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The Bargmann-Wigner Method in Galilean Relativity*

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Abstract. The equations of motion of a spin one particle as derived from Levy-Leblond's Galilean formulation of the Bargmann-Wigner equations are examined. Although such an approach is possible for the case of free particles, inconsistencies which closely parallel those encountered in the Bargmann-Wigner equations of special relaticity are shown to occur upon the introduction of minimal electromagnetic coupling. If, however, one considers the vector meson within the Lagrangian formalism of totally symmetric multispinors, it is found that the ten components which describe the vector meson in Minkowski space reduce to seven for the Galilean group and that in this formulation no difficulty occurs for minimal electromagnetic coupling.

More generally it is demonstrated that one can replace Levy-Leblond's version of the Bargmann-Wigner equations by an alternative set which leads to the correct number of variables for the vector meson. A final extension consists in the proof that for all values of the spin the (Lagrangian) multispinor formalism implies the Bargmann-Wigner equations. Thus the problem of special relativity of seeking a Lagrangian formulation of the Bargmann-Wigner set is found to have only a somewhat trivial counterpart in the Galilean case.

I. Introduction

A systematic study of the Galilean group in quantum mechanics has recently been carried out in a series of papers by Levy-Leblond [1–3]. While such an investigation might appear to be of little interest in an era in which special relativity has achieved virtually universal acceptance, it has nonetheless served the highly useful function of pointing out that certain of the predictions of quantum mechanics follow merely from Galilean invariance with no reference whatever to the Lorentz group. Of particular interest in this context is the case of the magnetic moment of the electron which has been shown by Levy-Leblond to have the same value in Galilean relativity as in the Dirac theory [3]. A closely related result is the fact that a nonrelativistic particle cannot possess intrinsic electromagnetic properties other than an electric charge and a magnetic dipole. This then at least suggests that a knowledge of the predictive powers of a non-relativistic theory is not to be disdained and can in fact serve as a basis for a more profound understanding of the impact of special relativity on quantum mechanics.

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In this paper the wave equations derived by Levy-Leblond from a Bargmann-Wigner approach [4] are considered for the case of a spin one particle interacting with an electromagnetic field. In Section 2 following a brief review of the spin one-half theory it is shown that the incorporation of minimal coupling into those equations of motion leads to an unacceptable increase in the number of degrees of freedom, thereby motivating a search for an alternative approach to the vector meson. In Section 3 it is demonstrated that the method of completely symmetric multispinors provides a simple framework for a theory of a spin one particle in which no inconsistency is found to occur upon inclusion of an electromagnetic coupling. The equations of motion in this approach are found to be a subset of those used by Levy-Leblond involving, however, only seven of the ten components used in his formulation. In the concluding section it is shown that a Bargmann-Wigner set which differs from that of Levy-Leblond can be derived from the multispinor formalism in much the same way as in the relativistic case with, however, the important advantage here that the derivation is found to generalize readily to the case of arbitrary spin.

II. The Minimally Coupled Vector Meson

In order to provide a convenient basis for the discussion of the vector meson it is useful to briefly summarize here Levy-Leblond's formulation of the Galilean invariant spin one-half theory [3]. One begins by noting that the restriction that the wave equation be of first order in all derivatives implies the form

$$\left(i\hbar A\frac{\partial}{\partial t}+\boldsymbol{B}\,\frac{\hbar}{i}\cdot\nabla+C\right)\psi=0$$

where A, $\boldsymbol{B}$ and C are numerical matrices. The further requirement that ψ satisfy the usual Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2}{2m}\,\nabla^2\right)\psi=0$$

leads to a set of conditions on these matrices which allows one to infer that ψ must be a four component spinor and that the above matrices can be taken to have the form

$$A=\frac{1}{2}(1+\varrho_3)=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

$$\boldsymbol{B}=\varrho_1\boldsymbol{\sigma}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix},$$

