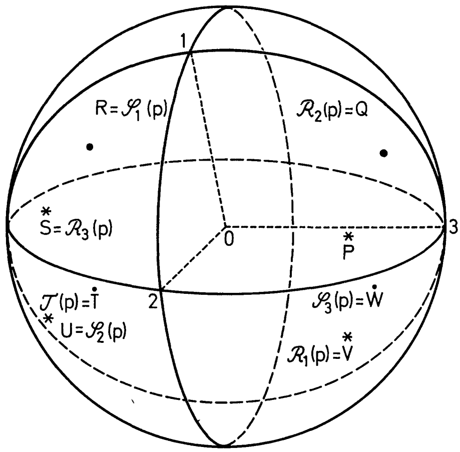

Fig. 1. The celestial sphere of observer O. Points marked * are on the outside of the sphere facing the reader. Points marked $\dot{}$ are seen through the sphere. The points 1, 2, 3 are the directions of the canonically defined tetrad axes. OP is a typical direction of observation (see Section 4)

observation. Thus *any fundamental observer will necessarily see these isotropies in all his observations on his celestial sphere.* In Fig. 1, if OP is a typical direction of observation, equivalent directions for the observer at O will be given by the following points: Class Aa, QRSTUVW; Class Ab, RUW; Class Ba, QRS; Class Bbi, R; Class Bbii, none.

We emphasize that these isotropies are in principle directly observable, requiring no interpretation regarding the physical nature of the sources other than that they are not local (i.e. that they have cosmological significance). Further, invariance under K, H or G, when it is the maximal isotropy group of observations by a fundamental observer, determines uniquely the directions of the covariantly-defined triad $\{e_\kappa\}$. When the invariance group is L, only the e_1-axis is thus determined.

If one examines the power series expressions of Section 3 one finds that n, ${}_\alpha K^\alpha = 0$ and

$$u_{a;b}K^aK^b = \theta_{\alpha\beta}K^\alpha K^\beta \tag{4.4}$$

are always invariant under G, while

$$\begin{aligned} &u_{a;bc}K^aK^bK^c \\ &= \left(-\frac{\theta^{\cdot}}{3}+\frac{2\theta^2}{9}\right)+(\sigma_{\alpha\beta}K^\alpha K^\beta)\left(2a_\delta K^\delta+\frac{4\theta}{3}\right)-2\sigma_{\delta\beta}K^\delta a^{\beta\bullet}+2\sigma^\nu{}_\gamma\sigma_{\nu\beta}K^\gamma K^\beta \\ &\quad -2\varepsilon_{\delta\alpha\tau}n^{\tau\gamma}\sigma^\delta{}_\beta K^\alpha K^\beta K_\gamma-(\partial_0\sigma_{\alpha\beta}-2\varepsilon_{(\alpha|\kappa\nu}\Omega^\kappa\sigma_{|\beta)}{}^\nu)K^\alpha K^\beta \end{aligned} \tag{4.5}$$

is invariant under exactly the groups specified in this section. Thus one sees from (3.5–10) that the invariance may be regarded as a second-order effect. Since it is difficult to assign an average value to the second-order coefficient from the redshift-magnitude relation using the (good) approximation that spacetime is locally like a Robertson-Walker universe (see e.g. [23, 27]), it is doubtful whether we could test for these isotropies by such measurements. However the microwave background radiation, on which isotropy measurements can be made with high precision [46] offers more hope.

While the isotropies so far discussed apply to all observational relations, certain relations may have more special invariance properties. In particular, observations dependent only on the behaviour of one geodesic, like the $z-t$ relation or black-body temperature, could be the same in two directions when observations depending on a small bundle of geodesics, like the $r_A - z$ relation, are not. We have found one case of some interest. (4.3) shows that the $z-t$ relation is the same in the $+\boldsymbol{e}_1$ and $-\boldsymbol{e}_1$ directions in all Class B models, including case Bbii, i.e. it is the same in the a^β direction as in the opposite direction.

One might hope that the isotropy group invariance would in itself give complete information about the contours on the celestial sphere of the value of the redshift of light from a particular surface $t=t_1$ (which would be isotherms of black-body radiation). We have found this is not so.

5. Analytic Integration of the Geodesic Equations

In a homogeneous universe, the observations must be the same for all observers in any hypersurface $\{t=\text{constant}\}$. Moreover use of the method of Section 3 is simplified by the existence of explicit first integrals of the geodesic equations. To see this, form the quantities

$$\pi_\nu = \xi_{\nu a} k^a; \tag{5.1}$$

then $(\pi_\nu)_{;c} k^c = \xi_{\nu a;c} k^a k^c + \xi_{\nu a} k^a{}_{;c} k^c = 0$, as the Killing vector $\xi_\nu{}^a$ satisfies Killing's equations $\xi_{\nu(a;b)} = 0$ and k^a is a geodesic vector. Therefore the quantities π_ν are constant along any geodesic[11]; the magnitude $k^a k_a$ of the geodesic vector is another first integral, since $(k^a k_a)_{;b} k^b = 2k^a(k_{a;b}k^b) = 0$ [12]. We consider only null geodesics, so that $k^a k_a = 0$.

[11] In fact the number of distinct scalar constants of geodesic motion which are linear (quadratic, cubic, etc.) in the momentum is equal to the number of independent Killing vectors (respectively second-rank tensors, third-rank tensors, etc.); cf. [47]. In the case of null geodesics, one can replace "Killing" by "conformally Killing" in this statement.

[12] This is a consequence of the fact that g_{ab} is a Killing tensor.

In the spaces we consider in this paper, one can find three such constants π_ν since one can find three independent Killing vectors $\{\xi_\nu\}$ by solving Eq. (1.6). A coordinate system adapted to the orthonormal tetrad in the Class A and $n^\beta{}_\beta = 0$ cases has been given in [1]. In Class A we introduce the function $c(x^2)$ which satisfies $\partial c/\partial x^2 = -\sqrt{1-N_1 N_3 c^2(x^2)}$ and $\lim_{x^2\to 0} c(x^2) = 0$, and choose regular coordinates (i.e. choose $\lim_{x^1\to 0} S(x^1) = 0$ and $\lim_{x^3\to 0} g(x^3) = 0$). Then

$$\left.\begin{aligned} &\xi_1 = -\sqrt{1-N_1 N_3 c^2(x^2)}\,\frac{\partial}{\partial x^1} + N_3 c(x^2)\, W, \qquad \xi_2 = -\frac{\partial}{\partial x^2}, \\ &\xi_3 = -N_1 c(x^2)\,\frac{\partial}{\partial x^1} - \sqrt{1-N_1 N_3 c^2(x^2)}\, W, \\ &\text{where} \\ &W := (1-N_2 N_3 S^2(x^1))^{-\frac{1}{2}}\left(N_2 S(x^1)\,\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\right), \end{aligned}\right\} \tag{5.2}$$

are three independent Killing vectors. In the cases where $n^\alpha{}_\alpha = 0$,

$$\left.\begin{aligned} &\xi_1 = -\frac{\partial}{\partial x^1} - (a_0+q_0)\, x^2\,\frac{\partial}{\partial x^2} - (a_0-q_0)\, x^3\,\frac{\partial}{\partial x^3} \\ &\xi_2 = -\frac{\partial}{\partial x^2}, \qquad \xi_3 = -\frac{\partial}{\partial x^3} \end{aligned}\right\} \tag{5.3}$$

are three independent Killing vectors.

