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Continuous Tensor Product States which are Translation Invariant but not Quasi-Free

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Abstract. We show how the theory of continuous tensor products can be used to construct, for commutation relations, translation invariant but not quasi-free states as continuous tensor products of states for systems with one degree of freedom.

Introduction

As was shown by R. T. Powers in [6] § 5.3 for the case of anticommutation relations, all translation invariant states which can be constructed as infinite tensor products of states for systems with a finite number of degrees of freedom are quasi-free and consequently not very interesting for physical applications; in this paper we show how the theory of continuous tensor products allows us to construct, in the case of commutation relations, translation invariant but not quasi-free states as continuous tensor products of states for systems with one degree of freedom; we consider only the nonrelativistic case since, unfortunately, we are not able to carry out the same construction in the relativistic case.

§ 1. The Algebras Associated with a Real Symplectic Space

We consider a real symplectic space (E, σ), i.e. a real vector space E with a non-degenerate symplectic form σ; we call *representation of* (E, σ) every mapping U of E into the unitary operators of a complex Hilbert space such that

(i) for each x in E the mapping $\mathbb{R} \ni h \mapsto U(hx)$ is strongly continuous

(ii) $U(x+y) = e^{i\sigma(x,y)} U(x)\, U(y)$.

With a real symplectic space one can associate several algebras:

1) The von Neumann algebra $\mathscr{A}_{E,\sigma}$ defined in [2], § 1.3; when E is finite dimensional $\mathscr{A}_{E,\sigma}$ is nothing but $\mathscr{L}(H)$ where H is the space

of the Schrödinger representation of (E, σ); in the general case $\mathscr{A}_{E,\sigma}$ is the von Neumann inductive limit of the algebras $\mathscr{A}_{F,\sigma}$ with F a finite dimensional subspace of E. There is a representation W of (E, σ) into $\mathscr{A}_{E,\sigma}$ which has the following universal property: given a Hilbert space H, the mapping $\pi \mapsto \pi \circ W$ is a bijection between the normal representations of $\mathscr{A}_{E,\sigma}$ in H and the representations of (E, σ) in H.

2) The Banach $*$-algebra $A_{E,\sigma}$ (which is similar to the algebra considered in [5]; $A_{E,\sigma}$ is the Banach space $l^1(E)$ whose elements are complex functions on E satisfying $\sum_{x\in E} |f(x)| < \infty$, equipped with the norm

$$\|f\| = \sum |f(x)|,$$

the multiplication

$$(fg)(z) = \sum_{x+y=z} e^{-i\sigma(x,y)} f(x)\, g(y)$$

and the involution

$$f^*(x) = \overline{f(-x)};$$

we denote by δ_x the unitary element of $A_{E,\sigma}$ defined by

$$\delta_x(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x; \end{cases}$$

then

$$\delta_{x+y} = e^{i\sigma(x,y)} \delta_x \delta_y;$$

given a Hilbert space H, the mapping $\pi \mapsto \pi \circ \delta$ is a bijection between the representations of $A_{E,\sigma}$ in H such that $h \mapsto \pi(\delta_{hx})$ is strongly continuous for each $x \in E$ and the representations of (E, σ) in H. In particular there exists a unique morphism $T: A_{E,\sigma} \mapsto \mathscr{A}_{E,\sigma}$ such that the diagramm

$$\begin{array}{ccc} A_{E,\sigma} & \xrightarrow{T} & \mathscr{A}_{E,\sigma} \\ {}_{\delta}\nwarrow & & \nearrow_{W} \\ & E & \end{array}$$

is commutative; $\operatorname{Im} T$ is strongly dense in $\mathscr{A}_{E,\sigma}$.

Concerning the states of $\mathscr{A}_{E,\sigma}$ and $A_{E,\sigma}$ there are bijective correspondences between

a) the complex functions ψ on E satisfying the following conditions

— $\psi(O) = 1$

— $\sum_{n,p} c_n \bar{c}_p e^{i\sigma(x_n, x_p)} \psi(x_n - x_p) \geqq 0 \quad \forall\, c_1, \ldots c_m \in \mathbb{C}, \quad x_1, \ldots x_m \in E$

– for each $x \in E$ the mapping $\mathbb{R} \quad h \mapsto \psi(hx)$ is continuous; such a function ψ will be called a *generating functional*;

b) the normal states φ of $\mathscr{A}_{E,\sigma}$;

c) the states χ of $A_{E,\sigma}$ satisfying: for each $x \in E$ the mapping $h \mapsto \chi(\delta_{hx})$ is continuous.