$$C=m(1-\varrho_3)=\begin{pmatrix}0&0\\0&2m\end{pmatrix}$$

where we use the two commuting sets of Pauli matrices ϱ_i and σ_i to span the 4×4 dimensional spinor space. The matrices σ_i transform as a spatial vector under pure rotations as can be readily verified from the complete transformation law

$$\psi'(\boldsymbol{x}', t') = e^{i\hbar^{-1} f(\boldsymbol{x}, t)} \Delta^{\frac{1}{2}}(\boldsymbol{v}, R)\, \psi(\boldsymbol{x}, t) \tag{2.1}$$

corresponding to the Galilean transformation

$$\boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}\,,$$
$$t' = t + b\,.$$

In writing (2.1) we have defined

$$f(x, t) = \frac{1}{2} m v^2 t + m\boldsymbol{v} \cdot R\boldsymbol{x}$$

and

$$\Delta^{\frac{1}{2}}(\boldsymbol{v}, R) = \begin{pmatrix} D^{\frac{1}{2}}(R) & 0 \\ \dfrac{-\boldsymbol{\sigma} \cdot \boldsymbol{v}}{2} D^{\frac{1}{2}}(R) & D^{\frac{1}{2}}(R) \end{pmatrix}$$

$$= \left(1 - \frac{\varrho_1 - i\varrho_2}{4} \boldsymbol{\sigma} \cdot \boldsymbol{v}\right) D^{\frac{1}{2}}(R)$$

where $D^{\frac{1}{2}}(R)$ is the usual two dimensional representation of spin one-half which acts in the space of the σ matrices. It is of interest to note that if one makes the decomposition of ψ into two component spinors ϕ and χ, i.e.

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

the spinor ϕ is found not to mix with χ under Galilean transformations. This implies the invariance of the bilinear form

$$\phi^* \phi = \psi^* \frac{1}{2}(1 + \varrho_3)\psi$$

which can alternatively be interpreted as expressing the fact that $\frac{1}{2}(1 + \varrho_3)$ behaves as a scalar under Galilean transformations. Although there exists another matrix – ϱ_2 – which is a scalar under all transformations continuously related to the identity, one can easily verify that under the parity operation

$$\psi'(-\boldsymbol{x}, t) = e^{i\eta} \varrho_3 \psi(\boldsymbol{x}, t)$$

and that ϱ_2 is actually the analogue of the pseudoscalar matrix γ_5 of the Dirac theory.

It follows immediately from the above discussion that the Lagrangian density can be written as

$$\mathscr{L} = i\hbar\psi^* \frac{1}{2}(1+\varrho_3)\frac{\partial}{\partial t}\psi + \psi^*\varrho_1\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla\psi + m\psi^*(1-\varrho_3)\psi\,.$$

In the case of a second quantized theory one infers from this form the commutation relations of the field operator ψ by the inversion of the matrix A in its nonsingular subspace. This would imply that

$$\frac{1}{2}(1+\varrho_3)\,\{\psi(\boldsymbol{x},t),\psi^*(\boldsymbol{x}',t)\}\,\frac{1}{2}(1+\varrho_3) = \frac{1}{2}(1+\varrho_3)\,\delta(\boldsymbol{x}-\boldsymbol{x}')$$

or

$$\{\phi(\boldsymbol{x},t),\phi^*(\boldsymbol{x}',t)\} = \delta(\boldsymbol{x}-\boldsymbol{x}')$$

where for the sake of definiteness we have assumed the usual Fermi-Dirac statistics for spin one-half particles despite the absence of a formal spin-statistics theorem for Galilean field theories [2]. As a final observation concerning the spin one-half theory it is of interest to consider minimal coupling for which one has the coupled equations

$$(E-e\varphi)\phi + \boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})\chi = 0\,,$$
$$\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})\phi + 2m\chi = 0$$

where we use the shorthand notation $(\boldsymbol{p},E)$ for $\left(\frac{h}{i}\nabla, ih\frac{\partial}{\partial t}\right)$. Upon elimination of the dependent components χ one obtains

$$(E-e\varphi)\phi - \frac{1}{2m}[\boldsymbol{\sigma}\cdot(\boldsymbol{p}-e\boldsymbol{A})]^2\phi = 0$$

or

$$(E-e\varphi)\phi - \frac{(p-eA)^2}{2m}\phi + \frac{e\hbar}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{H}\phi = 0$$

thereby displaying Levy-Leblond's previously quoted result for the magnetic moment.