Using the Killing vectors (5.2), Eqs. (5.1) show that the tetrad components k_α of a general geodesic in Class A spaces are

$$\left.\begin{aligned} k_1 = &\frac{-1}{X}\Big(\pi_1\big(N_1 N_2 N_3 S c g + \sqrt{1-N_1 N_2 g^2}\,\sqrt{1-N_1 N_3 c^2}\big) \\ &-\pi_2\sqrt{1-N_2 N_3 S^2}\, N_2 g - \pi_3 N_3\big(N_2 S g\sqrt{1-N_1 N_3 c^2} - c\sqrt{1-N_1 N_2 g^2}\big)\Big) \\ k_2 = &\frac{-1}{Y}\Big(N_1 \pi_1\big(g\sqrt{1-N_1 N_3 c^2} - N_3 S c\sqrt{1-N_1 N_2 g^2}\big) \\ &+\pi_2\sqrt{1-N_1 N_2 g^2}\,\sqrt{1-N_2 N_3 S^2} \\ &+\pi_3 N_3\big(N_1 c g + S\sqrt{1-N_1 N_3 c^2}\,\sqrt{1-N_1 N_2 g^2}\big)\Big) \\ k_3 = &\frac{-(1-N_2 N_3 S^2)^{\frac{1}{2}}}{Z}\left(-N_1 \pi_1 c - \frac{N_2 \pi_2 S}{\sqrt{1-N_2 N_3 S^2}} + \sqrt{1-N_1 N_2 c^2}\,\pi_3\right). \end{aligned}\right\} \tag{5.4}$$

(k^0 may be found from (4.1)). (Note that in [1], Eq. (4.9), g_{11} should read $X^2(1-N_1 N_2 g^2(x^3)) + Y^2 N_2{}^2 g^2(x^3)$.)

Similarly the Killing vectors (5.3) lead to the tetrad components

$$\left.\begin{aligned} k^1 &= \frac{1}{X}(-\pi_1 + (a_0+q_0)x^2\pi_2 + (a_0-q_0)x^3\pi_3 - f(x^0)\pi_2\exp(a_0+q_0)x^1) \\ k^2 &= -\frac{\pi_2}{Y}\exp(a_0+q_0)\,x^1; \quad k^3 = -\frac{\pi_3}{Z}\exp(a_0-q_0)\,x^1 \end{aligned}\right\} \quad (5.5)$$

for a general geodesic in spaces with $n^\alpha{}_\alpha = 0$.

For null geodesics only the ratios of the π_ν (i.e. two parameters) need be given. Any two parameters giving these ratios can then be used as (μ^1, μ^2) in the method of Section 3. One reasonable choice is to take direction cosines of the observation direction at A and parametrise by the angular coordinates (Θ, Ψ) as in spherical polars, cf. [37].

Eqs. (5.4, 5.5) represent the tetrad components of the geodesic tangent vector k^a in the form $k^a(x^i; \mu^1, \mu^2)$. To apply the method of Section 3, one has to find the coordinate components $k^j(v; \mu^1, \mu^2)$. One way to do this is to find explicitly the integral curve $x^i(v)$ of the vector field k^a which passes through G and A, and then eliminate the coordinates x^ν from (5.4, 5.5). Alternatively one might try to obtain the components $k^a(v; \mu^1, \mu^2)$ directly from the geodesic Eq. (4.2) or (4.3) (or to obtain the components $\tilde{k}^a(t; \mu^1, \mu^2)$ in which the time coordinate t is used as the curve parameter. It is in fact this choice we shall make later on.)

We wish to perform the integrations analytically as far as possible. (No problem arises in simultaneously integrating the geodesic equation and the first-order deviation equation by numerical methods.) The integration is greatly simplified when there exist non-trivial functions $g(k_\alpha, t)$ which are constant along any geodesic, since each such independent function g can be used to eliminate one of the k_α from the system of differential equations we have to solve, by setting g constant (equal to its initial value) on any particular geodesic. We investigate when this occurs as follows: we choose the triad $\{e_\alpha\}$ as a triad of shear eigenvectors, and define the lengths l_ν by [1] $(l_\sigma)^{\cdot}/l_\sigma = \theta_\sigma$[13] (so in the coords. above, $l_1 = X, l_2 = Y, l_3 = Z$). Quantities r_α are defined by[13] $r_\sigma := l_\sigma k_\sigma$; these are simply rescaled tetrad components of $\boldsymbol{k}$. Now we seek functions $g(r_\alpha)$ which are solutions of the equation $g_{,a}k^a = 0$, i.e. which are constant along the geodesics. By (4.2) and (1.3) this condition is

$$\frac{\partial g}{\partial r_\alpha} l_\alpha \left(\varepsilon_{\alpha\nu\tau}\Omega^\tau \frac{r^\nu}{l_\nu} k^0 + \gamma_{\nu\alpha\mu} \frac{r^\mu r^\nu}{l_\mu l_\nu} \right) = 0\,, \tag{5.6}$$

[13] Throughout this section we use a modified summation convention for brevity. Summation, in the obvious way, is implied by a triply-repeated index, while there will be no sum over the index σ wherever it appears.

that is,

$$\frac{\partial g}{\partial r_\alpha} l_\alpha \{\varepsilon_{\alpha\nu\tau}\Omega^\tau k^\nu k^0 + \varepsilon_{\alpha\nu\tau} n^\tau{}_\mu\, k^\mu k^\nu + (k^\nu k_\nu) a_\alpha - k_\alpha (a_\mu k^\mu)\} = 0 \tag{5.7}$$

(the variables r_α are used so as to eliminate the terms $\theta_{\alpha\nu} k^\nu k^0$ from these equations).

In Class A we can put $\Omega^\beta = 0$ and Eq. (5.7) is

$$\begin{aligned} \frac{1}{l_3{}^2}\left(\frac{N_2}{r_1}\frac{\partial g}{\partial r_1} - \frac{N_1}{r_2}\frac{\partial g}{\partial r_2}\right) + \frac{1}{l_2{}^2}\left(\frac{N_1}{r_3}\frac{\partial g}{\partial r_3} - \frac{N_3}{r_1}\frac{\partial g}{\partial r_1}\right) \\ + \frac{1}{l_1{}^2}\left(\frac{N_3}{r_2}\frac{\partial g}{\partial r_2} - \frac{N_2}{r_3}\frac{\partial g}{\partial r_3}\right) = 0 \end{aligned} \tag{5.8}$$

(remember the l_β are functions of time and the N_α are constants). In Class Aa (Bianchi I) all three r_α are independent solutions of (5.8). In Class Ab, $g = N_1 (r_1)^2 + N_2 (r_2)^2 + N_3 (r_3)^2$ is a solution. These are the only solutions for general geodesics (there exist further solutions for special geodesics) and arbitrary functions $l_\alpha(t)$; one can however obtain further solutions if $l_1(t) = l_2(t) = l_3(t)$, (a Robertson-Walker universe), or if $l_2(t) = l_3(t)$ in an L.R.S. space with $n_2 = n_3$. In the latter case, r_1 and $(r_2)^2 + (r_3)^2$ are independent solutions.

In Class B, we have been unable to obtain solutions of (5.7) in general. However we can deal fully with those cases in Class B, excepting Bbii, in which $n^\alpha{}_\alpha = 0$. The equations (5.7) take the form (as we can put $\Omega^\beta = 0$)

$$\begin{aligned} \left(l_1{}^2 \frac{1}{r_1}\frac{\partial g}{\partial r_1}\right)\left((r_2)^2 \frac{a_0 + q_0}{(l_2)^2} + (r_3)^2 \left(\frac{a_0 - q_0}{(l_3)^2}\right)\right) \\ - r_2 \frac{\partial g}{\partial r_2}(a_0 + q_0) - r_3 \frac{\partial g}{\partial r_3}(a_0 - q_0) = 0 \end{aligned} \tag{5.9}$$

where (by Eq. (6.3b) of [1]) $l_1 = (l_2 l_3)^{\frac{1}{2}} \left(\frac{l_2}{l_3}\right)^{q_0/2a_0}$. A general solution in all cases is $g = \left(\frac{r_3}{r_2}\right)(r_2 r_3)^{q_0/2a_0}$. One only obtains further solutions for general geodesics if $l_1(t) = l_2(t) = l_3(t)$, (a Robertson-Walker universe), or if $q_0 = a_0$ (an L.R.S. solution of Bianchi type III). In the latter case, r_3 and $(r_1)^2 + (r_2)^2$ are independent solutions.

Whenever there are two or more independent functions g which are solutions of (5.7), one can eliminate two of the k^α in terms of these constants and then hope to obtain the observational relations as simple integrals. Thus the cases one may expect to solve simply are the Robertson-Walker spaces, the Bianchi I spaces, and the L.R.S. cases. We shall not discuss the Robertson-Walker spaces since the observational relations in these cases are well-known (see [8] for a review). The other cases may be solved either directly from the geodesic equations and functions g obtained above, or by using the first integrals π_ν and resulting forms (5.4, 5.5) for the geodesic vectors. We consider these cases in turn.