These correspondences are given by $\psi = \varphi \circ W = \chi \circ \delta$, $\chi = \varphi \circ T$.

§ 2. A Particular Case of Real Symplectic Space

From now on we suppose E is a complex vector space of complex functions on $T=\mathbb{R}^n$ which are continuous and with compact support; and we set

$$\sigma(x,y)=\operatorname{Im}(x|y)=\operatorname{Im}\int x(t)\,\overline{y(t)}\,dt \qquad \forall\, x,y\in E;$$

we also suppose E is invariant under all translations. For every t in T we set

$$\begin{aligned} E_t&=\mathbb{C}, \\ \sigma_t(\alpha,\beta)&=\operatorname{Im}\alpha\overline{\beta} \qquad \forall\,\alpha,\beta\in E_t, \\ \mathscr{A}_t&=\mathscr{A}_{E_t,\sigma_t}, \\ W_t&=\text{the canonical mapping } E_t\to\mathscr{A}_t, \\ A_t&=A_{E_t,\sigma_t}. \end{aligned}$$

((E_t,σ_t) is the symplectic space corresponding to a system with one degree of freedom.)

Proposition 1. *$A_{E,\sigma}$ is isomorphic to the continuous tensor product of the algebras A_t; more precisely we have $A_{E,\sigma}\sim\widehat{\bigotimes_{t\in T}}{}^{\Gamma}A_t$ where Γ is the set of all families $t\mapsto\lambda(t)\,\delta_{x(t)}\in A_t$ with $\lambda\in C_0\cap L^1+1$ and $x\in E$.*

(We use the notations and definitions of [2], Ch. 3.)

First one must prove that $((A_t)_{t\in T},\Gamma)$ is a continuous family of Banach $*$-algebras in the sense of [2], § 3.4; the proof of the axiom (iii) of [2], § 3.2 is very similar to that of [3], prop. 12; the proof of the other axioms is trivial. Now the construction of the isomorphism is similar to that in [3], Prop. 12; we only emphasize the fact that this isomorphism F carries each element $\delta_x\in A_{E,\sigma}$ with $x\in E$, into the element $\otimes\,\delta_{x(t)}\in\hat{\otimes}^{\Gamma}A_t$; we also recall that for each λ in $\mathscr{C}_0\cap L^1+1$,

$$\otimes\,\lambda(t)\cdot\delta_{x(t)}=\Pi\,\lambda(t)\cdot\otimes\,\delta_{x(t)};$$

in particular if $x,y\in E$:

$$\begin{aligned} \otimes\,\delta_{x(t)}\cdot\otimes\,\delta_{y(t)}&=\otimes\,\delta_{x(t)}\delta_{y(t)} \\ &=\otimes\,e^{-ix(t)\overline{y(t)}}\delta_{x(t)+y(t)} \\ &=\Pi\,e^{-ix(t)\overline{y(t)}}\otimes\,\delta_{x(t)+y(t)} \\ &=e^{-i\sigma(x,y)}\otimes\,\delta_{x(t)+y(t)} \\ F^{-1}(\otimes\,\delta_{x(t)}\cdot\otimes\,\delta_{y(t)})&=e^{-i\sigma(x,y)}\delta_{x+y}=\delta_x\delta_y. \quad \text{QED.} \end{aligned}$$

Another Algebra Associated with (E, σ)

As explained in [2], § 3.6 we can also construct the continuous tensor product $\widehat{\bigotimes_{t \in T}}^{\Gamma'} \mathscr{A}_t$ where Γ' is the set of all families $t \mapsto \lambda(t) \cdot W_t(x(t))$ with $\lambda \in \mathscr{C}_0 \cap L^1 + 1$ and $x \in E$; we denote it by $\mathscr{A}'_{E,\sigma}$ and set

$$W'(x) = \otimes W_t(x(t)) \qquad \forall x \in E;$$

we have

$$W'(x+y) = e^{i\sigma(x,y)} W'(x) W'(y);$$

moreover there is a morphism $S: \mathscr{A}'_{E,\sigma} \to \mathscr{A}_{E,\sigma}$ such that the diagramm

$$\begin{array}{ccc} \mathscr{A}'_{E,\sigma} & \xrightarrow{S} & \mathscr{A}_{E,\sigma} \\ {\scriptstyle W'}\nwarrow & & \nearrow{\scriptstyle W} \\ & E & \end{array}$$

is commutative.