In selecting a technique for the generalization of the preceding approach to higher spin theories there are a number of options available. Levy-Leblond chooses a Bargmann-Wigner formulation which is conveniently described in terms of the multispinor $\psi_{a_1 a_2 \dots a_N}(\boldsymbol{x},t)$ $(a_i = 1,2,3,4)$ which is symmetric in the N variables a_i. Such an object clearly has

$$(N+3)!/N!3!$$

components. If one defines the Galilean invariant operator

$$G=\frac{1}{2}(1+\varrho_3)i\hbar\frac{\partial}{\partial t}+\varrho_1\boldsymbol{\sigma}\cdot\frac{\hbar}{i}\nabla+m(1-\varrho_3),$$

a Bargmann-Wigner set can be specified by

$$G_i\psi\equiv G_{a_i a_i'}\psi_{a_1\ldots a_{i-1}a_i'a_{i+1}\ldots a_N}=0\quad(i=1,2,\ldots N).\tag{2.2}$$

Levy-Leblond argues that since the Eq. (2.2) serves to express those components for which $a_i=3,4$ in terms of the remaining components, one has only the 2^N "upper" components as possible candidates for independent variables. Because of the symmetry of the multispinor, however, this number reduces to the desired $N+1$ independent components.

In order to examine the implications of this Bargmann-Wigner approach in the spin one case, one introduces the second rank spinor ϕ. Such an object has the representation

$$\phi=\left[\boldsymbol{U}\cdot\boldsymbol{\sigma}\frac{1}{4}(1+\varrho_3)-\frac{1}{m}\boldsymbol{L}\cdot\boldsymbol{\sigma}\frac{1}{8}(1-\varrho_3)-\frac{i}{4m}\boldsymbol{M}\cdot\boldsymbol{\sigma}\varrho_1+\frac{i}{4}\varrho_2 W\right]\sigma_2$$

which is the customary decomposition of a symmetric second rank spinor into three vectors ($\boldsymbol{U}, \boldsymbol{L}, \boldsymbol{M}$) and a scalar ($W$). Application of the Bargmann-Wigner Eqs. (2.2) now implies

$$\boldsymbol{M}=\boldsymbol{p}\times\boldsymbol{U},\tag{A}$$

$$\boldsymbol{p}\cdot\boldsymbol{U}=mW,\tag{B}$$

$$E\boldsymbol{U}+\frac{1}{2m}\boldsymbol{p}\times\boldsymbol{M}-\frac{1}{2}\boldsymbol{p}W=0,\tag{C}$$

$$\boldsymbol{p}\cdot\boldsymbol{M}=0,\tag{D}$$

$$\boldsymbol{L}=\frac{1}{2m}\boldsymbol{p}\times\boldsymbol{M}+\frac{1}{2}\boldsymbol{p}W,\tag{E}$$

$$E\boldsymbol{M}+\boldsymbol{p}\times\boldsymbol{L}=0,\tag{F}$$

$$\boldsymbol{p}\cdot\boldsymbol{L}=mEW.\tag{G}$$

There is clearly a considerable redundancy in this set (as in the corresponding relativistic case) and one can consequently consider some subset of these as defining the theory. Unlike the relativistic case, however, one does not have the tensorial representations which imply an essentially unique choice for the defining equations. Levy-Leblond chooses the set