The L.R.S. spaces of Bianchi type III ($n^\alpha{}_\alpha = 0$) are the Kantowski-Sachs Case II spaces [5]. The observational relations in these spaces have already been obtained by Tomita [40]. In our notation these are the cases $a_0 = q_0$, $X = Y$. Using (5.5) the explicit form of (3.2) is

$$\left.\begin{aligned} \frac{dx^1}{dt} &= -\frac{(\pi_1 - 2a_0\pi_2 x^2)}{KX^2}, \\ \frac{dx^2}{dt} = -\frac{\pi_2 e^{4a_0x^1}}{KX^2}, &\quad \frac{dx^3}{dt} = -\frac{\pi_3}{KZ^2}, \end{aligned}\right\} \tag{5.10}$$

where

$$K^2(t) := (k^0)^2 = \left(\frac{X_A^2}{X^2}\left((\pi_1 - 2a_0\pi_2x^2)^2 + \pi_2{}^2 e^{4a_0x^1}\right) + \frac{Z_A^2}{Z^2}\pi_3^2\right)K_A^2. \tag{5.11}$$

We parametrise as suggested above, setting $K_A = 1$, $-\pi_3 = Z_A \cos\Theta$, $-\pi_2 = X_A \sin\Theta \sin\Psi$, $-\pi_1 = X_A \sin\Theta\cos\Psi$. One obtains

$$(1+z) = \left(\frac{X_A^2}{X^2}\sin^2\Theta + \frac{Z_A^2}{Z^2}\cos^2\Theta\right)^{\frac{1}{2}} \tag{5.12}$$

on the geodesics with $\pi_2 = x^2 = \Psi = 0$; as the space is L.R.S. one need only consider these geodesics. (The observer is taken to be at the origin.) Differentiating, one can explicitly evaluate (3.3b) for these geodesics: using t as the parameter λ

$$\left.\begin{aligned} \frac{\partial^2 x^1}{\partial\Theta\partial t} &= \frac{X_A Z_A^2\cos\Theta}{X^2Z^2(1+z)^3}; \quad \frac{\partial^2 x^3}{\partial\Theta\partial t} = \frac{-Z_A X_A^2\sin\Theta}{X^2Z^2(1+z)^3}; \\ \frac{\partial^2 x^2}{\partial\Theta\partial t} &= \frac{\partial^2 x^1}{\partial\Psi\partial t} = \frac{\partial^2 x^3}{\partial\Psi\partial t} = 0; \quad \frac{\partial^2 x^2}{\partial\Psi\partial t} = \frac{X_A\sin\Theta\, e^{4a_0x^1}}{X^2(1+z)}. \end{aligned}\right\} \tag{5.13}$$

Integrating (5.10) and (5.13) one finds

$$\left.\begin{aligned} x_G^1 &= \int_G^A \frac{X_A\sin\Theta\, dt}{X^2(1+z)}; \quad x_G^2 = 0; \quad x_G^3 = \int_G^A \frac{Z_A\cos\Theta\, dt}{Z^2(1+z)} \\ p_\Theta^1 &= \frac{\partial x^1}{\partial\Theta} = \int_G^A \frac{X_A Z_A^2\, dt\cos\Theta}{X^2Z^2(1+z)^3}; \quad p_\Theta^3 = \frac{\partial x^3}{\partial\Theta} = -\int_G^A \frac{Z_A X_A^2\sin\Theta\, dt}{X^2Z^2(1+z)^3} \\ p_\Theta^2 &= \frac{\partial x^2}{\partial\Theta} = p_\Psi^1 = \frac{\partial x^1}{\partial\Psi} = p_\Psi^3 = \frac{\partial x^3}{\partial\Psi} = 0; \quad p_\Psi^2 = \frac{\partial x^2}{\partial\Psi} = \frac{(e^{4a_0x_G^1} - 1)}{4a_0}. \end{aligned}\right\} \tag{5.14}$$

Now use of (3.4) with $g_{ij} = \mathrm{diag}(-1, X^2, X^2 e^{-4a_0 x^1}, Z^2)$ gives

$$r_A^2 = \frac{l_A^3 l_G^3 (1+z) \sinh 2a_0 u}{2a_0 X_A \sin\Theta} \left(\int_G^A \frac{dt}{X^2 Z^2 (1+z)^3} \right) \tag{5.15}$$

where $(1+z)$ is given by (5.12) and

$$u := \int_G^A \frac{X_A \sin\Theta \, dt}{X^2(1+z)}; \qquad l^3 := X^2 Z.$$

To evaluate (5.15) when $\Theta = 0$ we take the obvious limit as $\sin\Theta \to 0$. (We are bound to have some coordinate singularity in parametrising directions about A unless we use more than one coordinate patch.) One can find the distortion from (5.14) by the method of Section 3.

Similarly we can use (5.4) or (5.5) in Bianchi type I, where $N_1 = N_2 = N_3 = 0 = S(x^1) = c(x^2) = g(x^3)$, yielding coordinate components

$$k^\mu = \frac{\partial x^\mu}{\partial v} = \frac{\pi_\mu}{(l_\mu)^2}. \tag{5.16}$$

(This simply expresses the constancy of the three solutions of (5.7).) One may again take $k_A^0 = 1$ and parametrise by (Θ, Ψ) although in this case we will express the result in a form independent of the parametrisation. By a calculation similar to that above one finds

$$1 + z = \left(\sum_\mu \left(\frac{\pi_\mu}{l_\mu} \right)^2 \right)_G^{\frac{1}{2}}, \tag{5.17}$$

$$r_A^2 = (1+z)\, l_A^3 l_G^3 \Delta, \tag{5.18}$$

where

$$\Delta = (\pi_1^2 I_2 I_3 + \pi_2^2 I_1 I_3 + \pi_3^2 I_1 I_2),$$

and I_ν are defined by cyclic interchange from

$$I_1 := \int_G^A \frac{dt}{l_2{}^2 l_3{}^2 (1+z)^3},$$

$l^3 := l_1 l_2 l_3$ and $\sum_\mu \left(\frac{\pi_\mu}{l_\mu} \right)_A^2 = 1$. These relations have been obtained previously by Saunders [48].

In the case of L.R.S. Class A solutions, the functions $r_1, (r_2)^2 + (r_3)^2$ are constant along the geodesics and so (normalising $k^0|_A = 1$) the tetrad components k_α satisfy

$$k_1 = \frac{X_A \cos\Theta}{X(t)}, \qquad (k_2)^2 + (k_3)^2 = \frac{Y_A^2 \sin^2\Theta}{Y^2(t)}$$

where $l_1(t)=X(t)$, $l_2(t)=l_3(t)=Y(t)$ and Θ is a constant. Thus one finds

$$(1+z)=\left(\frac{X_A^2\cos^2\Theta}{X^2(t)}+\frac{Y_A^2\sin^2\Theta}{Y^2(t)}\right)^{\frac{1}{2}}. \tag{5.19}$$

One can solve the geodesic equation by setting $k_2=Y_A\sin\Theta\cos\Phi(t)/Y(t)$, $k_3=Y_A\sin\Theta\sin\Phi(t)/Y(t)$ with

$$\Phi(t)=\int_G^A\frac{X_A}{(1+z)}\left(\frac{N_1}{Y^2}-\frac{N_2}{X^2}\right)dt\cos\Theta+\Psi,$$

where Ψ is a constant, and so obtain the tetrad components $\tilde{k}^a(t;\Theta,\Psi) = \frac{1}{(1+z)}k^a(t;\Theta,\Psi)$. However, to integrate Eq. (3.3b) we need the *coordinate* components of $\tilde{k}^i$, which are easily evaluated using (4.4) of [1], to have the form $k^i(t,\Theta,\Psi)$, i.e. we do have to explicitly eliminate the functions $x^\nu(t)$ occurring in these coordinate components. We have been unable to do this in the L.R.S. cases of Bianchi types VIII and IX.