$$\boldsymbol{M} = \boldsymbol{p} \times \boldsymbol{U}, \tag{A'}$$

$$\boldsymbol{p} \cdot \boldsymbol{L} = mEW, \tag{B'}$$

$$\boldsymbol{L} = \boldsymbol{p}W - E\boldsymbol{U}, \tag{C'}$$

$$\boldsymbol{p} \times \boldsymbol{M} = m\boldsymbol{L} - mE\boldsymbol{U}, \tag{D'}$$

where (A) and (A′) are identical, and (B′) is the same as (G). Equations (C′) and (D′) are evidently the sum and difference of (C) and (E). The remaining equations (B), (D), and (F) are a consequence of the primed set as (A′) clearly implies (D) while (A′) and (C′) yield (F). Finally (B′) and (D′) are seen to give the result

$$E(\boldsymbol{p} \cdot \boldsymbol{U} - mW) = 0, \tag{2.3}$$

i.e. the time derivative of (B).

It is apparent that the choice made in writing the primed set of equations is quite arbitrary and although it is virtually impossible to get any ill effects for free particles there is considerable danger that the introduction of interactions could destroy the consistency of the theory. Before demonstrating that this does indeed happen in the present formulation it is instructive to count the number of degrees of freedom in the theory. Clearly (A′) merely defines $\boldsymbol{M}$ in terms of $\boldsymbol{U}$ (since no time derivative occurs in that equation) so that $\boldsymbol{M}$ can be considered a dependent variable. If one looks at (C) and (E) (rather than their linear combinations (C′) and (D′)) one sees further that $\boldsymbol{L}$ is defined by (E) in terms of $\boldsymbol{M}$ and W. Thus one has the true equation of motion (A) for the vector $\boldsymbol{U}$ which means that there are at least the desired three degrees of freedom in the theory. Furthermore if W can be expressed in terms of $\boldsymbol{U}$ by a constraint no unwanted degrees of freedom will occur. The Eq. (2.3) would provide such a constraint provided that the time derivative can be removed. Rather than dwell on the possibility of extracting (B) from (2.3) it will merely be pointed out here that whatever is the case for the free particle the function W is not constrained upon the introduction of an electromagnetic coupling. This result immediately follows upon inclusion of minimal coupling into the set (A′) through (D′). If for convenience one drops the scalar potential and defines $\boldsymbol{\Pi} = \boldsymbol{p} - e\boldsymbol{A}$, Eq. (2.3) becomes

$$\boldsymbol{\Pi} \cdot (\boldsymbol{\Pi} \times \boldsymbol{M}) = m(mEW - \boldsymbol{\Pi} \cdot E\boldsymbol{U})$$

or

$$i\frac{e}{m}\boldsymbol{H} \cdot \boldsymbol{M} = mEW - \boldsymbol{\Pi} \cdot E\boldsymbol{U}$$

which is now an equation of motion for W. Thus one concludes that the Levy-Leblond set is not consistent for electromagnetic coupling in that it does not represent a theory with the correct number of dynamical variables. This has as a practical consequence the fact that one cannot disentangle the equations of motion for $\boldsymbol{U}$ and W and it is therefore impossible to identify the magnetic moment. We now turn to an alternative formulation of the Galilean vector meson which allows an escape from these difficulties.

III. A Lagrangian Approach

The difficulties in Levy-Leblond's Bargmann-Wigner approach, while not identical to, are nonetheless quite reminiscent of the inconsistencies which characterize the minimally coupled Bargmann-Wigner equations of special relativity. Since these latter complications are found not to occur in a strictly Lagrangian formulation, this tends to suggest that a fully consistent Galilean theory of vector mesons could conceivably be obtained within a Lagrangian formalism. More explicitly one considers the symmetric bispinor ψ_{ab} (which transforms according to (2.1) in each spinor index) and seeks a Lagrangian of the form

$$\mathscr{L} = i\hbar\psi^*_{ab}A_{aa'bb'}\frac{\partial}{\partial t}\psi_{a'b'} + \psi^*_{ab}\boldsymbol{B}_{aa'bb'}\cdot\frac{\hbar}{i}\nabla\psi_{a'b'} + m\psi^*_{ab}C_{aa'bb'}\psi_{a'b'} \tag{3.1}$$

where $A_{aa'bb'}$, $\boldsymbol{B}_{aa'bb'}$ and $C_{aa'bb'}$ are to be determined so as to guarantee the Galilean invariance of (3.1). Using the fact that $\frac{1}{2}(1+\varrho_3)$ is a scalar matrix it is clear that one Galilean invariant possibility for these matrices is

$$\begin{aligned}
A_{aa'bb'} &= \frac{1}{4}(1+\varrho_3)_{aa'}(1+\varrho_3)_{bb'} \\
\boldsymbol{B}_{aa'bb'} &= (\varrho_1\boldsymbol{\sigma})_{aa'}\frac{1}{4}(1+\varrho_3)_{bb'} + \frac{1}{4}(1+\varrho_3)_{aa'}(\varrho_1\boldsymbol{\sigma})_{bb'}, \\
C_{aa'bb'} &= \frac{1}{4}(1+\varrho_3)_{aa'}(1-\varrho_3)_{bb'} + \frac{1}{4}(1-\varrho_3)_{aa'}(1+\varrho_3)_{bb'}
\end{aligned} \tag{3.2}$$

or

$$\mathscr{L} = \frac{1}{2}\psi^*_{ab}[G_{aa'}\Gamma_{bb'} + \Gamma_{aa'}G_{bb'}]\psi_{a'b'} \tag{3.3}$$

where we have introduced the shorthand notation

$$\Gamma = \frac{1}{2}(1+\varrho_3)$$

for the invariant matrix together with the Galilean operator G as defined in the preceding section. The task of actually demonstrating that (3.3) is the most general possible form of $\mathcal{L}$ is a considerably more formidable task. In fact a tedious calculation shows that if one requires only invariance under orthochronous Galilean transformations, one has two possible additional terms in $\mathcal{L}$, i.e. one can add to the matrices $\boldsymbol{B}$ and C the following combinations

$$\boldsymbol{B} = \lambda_1 \left\{ [\varrho_2]_{aa'} \left[\frac{1}{2}(1+\varrho_3)\boldsymbol{\sigma}\right]_{bb'} + \left[\frac{1}{2}(1+\varrho_3)\boldsymbol{\sigma}\right]_{aa'} [\varrho_2]_{bb'} \right\},$$

$$C = \lambda_1 \{[\varrho_1]_{aa'} [\varrho_2]_{bb'} + [\varrho_2]_{aa'} [\varrho_1]_{bb'}\} + \lambda_2 [\varrho_2]_{aa'} [\varrho_2]_{bb'} .$$

Although the terms proportional to λ_1 are eliminated by imposing the requirement of time reversal invariance there is no corresponding argument which allows one to discard the λ_2 part of C. It will be seen, however, that a nontrivial theory is compatible only with the choice $\lambda_2 = 0$.

One can now introduce an explicit representation for the symmetric spinor ψ by writing

$$\psi = \left[\boldsymbol{X} \cdot \boldsymbol{\sigma} \frac{1}{2}(1+\varrho_3) + \frac{1}{2} \boldsymbol{Y} \cdot \boldsymbol{\sigma} \varrho_1 + \frac{1}{2} \boldsymbol{L} \cdot \boldsymbol{\sigma}(1-\varrho_3) + \frac{1}{2} Z \varrho_2 \right] \sigma_2 .$$