In the L.R.S. case of Bianchi type II, the coordinate components take the form

$$\begin{aligned}
\tilde{k}^1&=\frac{1}{1+z}\left(\frac{X_A\cos\Theta}{X^2}-\frac{N_1Y_Ax^3\sin\Theta\cos\Phi(t)}{Y^2}\right),\\
\tilde{k}^2&=\frac{1}{1+z}\left(\frac{Y_A\sin\Theta\cos\Phi(t)}{Y^2}\right),\\
\tilde{k}^3&=\frac{1}{1+z}\left(\frac{Y_A\sin\Theta\sin\Phi(t)}{Y^2}\right),
\end{aligned} \tag{5.20}$$

where $\Phi=\int_G^A\frac{X_AN_1\cos\Theta\,dt}{Y^2(1+z)}+\Psi$, Ψ is constant and

$$x^3=-\int_G^A\frac{\sin\Phi(t)\,dt}{Y^2(1+z)}Y_A\sin\Theta.$$

Now we can again obtain

$$p_\Theta^\nu=\int\frac{\partial\tilde{k}^\nu}{\partial\Theta}dt;\qquad p_\Psi^\nu=\int\frac{\partial\tilde{k}^\nu}{\partial\Psi}dt \tag{5.21}$$

and hence find r_A and d.

The forms (5.12, 15, 17–19) can be substituted in (2.13–18) to calculate the observational relations at any time t_A. We may note that (5.15) and (5.18) are clearly independent of the various rescalings (e.g. rescaling of l and μ^M).

It seems unlikely that one can obtain such simple expressions in the remaining cases in Class A and with $n^{\alpha}{}_{\alpha}=0$, when there is only one solution g of (5.7), unless there exists a better choice of tetrad in these spaces than that used above. The only cases in which we are aware that such a tetrad exists are the L.R.S. cases in which a tetrad may be chosen (cf. [4.49]) to fit the multiply-transitive group rather than a simply-transitive subgroup. It is probable that by use of such a tetrad, one can obtain simple expressions for the observational relations in the L.R.S. cases of types VIII and IX. It is relevant to note that Tomita [40] has obtained the observational relations in the case in which there is no simply-transitive subgroup G_3 (this space, the Kantowski-Sachs Case I, is very similar to the L.R.S. space of type III discussed above).

The existence of solutions of Eq. (5.7) is closely related to the existence of homogeneous constants of motion. Suppose that a vector field $\boldsymbol{k}$ is a homogeneous vector field [14], i.e. has tetrad components $k^{\alpha}=k^{\alpha}(t)$. The quantities π_{ν} defined by (5.1) will in general not be constant in a surface $\{t=\text{constant}\}$. However there may be some functions of the π_{ν} which are constant in these surfaces, such functions being called *homogeneous constants of motion.* They are therefore functions $f(\pi_{\nu})$ which are invariant when the geodesic is dragged along by the simply-transitive group of motions [15]. It follows [50] that they are solutions of the equation

$$\frac{\partial f}{\partial \pi_{\nu}} C^{\kappa}{}_{\nu\mu}\pi_{\kappa}=0\,. \tag{5.22}$$

On choosing a Killing vector basis $\xi_{\nu} \stackrel{*}{=} -\boldsymbol{e}_{\nu}$ one finds $C^{\nu}{}_{\mu\kappa} \stackrel{*}{=} \gamma^{\nu}{}_{\mu\kappa}$ (cf. Section 1), so each solution of (5.22) will, when $\Omega^{\alpha}=0$, imply a closely corresponding solution of (5.6). In fact, in Class A the solutions of (5.22) are π_1, π_2, π_3 for a group of type I and $N_1(\pi_1)^2+N_2(\pi_2)^2+N_3(\pi_3)^2$ otherwise; in Class B cases with $n^{\alpha}{}_{\alpha}=0$, these equations have the solution $\left(\frac{\pi_3}{\pi_2}\right)(\pi_2\pi_3)^{q_0/2a_0}$. We have been unable to find solutions in the remaining Class B cases. Thus these solutions correspond precisely to the solutions of Eq. (5.6) found above, except in the case of L.R.S. spaces. To deal with the L.R.S. spaces we would have to distinguish the constants invariant under the various simply-transitive subgroups and those invariant under the isotropy group of a point; for our purposes direct use of (5.6) is simpler.

[14] With respect to a given simply-transitive subgroup: this definition has invariant meaning except when the space is L.R.S. (when one could choose different simply-transitive subgroups; $\boldsymbol{k}$ would not be homogeneous with respect to all of them).

[15] And so are constants of the motion which are invariant under the automorphisms of the Killing Lie algebra induced by the action of the group on itself.

However a systematic use of homogeneous constants of motion is probably the best way of solving the Liouville equation of relativistic kinetic theory in these spaces (cf. [24, 50])[16].

6. Observations Down the Axes and Further Properties

At any point in spacetime, one can in principle determine the shear eigenvectors by observing the anisotropies in the first order Hubble law (i.e. in the term $u_{a;b}K^a K^b$ in (3.8)). If there is a continuous isotropy group (i.e. if the space is L.R.S.) one can find many orthonormal triads $\{e_\alpha\}$ of shear eigenvectors; in particular, one can choose triads of shear eigenvectors which commute with Killing vectors $\{\xi_\nu\}$ generating a simply-transitive subgroup G_3 of isometries[17]. If one does so, these spaces may be assumed to be special cases of those discussed in the rest of this section: we shall now assume, unless otherwise stated, that the spacetime is not L.R.S. Then there is only a discrete isotropy group and the shear eigenvectors will, except in one special case, determine a unique[18] triad of vectors $\{e_\alpha\}$ which are invariant under the discrete isotropy group. The special case is a space of type VI_0 with $n^\alpha{}_\alpha = 0$ and $\theta_2 = \theta_3$; in this rather exceptional case, however, a unique triad of shear eigenvectors is determined by the discrete isotropy group.

In practice, it would probably be easier to determine the discrete isotropy group than the shear eigenvectors, since an accurate measurement of microwave radiation isotherms in the sky would immediately limit severely the possible isotropy groups, while the shear might be very small at the present time. In a Class A model, the discrete isotropy group will determine a unique triad of shear eigenvectors. It follows from the discrete isotropies that a geodesic which is initially directed down one of these canonically defined axes will have this property at every point; this also follows directly from (4.3), which has the solutions

$$k^\alpha = 0 \ (\alpha \neq \beta), \qquad k^\beta = \frac{C}{l_\beta} \tag{6.1}$$

for any constant C and for $\beta = 1, 2, 3$. Thus *one can look down the principal axes of shear right back to the singularity* (or rather, until absorption

[16] At a point of emission p, at time t_1 say, an isotropic distribution can be expressed as $f(p, \pi_\kappa)$. At a later time t_2 an observer at a point p' is sampling the emission from a two-dimensional set of points in $t = t_1$ and $f(p, \pi_\kappa)$ will not have the same form as a function of π_κ at all these points unless it can be expressed as a function of homogeneous constants of motion alone. In general this would require the existence of three independent such constants (cf. [51]).

[17] Except in the Kantowski-Sachs spaces of Case I, cf. footnote 1.

[18] "Unique" is understood to mean "unique up to a sign and renumbering".

becomes appreciable) in these models. Since the directions of the principal axes of shear are directions which are locally fixed in a local inertial rest frame, the galaxies in these directions appear to be in fixed positions in the sky (this again follows from the discrete isotropies). The redshift relation $z(t)$ for these geodesics is (by (4.1), (6.1))

$$1+z=\frac{(l_\beta)_A}{(l_\beta)_G} \quad \text{(no sum).} \tag{6.2}$$

If one knows that particular radiation sources in these directions were emitting at the same time, one can use this relation to find directly the (integrated) distortion of the universe since that time from the redshifts of the sources; in particular, it can be applied to determine the distortion of the universe since the time of decoupling, by measuring the temperature of primeval black-body radiation in these directions. Detailed knowledge of the functions $r_A(t)$ for these axes would enable one to find the functions $l_\alpha(t)$ from observations in these directions alone.