The insertion of this form into the Lagrangian can readily be shown to yield

$$\mathcal{L} = \boldsymbol{X}^* \cdot i\hbar \frac{\partial}{\partial t} \boldsymbol{X} + \hbar [\boldsymbol{X}^* \cdot \nabla Z - Z^* \nabla \cdot \boldsymbol{X} + \boldsymbol{X}^* \cdot \nabla \times \boldsymbol{Y} + \boldsymbol{Y}^* \cdot \nabla \times \boldsymbol{X}]$$
$$+ 2m(\boldsymbol{Y}^* \cdot \boldsymbol{Y} + Z^* Z) + \lambda_2 m [\boldsymbol{Y}^* \cdot \boldsymbol{Y} - Z^* Z - 2(\boldsymbol{L}^* \cdot \boldsymbol{X} + \boldsymbol{X}^* \cdot \boldsymbol{L})]$$

where it is to be noted that $\boldsymbol{L}$ appears only in the λ_2 term. Since variation of $\boldsymbol{L}$ leads to the condition $\lambda_2 \boldsymbol{X} = 0$, this immediately illustrates the asserted absence of nontrivial solutions for λ_2 different from zero. We consequently take $\lambda_2 = 0$ and obtain the equations

$$i\hbar \frac{\partial}{\partial t} \boldsymbol{X} + \hbar \nabla Z + \hbar \nabla \times \boldsymbol{Y} = 0 , \qquad (3.4\text{a})$$

$$\hbar \nabla \times \boldsymbol{X} + 2m \boldsymbol{Y} = 0 , \qquad (3.4\text{b})$$

$$-\hbar \nabla \cdot \boldsymbol{X} + 2mZ = 0 . \qquad (3.4\text{c})$$

Since (3.4b) and (3.4c) serve to define $\boldsymbol{Y}$ and Z respectively in terms of the curl and divergence of $\boldsymbol{X}$, it is clear that (3.4a) is the only true equation of motion. Thus this theory describes vector mesons with the appropriate three degrees of freedom. It is also important to note that the ten components which are relevant to the description of the vector meson in special relativity and in Levy-Leblond's Galilean theory have been reduced to seven in the present formulation.

It is now easy to verify that the set (3.4) implies the Schrödinger equation for all seven components. Furthermore in the presence of a magnetic field $\boldsymbol{H}$ the minimally coupled equations

$$EX + i\Pi Z + i\boldsymbol{\Pi} \times \boldsymbol{Y} = 0\,,$$
$$i\boldsymbol{\Pi} \times \boldsymbol{X} + 2m\boldsymbol{Y} = 0\,,$$
$$-i\boldsymbol{\Pi} \cdot \boldsymbol{X} + 2mZ = 0$$

are readily combined to yield

$$\left(E - \frac{1}{2m}\Pi^2\right)X_i + i\varepsilon_{ijk}\frac{e}{2m}H_j X_k = 0$$

or

$$\left(E - \frac{1}{2m}\Pi^2\right)\boldsymbol{X} + \frac{e}{2m}(\boldsymbol{t} \cdot \boldsymbol{H})\boldsymbol{X} = 0$$

where $\boldsymbol{t}$ is the usual set of spin one matrices. This clearly gives the same intrinsic magnetic moment as predicted in the relativistic theory and has the further advantage of failing to give the additional derivative coupling terms found by Levy-Leblond [3].

Finally it is of interest to note that if one identifies $\boldsymbol{X}$, $\boldsymbol{Y}$ and Z with $\boldsymbol{U}$, $\boldsymbol{M}$ and W of Levy-Leblond's formulation, Eqs. (3.4) are essentially identical to the Eqs. (A), (B) and (C) of the preceding section. Thus the failure of his approach to adequately describe electromagnetic coupling is in essence a consequence of the difficulty of identifying the appropriate minimal set of defining equations for the vector meson. This is, of course, a feature of the relativistic Bargmann-Wigner method as well although in that case the existence of the tensorial representations enormously simplifies the task of selecting the appropriate subset of the complete Bargmann-Wigner equations. In the following section it is shown that there exists an alternative formulation of the Bargmann-Wigner method in the Galilean case which aids considerably in the difficult process of extracting the defining equations of the theory.

IV. The Bargmann-Wigner Equations

In order to motivate the derivation of an appropriate set of Bargmann-Wigner equations it is of interest to consider briefly the more familiar relativistic case. In that example one considers a symmetric multispinor

$\psi_{a_1 a_2 \dots a_N}$ and writes as a possible Lagrangian the form

$$\mathscr{L} = \psi^*_{a_1 \dots a_N} \left[\sum_{i=1}^{N} \beta_{a_1 a'_1} \cdots \beta_{a_{i-1} a'_{i-1}} \beta_{a_{i+1} a'_{i+1}} \cdots \beta_{a_N a'_N} (\beta D)_{a_i a'_i} \right] \psi_{a'_1 \dots a'_N} \tag{4.1}$$

where D is the Dirac operator

$$D = \gamma p + m .$$

The above structure is clearly Lorentz invariant and one can consequently infer the covariant equation

$$\sum_{i=1}^{N} D_{a_i a'_i} \psi_{a_1 \dots a_{i-1} a'_i a_{i+1} \dots a_N} = 0 \tag{4.2}$$

where essential use has been made of the fact that β is a nonsingular matrix. For the case $N = 2$ one can show that (4.2) implies

$$D_{a_1 a'_1} \psi_{a'_1 a_2} = D_{a_2 a'_2} \psi_{a_1 a'_2} = 0$$

a result which is suggestive of the general Bargmann-Wigner set

$$D_{a_i a'_i} \psi_{a_1 \dots a_{i-1} a'_i a_{i+1} \dots a_N} = 0 . \tag{4.3}$$

It is, however, important to note that except for the case $N = 2$, Eq. (4.3) cannot be derived from a local Lagrangian such as (4.1) unless one is willing to allow the introduction of auxiliary fields [5].

Proceeding now to the Galilean case it is clear that the extension of (3.3) to a general symmetric multispinor is

$$\mathscr{L} = \frac{1}{N} \psi^*_{a_1 \dots a_N} \left[\sum_{i=1}^{N} \Gamma_{a_1 a'_1} \cdots \Gamma_{a_{i-1} a'_{i-1}} G_{a_i a'_i} \Gamma_{a_{i+1} a'_{i+1}} \cdots \Gamma_{a_N a'_N} \right] \psi_{a'_1 \dots a'_N} .$$

The Galilean invariance of this form is readily verified and one thus has the equations

$$\sum_{i=1}^{N} \Gamma_{a_1 a'_1} \cdots \Gamma_{a_{i-1} a'_{i-1}} G_{a_i a'_i} \Gamma_{a_{i+1} a'_{i+1}} \cdots \Gamma_{a_N a'_N} \psi_{a'_1 \dots a'_N} = 0 \tag{4.4}$$

as the Galilean analogue of (4.2). There is however, the crucial difference here that the scalar matrix Γ which appears in (4.4) (unlike the matrix β of the relativistic case) is singular and one cannot eliminate it from the equations of motion. One can, however, infer from (4.4) a Bargmann-Wigner form by noting that with the definition

$$\begin{aligned} \tilde{G} &= \varrho_2 G \varrho_2 \\ &= \frac{1}{2}(1 - \varrho_3) E - \varrho_1 \boldsymbol{\sigma} \cdot \boldsymbol{p} + m(1 + \varrho_3) \end{aligned}$$

one has

$$\tilde{G}G = 2m(E - p^2/2m)$$
$$= 2mS$$

where we have introduced the Schrödinger operator $S \equiv E - p^2/2m$. For the case $N = 2$ multiplication by $\Gamma\, G$ yields

$$[2mS\Gamma_{aa'}\Gamma_{bb'} + (\Gamma\,\tilde{G}\,\Gamma)_{aa'}G_{bb'}]\,\psi_{a'b'} = 0$$

or

$$[S\Gamma_{aa'}\Gamma_{bb'} + \Gamma_{aa'}G_{bb'}]\,\psi_{a'b'} = 0\,. \tag{4.5}$$