In case Ba a unique triad of shear eigenvectors is again determined by the discrete isotropies. However in the Bbi cases only the $\pm\boldsymbol{e}_1$ axis (i.e. the $\boldsymbol{a}$ axis) is determined by these isotropies, and even that is not true in the Bbii cases (when there are no discrete isotropies). In the Ba and Bbi cases, a null geodesic initially directed down the $\boldsymbol{e}_1$ axis will always have this property; this follows from the discrete isotropies, or directly from the geodesic equation which has the solution (6.1) with $\beta=1$. Thus *one can look back down the* $\boldsymbol{e}_1$ *axis to the singularity in Ba or Bbi cases.* This is not true for the other two principal shear directions (a null geodesic initially down these directions deviates towards the $-\boldsymbol{e}_1$ direction) in cases Ba or Bbi. It is true in case Bbii if we define $\boldsymbol{e}_1$ not as a shear eigenvector but as the $\boldsymbol{a}$ axis (only in case Bbii are the two definitions not equivalent); *one cannot look back down any principal axis of shear in case Bbii.* Thus although the motion of matter in this space is strictly ordered, it appears (since the geodesics deviate from the principal shear directions) to be rather disordered.

In the Ba and Bbi cases, the redshift relation for the $\boldsymbol{e}_1$ axis is again (6.2). Since (6.1) holds for both positive and negative values of C, i.e. (6.2) holds for geodesics in both the $\boldsymbol{e}_1$ and $-\boldsymbol{e}_1$ directions, *the black-body temperature is the same in the* $\boldsymbol{e}_1$ *direction and the opposite* (i.e. $-\boldsymbol{e}_1$) *direction.* This last is also true in case Bbii. In fact, unless there is some accidental cancellation, one would expect that the $\boldsymbol{e}_1$ direction, and (except in case Bbii) the directions in the plane perpendicular to the $\boldsymbol{e}_1$ direction are the only directions for which this is true. This equality of the black-body temperature in the $\boldsymbol{e}_1$ and $-\boldsymbol{e}_1$ directions offers a way of observationally determining the $\boldsymbol{e}_1$ axis in case Bbii. If one can find

$r_A(t)$ for the e_1 direction, the observations in this direction will determine $l_1(t)$, except in case Bbii. (In case Ba, $l_1(t)$ is just the average length scale $l(t)$.)

We have seen that one can obtain partial information on the expansion and shear in Class B, and complete information in Class A, merely by observing the $r_A - z$ relations for certain canonically defined directions (namely those for which (6.1) holds). One can in fact use the methods of Section 3 to calculate the $r_A(t)$ relations explicitly for these axes in Class A and in those Class B cases where $n^\alpha{}_\alpha = 0$; combining these relations with (6.2) one obtains the corresponding $r_A - z$ relations.

To obtain $r_A - z$ relations in Class A, we use regular coordinates $(S(0) = g(0) = c(0) = 0)$ with the observer at the origin at the time t_A, and parametrise the geodesics by (Θ, Ψ) so that the constants (5.1) are $-\pi_1 = X_A \sin\Theta \cos\Psi$, $-\pi_2 = Y_A \sin\Theta \sin\Psi$, $-\pi_3 = Z_A \cos\Theta$ for a geodesic with $k_A^0 = 1$. We will only perform the derivation for one of the three cases, that of the e_1 axis, the results for the other axes following by suitable cyclic permutation. On this axis $0 = \Psi = x^2 = x^3 = g = S$, $\Theta = \pi/2$. We find from (5.4)

$$\left.\begin{aligned}
&\frac{dt}{dv} = k^0 = 1 + z = X_A/X;\\
&\frac{dx^1}{dt} = \frac{1}{X}; \quad \frac{dx^2}{dt} = \frac{dx^3}{dt} = 0;\\
&\frac{\partial^2 x^1}{\partial t\,\partial\Theta} = \frac{\partial^2 x^1}{\partial\Psi\,\partial t} = 0; \quad \frac{\partial^2 x^2}{\partial\Theta\,\partial t} = -\frac{N_3 Z_A S(x^1)}{Y^2(1+z)\sqrt{1 - N_2 N_3 S^2}}\\
&\frac{\partial^2 x^3}{\partial\Theta\,\partial t} = -\frac{\sqrt{1 - N_2 N_3 S^2}\, Z_A}{(1+z)\, Z^2} - \frac{Z_A N_3{}^2 S^2}{(1+z)\, Y^2 \sqrt{1 - N_2 N_3 S^2}};\\
&\frac{\partial^2 x^2}{\partial\Psi\,\partial t} = \frac{Y_A}{Y^2(1+z)}; \quad \frac{\partial^2 x^3}{\partial\Psi\,\partial t} = -\frac{N_2 Y_A S}{Z^2(1+z)} + \frac{N_3 Y_A S}{Y^2(1+z)}
\end{aligned}\right\} \tag{6.3}$$

whence

$$\begin{aligned}
r_G^2 = Z_A Y_A Z Y (1 - N_2 N_3 S^2) \Bigg[&\int_G^A \frac{dt}{Y^2(1+z)} \int_G^A \frac{dt \sqrt{1 - N_2 N_3 S^2(t)}}{(1+z)\, Z^2}\\
&+ N_2 N_3 \int_G^A \frac{S(t)\, dt}{Z^2(1+z)} \int_G^A \frac{S(t)\, dt}{Y^2(1+z)\sqrt{1 - N_2 N_3 S^2}}\\
&+ N_3^2 \Bigg(\int_G^A \frac{dt}{Y^2(1+z)} \int_G^A \frac{S^2(t)\, dt}{Y^2(1+z)\sqrt{1 - N_2 N_3 S^2}}\\
&- \int_G^A \frac{S\, dt}{Y^2(1+z)} \int_G^A \frac{S\, dt}{Y^2(1+z)\sqrt{1 - N_2 N_3 S^2}} \Bigg) \Bigg]
\end{aligned} \tag{6.4}$$

where $\frac{\partial S}{\partial t} = \frac{1}{(1+z)} \sqrt{1-N_2 N_3 S^2}$ and $S(t_A)=0$. It is clear from this calculation that the geodesic deviation vector twists relative to the covariantly-defined tetrad as one moves along the geodesic, if N_2 or N_3 is non-zero.

In case Ba and Bbi models with $n^\alpha{}_\alpha=0$ Eq. (5.5) shows that

$$\left.\begin{aligned}
&\frac{\partial x^1}{\partial t} = \frac{X_A}{X^2(1+z)}; \quad \frac{\partial x^2}{\partial t} = \frac{\partial x^3}{\partial t} = 0; \ (1+z) = k^0 = \frac{X_A}{X},\\
&\frac{\partial^2 x^1}{\partial\Theta\,\partial t} = \frac{\partial^2 x^2}{\partial\Theta\,\partial t} = \frac{\partial^2 x^1}{\partial\Psi\,\partial t} = \frac{\partial^2 x^3}{\partial\Psi\,\partial t} = 0,\\
&\frac{\partial^2 x^3}{\partial\Theta\,\partial t} = -\frac{Z_A \exp 2(a_0-q_0)x^1}{Z^2(1+z)}; \quad \frac{\partial^2 x^2}{\partial\Psi\,\partial t} = \frac{Y_A \exp 2(a_0+q_0)x^1}{Y^2(1+z)}
\end{aligned}\right\} \tag{6.5}$$

hold for the geodesic along the $\boldsymbol{e}_1$ axis, using the same parametrisation as in Class A for the geodesics. Thus we find (the metric being $\operatorname{diag}(-1, l_1{}^2, l_2{}^2 \exp -2(a_0+q_0)\,x^1, l_3{}^2 \exp -2(a_0-q_0)\,x^1)$)

$$r_A{}^2 = \frac{(l_2 l_3)_A (l_2 l_3)_G}{(l_1)^2{}_A} e^{-2a_0 u} \left(\int_G^A \frac{l_1 e^{2(a_0-q_0)u} dt}{(l_3)^2} \int_G^A \frac{l_1 e^{2(a_0+q_0)u} dt}{(l_2)^2}\right) \tag{6.6}$$

where $u = \int \frac{dt}{l_1}$ and, by (6.3b) of [1], $l_1{}^2 = (l_2 l_3)^{\frac{1}{2}} (l_2/l_3)^{q_0/2a_0}$. This applies to a geodesic in the positive x^1 direction. The opposite direction yields the same formulae with u replaced by $-u$.

(We note that in the type V case we may use (7.10, 16) of [1] so that $l_1 = X = l$ and $l_2 = Y$, $l_3 = Z$ may be written as

$$l_\beta(t) = l(t) \exp\left\{(-1)^\beta \Sigma \int \frac{dt}{l^3}\right\} \quad (\beta = 2, 3) \tag{6.7a}$$

where Σ is a constant and

$$3\,l'^2 = \Sigma^2 l^{-4} + (\mu\, l^2) + \Lambda\, l^2 + 3\, a_0{}^2\,.) \tag{6.7b}$$

The methods used here may be used to find $r_A(t)$ along any geodesic which can be solved completely. One can also approximately solve the geodesic equations for nearby geodesics; for example, in the solutions with $n^\alpha{}_\alpha = 0$ (except Bbii) a general geodesic very nearly down the $\boldsymbol{a}$ axis has tetrad components

$$k_1 = \frac{\cos\Theta}{l_1(t)}, \quad k_2 = \frac{\sin\Theta\cos\Phi}{l_2(t)} (g(t))^{a_0+q_0}, \quad k_3 = \frac{\sin\Theta\sin\Phi}{l_3(t)} (g(t))^{a_0-q_0} \tag{6.8}$$

where $g(t) := \exp \int_t^A \frac{dt}{l_1(t)}$ and Θ, Φ are constants. This solution is valid when $|k_2 l_2| \ll 1$, $|k_3 l_3| \ll 1$, i.e. there are small cones about $\Theta = 0$ and $\Theta = \pi$ for which it is valid. Substituting into (4.1) one obtains an approximate expression for the redshift z as a function of t, Θ, Φ along these geodesics. One finds in this way that, in that part of the cone about $\Theta = 0$ for which $|k_2| \ll |k_1|$, $|k_3| \ll |k_1|$ hold at the time of decoupling, the temperature of the primeval black-body radiation would have the angular dependence $A\,\Theta^2(1 + C\cos 2\,\Phi)$ where A, C are constants; and that the temperature in the opposite directions would have the same value to this order of approximation.

It follows from the form taken by the terms $u_{a;b}K^a K^b$ and $u_{a;bc}K^a K^b K^c$ (see (4.4, 5)) that if one could determine the second-order term in the $m - z$ relation (i.e. the term $(4 - (u_{a;bc}K^a K^b K^c)/(u_{d;e}K^d K^e)^2)$ in (3.8)) one would completely determine the cosmological model. In fact in most of the models one could determine all the parameters of the spacetime directly from observations of the first and second order terms in the principal shear directions (the preferred axes mentioned at the beginning of this Section) only. However there is an exception to this: in the case Ba (type V) models the terms $u_{a;b}K^a K^b$ and $\left(4 - \frac{(u_{a;bc}K^a K^b K^c)}{(u_{d;e}K^d K^e)^2}\right)$ take the form

$$H_0 + \sigma_0 \sin^2\Theta \cos 2\,\Phi \tag{6.9a}$$

and

$$\begin{aligned}\Big\{(1-q_0) + \frac{\sigma_0}{H_0}\sin^2\Theta\cos 2\Phi\left(7 - \frac{2a_0}{H_0}\cos\Theta\right) + \left(\frac{2\sigma_0}{H_0}\sin^2\Theta\cos 2\Phi\right)^2 \\ - \frac{2\sigma_0^2}{H_0^2}\sin^2\Theta\Big\}\left(1 + \frac{\sigma_0}{H_0}\sin^2\Theta\cos 2\Phi\right)^{-2}\end{aligned} \tag{6.9b}$$

where $K^\nu = (\cos\Theta, \sin\Theta\cos\Phi, \sin\Theta\sin\Phi)$, σ_0 and a_0 are the values at time t_0 of the quantities σ and a, and $H_0 = \left.\frac{l^{\cdot}}{l}\right|_0$, $q_0 = \left.\frac{-l^{\cdot\cdot}}{lH_0^2}\right|_0 = \left.\frac{-(\theta^{\cdot} + \theta^2/3)}{3H_0^2}\right|_0$. (Note an error in this expression in [1].)

This shows that down the $\boldsymbol{a}$ axis, the first terms of the $m - z$ relation are *precisely the same as in a Robertson-Walker universe*, i.e. (3.9) holds for this axis. (However q_0 is related to μ_0, H_0, $R^* = 6a^2$ and Λ by the relations in Section 3 of [1] instead of the corresponding Robertson-Walker relations in which $\sigma = 0$.) Further it is clear that one gets no direct information as to the magnitude a_0 from the second order $m - z$ relations down any of the principal axes. It would be easiest to determine

a_0 from second-order $m-z$ terms by looking in the directions with $\Phi=0$, $\pi/2$ and $\cos\Theta = \pm \frac{1}{\sqrt{3}}$. In practice, use of the power series expressions to determine the model parameters may not be possible in any direction, or if possible may be inaccurate (cf. Sections 3 and 4).

One can also obtain the observational relations approximately in any spacetime which is nearly the same as a spacetime in which one can solve these equations exactly. For example, one can find the black-body radiation temperature observed in a type V universe model which is almost isotropic (cf. [52]), i.e. in a low-density anisotropic universe that is almost a Robertson-Walker universe. To do so, note that Eqs. (4.2, 3) show that $\frac{dk^0}{dv} = -\theta_{\mu\nu}k^\mu k^\nu$; this can be solved in the form

$$k^0(t) = \frac{1}{l(t)} \exp\left\{-\int \frac{\sigma_{\kappa\nu}(t)\, k^\kappa(t)\, k^\nu(t)\, dt}{k^\alpha(t)\, k_\alpha(t)}\right\}$$

which (by Eq. (4.1)) determines the redshift in any of our universe models. Using the canonical tetrad defined in [1], this equation takes the form

$$k^0 = \frac{1}{l} \exp\left\{-\Sigma \int \frac{(k_2)^2 - (k_3)^2\, dt}{((k_1)^2 + (k_2)^2 + (k_3)^2)\, l^3}\right\} \tag{6.10}$$

in a type V universe. Now in a Robertson-Walker universe of type V (i.e. $\Sigma = 0$) a general null geodesic is given by

$$k^0 = \pm \frac{1}{l(t)},\quad k^1 = \frac{\cos\Theta(t)}{l(t)},\quad k^2 = \frac{\sin\Theta(t)\sin\Phi}{l(t)},\quad k^3 = \frac{\sin\Theta(t)\cos\Phi}{l(t)} \tag{6.11a}$$

where

$$\cot\frac{\Theta(t)}{2} = (\cot\Psi/2)\exp\left(-a_0 \int^t \frac{dt}{l(t)}\right), \tag{6.11b}$$

Ψ, Φ are constants and l is normalised so that $l=1$ when $|k^0|=1$. Eq. (6.7b) shows $\dot{l} = \sqrt{a_0^2 + \mu l^2/3 + \Lambda l^2/3}$ so the integral (6.11b) can be written as a simple integral in l. One can now obtain the approximate form of the metric in an almost isotropic type V space from (6.7a) on using the value of $l(t)$ for the Robertson-Walker model on the right hand side, thus determining $l_\beta(t)$, and find $k^0(t)$ in this space (cf. [52]) by using (6.11) as the form of $k_\alpha(t)$ in the integral (6.10), which can again be expressed as a simple integral in l alone. This then determines the black-body radiation temperature in these models from the expression (cf. (2.17))

$$T = \frac{T_e}{1+z} = T_e \frac{k^0(l_0)}{k^0(l_e)}$$

where T_e is the temperature of the black-body radiation on emission ($\sim 3000^0\,K$) and l_e is the length scale l at the time of emission of this radiation (so $l_e/l_0 \sim 1/1000$ if there is negligible intergalactic matter).