The corresponding result on the second index of ψ allows one to infer

$$\Gamma_{aa'}G_{bb'}\psi_{a'b'} = \Gamma_{bb'}G_{aa'}\psi_{a'b'}$$

and the desired Bargmann-Wigner result

$$\Gamma_{aa'}G_{bb'}\psi_{a'b'} = \Gamma_{bb'}G_{aa'}\psi_{a'b'} = 0\,. \tag{4.6}$$

In view of the existence of the corresponding relativistic equations for $N = 2$ it is not at all surprising that a result of the form (4.6) can be obtained. It is, however, interesting to note that the above can be extended to all N. To this end one notes that the generalization of (4.5) to arbitrary N is

$$\left\{\begin{matrix} 2mS\Gamma_{a_1a'_1}\cdots\Gamma_{a_Na'_N} + (\Gamma\,\tilde{G}\,\Gamma)_{a_ia'_i} \\ \cdot\sum\limits_{j\neq i}\Gamma_{a_1a'_1}\cdots\Gamma_{a_{j-1}a'_{j-1}}G_{a_ja'_j}\Gamma_{a_{j+1}a'_{j+1}}\cdots\Gamma_{a_Na'_N}\end{matrix}\right\}\psi_{a'_1\cdots a'_N}$$

or

$$\left\{\begin{matrix} S\Gamma_{a_1a'_1}\cdots\Gamma_{a_Na'_N} \\ +\sum\limits_{j\neq i}\Gamma_{a_1a'_1}\cdots\Gamma_{a_{j-1}a'_{j-1}}G_{a_ja'_j}\Gamma_{a_{j+1}a'_{j+1}}\cdots\Gamma_{a_Na'_N}\end{matrix}\right\}\psi_{a'_1\ldots a'_N}\,. \tag{4.7}$$

By considering two different values of the index i one readily infers from (4.7) the result

$$\Gamma_{a_1a'_1}\cdots\Gamma_{a_{i-1}a'_{i-1}}G_{a_ia'_i}\Gamma_{a_{i+1}a'_{i+1}}\cdots\Gamma_{a_Na'_N}\psi_{a'_1\ldots a'_N}$$
$$= \Gamma_{a_1a'_1}\cdots\Gamma_{a_{j-1}a'_{j-1}}G_{a_ja'_j}\Gamma_{a_{j+1}a'_{j+1}}\cdots\Gamma_{a_Na'_N}\psi_{a'_1\ldots a'_N}$$

and one has as the alternative to Levy-Leblond's form of the Bargmann-Wigner set the equations

$$\Gamma_{a_1a'_1}\cdots\Gamma_{a_{i-1}a'_{i-1}}G_{a_ia'_i}\Gamma_{a_{i+1}a'_{i+1}}\cdots\Gamma_{a_Na'_N}\psi_{a'_1\ldots a'_N} = 0\,. \tag{4.8}$$

Since this form is derived from the Lagrangian formalism it has the enormous advantage that the number of components in this set will be precisely the same as in the strictly Lagrangian approach. This can be readily verified for the case of the vector meson by noting that the Eqs. (E), (F) and (G) of Section 2 do not occur when one uses (4.8) and the only additional equation relative to the approach of the preceding section is (D) (which of course, is a consequence of (A)). Thus one again has a theory of the vector meson involving only the seven components appropriate to Galilean relativity.

As a final comment it is perhaps useful to point out that the above derivation of the Bargmann-Wigner equations cannot accommodate minimal coupling. In that case one has the choice of using the entire Lagrangian formalism or extracting the minimal set of equations from the Bargmann-Wigner set before the introduction of the electromagnetic coupling. (In the case of the vector meson, this merely requires the discarding of (D).) However, even this latter approach has been reduced to manageable proportions by the derivation of the form (4.8) for the Bargmann-Wigner equations since one no longer has the problem generated by the occurrence of spurious components in the wave equations.

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