This calculation is fairly tractable if $\Lambda = 0$ and the universe is filled with a non-interacting mixture of dust and radiation, i.e. if $\mu = \frac{M}{l^3} + \frac{R}{l^4}$ where M and R are constants. An exceptionally simple case arises when $R = M^2/12\,a_0^2$, i.e. when the relation

$$(\mu_m)^2 = 2\,\mu_r |R^*| \tag{6.12}$$

is valid [19] (μ_m being the energy density of the matter, μ_r that of the radiation, and R^* the scalar curvature of the three-spaces $\{t = \text{constant}\}$, cf. [1]). We may note that this relation, which implies $q_0{}^2 = \mu_r/3\,H_0{}^2$, seems to give a good description of a realistic low-density Robertson-Walker universe with $\Lambda = 0$, for the total energy density μ_0 in the universe at the present time t_0 is almost the same as $\mu_m|_0$ and so at the present time (6.12) would be [20]

$$\mu_0{}^2 \simeq 12\,H_0{}^2 \mu_r\,. \tag{6.13}$$

Substituting in (6.13) the values $\mu_r \simeq 10^{-33}$ gm/cc, $3\,H_0^2 \simeq 10^{-29}$ gm/cc one finds $\mu_0 \simeq 2.10^{-31}$ gm/cc, in very close agreement with the observed density of luminous matter in the universe. When (6.12) is valid, the approximate solution of (6.10) is

$$\begin{aligned} k^0(l) = \frac{1}{l}\exp\Bigg\{ -2\Sigma c^2 \cos 2\Phi \Bigg[&\frac{c^2 - 3b^2}{(b^2+c^2)^3} \log \frac{l^2}{((l+b)^2+c^2)} \\ &+ \frac{2b(c^2+5b^2)}{c(c^2+b^2)^3} \tan^{-1}\left(\frac{l+b}{c}\right) - \frac{(4bl^2 + (7b^2-c^2)l + 4b^3 - 2bc^2)}{((l+b)^2+c^2)(b^2+c^2)\,l} \Bigg]\Bigg\} \end{aligned} \tag{6.14}$$

where for brevity we have written $c := \cot \Psi/2$, $b := M/6a_0^2$. This is therefore the formula determining the black-body temperature in a low-density type V universe which is almost isotropic. It follows that to first order in Σ the black-body temperature has the angular dependence $f(\Theta)$ $\cos 2\Phi$ where $f(\Theta)$ is sharply peaked at small values of Θ. These "hot spots" near the $\boldsymbol{a}$ direction result from the way the geodesics in these

[19] This is the simplest family of Robertson-Walker models with $R^* < 0$, because it is precisely the family in which the expression for $l^\cdot$ does not involve a square root $\left(\text{in fact, } l^\cdot = a_0 + \frac{M}{6\,l a_0}\right)$.

[20] Where we have used the relations at the end of Section 3 of [1], and made the approximation $\mu_r \ll 3H_0^2$. More generally, (6.13) would take the form

$$\mu_m = 2(\sqrt{3H_0^2\mu_r} - \mu_r)\,.$$

spaces tend to the $-\boldsymbol{a}$ direction; this geometrical effect might lead us to expect such observational effects in all Class B models. (Note however that no such measurable effects occur in the L.R.S. Class B models, in which the $\boldsymbol{a}$ direction has no invariant significance.) Calculations similar to that above have been given by Novikov [53] and Matzner [51] (and, by various authors, for the more general models of Bianchi type V in which the fluid flow is not orthogonal to the surfaces of homogeneity [52, 54]).

Finally, we note that (6.2) shows that as one looks back towards the singularity (where $l \to 0$) one would see a large blueshift for objects near the singularity in the direction of any axis for which $l_\beta \to \infty$ as $l \to 0$. If l_β tends to a finite number as $l \to 0$, objects in that direction would be seen to have a finite maximum redshift. If $l_\beta \to 0$ as $l \to 0$, the redshift for objects in that direction would go to infinity as $l \to 0$. In fact all these behaviours can occur, for in a type I model with $-\frac{\pi}{6} < \alpha < \frac{\pi}{2}$ one would see infinite redshifts in two axis directions and infinite blueshifts in the third (a "cigar" singularity would occur, cf. [55]) while if $\alpha = \frac{\pi}{2}$ one would see infinite redshifts in one axis direction and finite maximum redshifts in the other two directions[21] (a "pancake" singularity would occur), where the angle α is as in (7.14) of [1]. The behaviour of the other models near the singularity is discussed in a subsequent paper; one finds behaviour like the type I "cigar" case in many cases, but more complex behaviour can occur (cf. [56, 57]). In practice, of course, one would not expect to see the behaviour near the singularity as the universe is opaque to light and radio waves at early times; however one might expect to see related deviations from an isotropic $z-t$ relation, in which the observational curves turn over (cf. [48]).

7. Discussion

In this paper we have reviewed ways of calculating observational relations in general cosmological models and applied the simplest of these to the class of spacetimes studied in a previous paper [1]. The form taken by the observational relations in the Robertson-Walker models is well known; closed form expressions have also been given in Bianchi I spaces [48] and in Kantowski-Sachs spaces [40]. We show in Section 5 that such expressions can also be found in L.R.S. type II spaces, and that apart from the remaining L.R.S. spaces (of types VIII

[21] Providing that the behaviour of the matter and radiation is reasonable, e.g. $\mu/3 \geqq p \geqq 0$ and $\mu \neq 0$.

and IX) it is unlikely that one will be able to find closed form expressions for observational relations in the other models discussed in [1]. However no difficulty arises in integrating the observational relations numerically in these spaces (cf. [37]). Further we show in Section 6 that one can obtain closed-form expressions for the observational relations down the principal axes of shear in many cases (in particular, in all Class A models); in principle one could obtain complete information about the history of these universes from observations in these directions alone. In fact, one could determine the world model from the coefficients of the first two powers of z in the $m-z$ relations down these axes alone.

The variation of the observed temperature of primeval black-body radiation over the sky would give a measure (at least in Class A models) of the overall distortion of the universe since the time of last scattering of this radiation. In most of the models, large black-body temperatures could occur in certain directions with accompanying anomalous behaviour of the other observational relations for these directions. In Section 6 we calculate explicitly the observed black-body radiation temperature in a (type V) universe which is, since the time of decoupling, nearly the same as a low-density Robertson-Walker universe with $\Lambda = 0$. Incidentally, we show that there is a unique simplest such Robertson-Walker universe defined by the Hubble constant H_0 and the present value μ_r of the density of radiation in the universe: it is that one in which the density μ_m of matter is given by $\mu_m = 2(\sqrt{3H_0{}^2\mu_r} - \mu_r)$. (This value of μ_m is very close to the observed density of luminous matter in the universe.)

The universe models we consider are homogeneous in a strict mathematical sense: they admit a three-dimensional continuous group of isometries. One might ask whether alternative definitions of homogeneity might in fact correspond better to those universes an observer would regard as homogeneous. For example, Grishchuk [58] has suggested that one should regard as homogeneous only those universe models in which the spatial parts [22] of the covariant derivatives of the spatial parts of all geometrically or physically defined quantities vanish; the spaces satisfying this criterion are the Robertson-Walker spaces, the Bianchi I spaces discussed in [1] and the Kantowski-Sachs spaces (and so are among the spaces in which explicit forms of the observational relations can be obtained).

We show in Section 4 that the spaces considered here are such that (except in case Bbii) all observational relations at any point are invariant under a discrete group of isotropies. Thus one may regard the existence

[22] By "spatial parts" we mean the projection of these tensors perpendicular to the average velocity vector u^a.

of isotropies in astronomical observations as an observational test for homogeneity: the existence of a continuous group of isotropies implying the invariance of the space under a multiply-transitive group of isometries (cf. [4, 49]) and discrete isotropies implying the existence of a simply-transitive group of isometries. To determine a minimal dimension for the orbits of the group of isometries, one can consider the directions $\boldsymbol{e}$ in the sky such that observational relations in the $\boldsymbol{e}$ direction are identical with those in the $-\boldsymbol{e}$ direction, and apply the following criterion[23]: if there are at least two independent such directions, the orbits of the group are at least two-dimensional; if there are at least three independent such directions (i.e. there is a third such direction which does not lie in the plane defined by the first two) then the orbits of the group are at least three-dimensional, and so the spacetime is spatially homogeneous. This then provides observational criteria which are sufficient to enable one to state that a cosmological model is spatially homogeneous (the spaces so defined include all L.R.S. subcases and all Class A spaces; they therefore include all the spaces satisfying Grishchuk's criterion). This criterion does not include all the spaces satisfying the conditions imposed in [1]; however in most of these exceptional cases (i.e. all except Bbii) one might be able to determine the spatial homogeneity by noting that there must exist a third direction $\boldsymbol{e}$, orthogonal to the first two, such that the black-body temperature is necessarily the same in the $\boldsymbol{e}$ and $-\boldsymbol{e}$ directions. Thus, with the exception of case Bbii it is possible one could use the observational isotropies to characterise spatial homogeneity in all of these spaces. One suspects that any other way of trying to prove observationally that these models were spatially homogeneous would be rather more difficult to carry out both in principle and in practice.

The nature of the singularities in these spacetimes, and the behaviour far from them, will be discussed in a subsequent paper.

Acknowledgements. We are indebted to our colleagues Drs. S. W. Hawking and B. G. Schmidt for many helpful discussions, and to Drs. D. W. Dewhirst and J. M. Stewart and Profs. J. Ehlers, C. W. Misner and R. K. Sachs for stimulus and enlightenment at various stages of the work. During the course of this work one of us (M. A. H. M.) was in receipt of a grant from the U.K. Science Research Council, for which he thanks that body.

References

1. Ellis, G. F. R., MacCallum, M. A. H.: Commun. Math. Phys. **12**, 108 (1969).
2. Schmidt, B. G.: Commun. Math. Phys. **15**, 329 (1969).
3. MacCallum, M. A. H., Stewart, J. M., Schmidt, B. G.: Commun. Math. Phys. **17**, 343 (1970).

[23] A formal proof of these statements would use Kristian and Sachs' results [18] and would constitute a generalisation in the same spirit as Carter and Schmidt [2, 59] of the theory of symmetric spaces.

4. Stewart, J. M., Ellis, G. F. R.: J. Math. Phys. **9**, 1072 (1968).
5. Kantowski, R., Sachs, R. K.: J. Math. Phys. **7**, 443 (1966).
6. Bianchi, L.: Lezioni sulla teoria dei gruppi continui finite transformazioni. Pisa: Spoerri 1918.
7. Estabrook, F. B., Wahlquist, H. D., Behr, C. G.: J. Math. Phys. **9**, 497 (1968).
8. Ellis, G. F. R.: Lectures at the International School of Physics, Varenna. Corso XLVII (1969), to be published.
9. Trautmann, A.: In: Recent developments in general relativity (Infeld Festschrift volume). Oxford: Pergamon; and Warsaw: PWN 1962.
10. Dehnen, H.: Z. Astrophys. **68**, 190 (1968).
11. Ehlers, J.: Z. Naturforsch. **22a**, 1328 (1967).
12. Isaacson, R. A.: Phys. Rev. **166**, 1263 (1968).
13. Jordan, P., Ehlers, J., Sachs, R. K.: Akad. Wiss. Lit. (Mainz), Abhandl. Math.-Nat. Kl. No. 1 (1961).
14. Sachs, R. K.: Proc. Roy. Soc. (London) **264**, A309 (1961).
15. McVittie, G. C.: General relativity and cosmology. London: Chapman and Hall 1956.
16. Etherington, I. M. H.: Phil. Mag. **15**, 761 (1933).
17. Penrose, R.: In: Perspectives in geometry and relativity (Hlavaty Festschrift), ed. B. Hoffmann. Indiana: Indiana U.P. 1966.
18. Kristian, J., Sachs, R. K.: Astrophys. J. **143**, 379 (1966).
19. — Astrophys. J. **147**, 864 (1967).
20. Newman, E. T., Goldberg, J. N.: Phys. Rev. **114**, 1391 (1959).
21. Bertotti, B.: Proc. Roy. Soc. (London) **294**, A195 (1966).
22. Hoyle, F.: Lectures at International School of Physics, Varenna, Corso XX. New York: Academic Press 1962.
23. Sandage, A.: Astrophys. J. **133**, 355 (1961).
24. Sachs, R. K.: Lectures at the Brandeis Summer Institute (1968), to be published.
25. Rees, M. J.: Astrophys. J. **153**, L1 (1968); Weymann, R.: Astrophys. J. **145**, 560 (1966).
26. Longair, M. S., Scheuer, P. A. G.: Nature **215**, 919 (1967).
27. Sandage, A.: Observatory **88**, 91 (1968).
28. Stock, J., Schücking, E.: Astron. J. **62**, 98 (1957).
29. Jauncey, D. L.: Nature **216**, 877 (1967).
30. Refsdal, S.: Astrophys. J. **153**, 373 (1969).
31. Kantowski, R.: Astrophys. J. **155**, 89 (1969).
32. Saunders, P. T.: Ph. D. thesis, King's College, London (1967).
33. Temple, G.: Proc. Roy. Soc. (London) **168**, A122 (1938).
34. Joseph, V.: Monthly Notices Roy. Astron. Soc. **118**, 631 (1958).
35. Synge, J. L.: Relativity: the general theory. Amsterdam: North-Holland 1960.
36. Zipoy, D. M.: Phys. Rev. **142**, 825 (1966).
37. Saunders, P. T.: Monthly Notices Roy. Astron. Soc. **141**, 427 (1968).
38. Solheim, J.-E.: Monthly Notices Roy. Astron. Soc. **133**, 321 (1966).
39. Mattig, W.: Astron. Nachr. **284**, 109 (1958).
40. Tomita, K.: Prog. Theor. Phys. **40**, 264 (1968).
41. Mattig, W.: Astron. Nachr. **285**, 1 (1959).
42. Bondi, H.: Cosmology. Cambridge: C.U.P. 1960.
43. Pooley, G. G., Ryle, M.: Monthly Notices Roy. Astron. Soc. **139**, 515 (1968).
44. Zwicky, F.: Morphological astronomy. Berlin, Göttingen, Heidelberg: Springer 1957.
45. Sandage, A.: Astrophys. J. **136**, 319 (1962).
46. Partridge, R. B., Wilkinson, D. W.: Phys. Rev. Letters **18**, 557 (1967).
47. Katzin, G. H., Levine, J.: J. Math. Phys. **9**, 8 (1968).
48. Saunders, P. T.: Monthly Notices Roy. Astron. Soc. **142**, 213 (1969).

49. Ellis, G. F. R.: J. Math. Phys. **8**, 1171 (1967).
50. Ehlers, J.: Lectures at the International School of Physics, Varenna. Corso XLVII (1969), to be published.
51. Matzner, R. A.: University of Texas report (1969); Astrophys. J. **157**, 1085 (1969).
52. Hawking, S. W.: Monthly Notices Roy. Astron. Soc. **142**, 129 (1969).
53. Novikov, I. D.: Soviet Astron.-A.J. **12**, 427 (1969); Russian orig., Astron. Zh. **45**, 538 (1968).
54. Grishchuk, L. P., Doroshkevich, A. G., Novikov, I. D.: J.E.T.P. **28**, 1210 (1969), Russian orig. Z.E.T.F. **55**, 2281 (1968); Ruzmaikina, T. V., Ruzmaikin, A. A.: J.E.T.P. **29**, 934 (1969); Russian orig. Z.E.T.F. **56**, 1742 (1969).
55. Jacobs, K. C.: Astrophys. J. **153**, 661 (1968) and **155**, 379 (1969); Ph. D. thesis, Caltech. 1969.
56. Misner, C. W.: Phys. Rev. Letters **22**, 1071 (1969); Phys. Rev. **186**, 1319 and 1328 (1969); University of Maryland Tech. Rpt. 70-039 (1969).
57. Belinsky, V. A., Khalatnikov, I. M.: Z.E.T.F. **56**, 1700 (in Russian) (1969) and **57**, 2163; Khalatnikov, I. M., Lifshitz, E. M.: Phys. Rev. Letters **24**, 76 (1970).
58. Grishchuk, L. P.: Soviet Astron.-A.J. **11**, 881 (1968); Russian orig., Astron. Zh. **44**, 1097 (1967).
59. Carter, B.: J. Math. Phys. **10**, 70 (1969).

M. A. H. MacCallum and G. F. R. Ellis
Dept. of Applied Mathematics
and Theoretical Physics
Silver Street
Cambridge, England