Automorphisms of the Above Algebras Induced by Translations

Every element τ of T determines an automorphism of (E, σ):

$$x \mapsto x_\tau \quad \text{with} \quad x_\tau(t) = x(t - \tau);$$

this automorphism determines in turn, as easily seen, automorphisms $\alpha_\tau, \beta_\tau, \gamma_\tau$ of $\mathscr{A}_{E,\sigma}, A_{E,\sigma}, \mathscr{A}'_{E,\sigma}$ respectively, such that

$$\alpha_\tau(W(x)) = W(x_\tau)$$
$$\beta_\tau(\delta_x) = \delta_{x_\tau}$$
$$\gamma_\tau(W'(x)) = W'(x_\tau);$$

recalling that $A_{E,\sigma}$ and $\mathscr{A}'_{E,\sigma}$ are continuous tensor products, β_τ and γ_τ take the simpler forms:

$$\beta_\tau(\otimes \delta_{x(t)}) = \otimes \delta_{x(t-\tau)},$$
$$\gamma_\tau(\otimes W_t(x(t))) = \otimes W_{t-\tau}(x(t-\tau)).$$

These automorphisms are compatible with the canonical mappings $A_{E,\sigma} \to \mathscr{A}_{E,\sigma}$ and $\mathscr{A}'_{E,\sigma} \to \mathscr{A}_{E,\sigma}$.

§ 3. Continuous Tensor Products of States

Consider a generating functional ψ on (E, σ) of the form

$$\psi(x) = \exp\left[\int F_t(x(t))\, dt\right]$$

where each F_t is a continuous complex function on $E_t = \mathbb{C}$ with the following properties:

(i) $F_t(O) = O$;

(ii) the function $\alpha \mapsto \psi_t(\alpha) = e^{F_t(\alpha)}$ is a generating functional on (E_t, σ_t);

(iii) for every $x \in E$ the function $t \mapsto F_t(x(t))$ belongs to $\mathscr{C}_0 \cap L^1$.

The state χ of $A_{E,\sigma}$ associated with ψ is the continuous tensor product of the states χ_t (of A_t) associated with ψ_t; in fact for $x \in E$

$$\begin{aligned}\chi(\delta_x) = \psi(x) &= \exp[\int F_t(x(t))\, dt] \\ &= \prod_{t \in T} \exp[F_t(x(t))] = \prod_{t \in T} \psi_t(x(t)) \\ &= \prod_{t \in T} \chi_t(\delta_{x(t)});\end{aligned}$$

but as we know δ_x is identified with $\otimes\, \delta_{x(t)}$ (cf. Prop. 1). (There is a similar result for the state of $\mathscr{A}'_{E,\sigma}$ associated with ψ). Moreover the representation associated with ψ is a continuous tensor product in the sense of [4].

If moreover F_t is equal to some function F independent of t, the state χ is obviously translation invariant, i.e. invariant under all the automorphisms β_τ.

Examples. Let F^0 be a complex continuous function on $\mathbb{C}$ verifying

a) $F^0(O) = O$,

b) $\exp F^0$ is positive definite,

c) the function ψ^0 on E defined by $\psi^0(x) = \exp[\int F^0(x(t))\, dt]$ is positive definite;

set

$$F(\alpha) = -\tfrac{1}{2}|\alpha|^2 + F^0(\alpha) \qquad \forall\, \alpha \in \mathbb{C};$$

then conditions (i) and (iii) above are trivially satisfied; as for condition (ii), it is known and easily verified that $\alpha \mapsto \exp(-\tfrac{1}{2}|\alpha|^2)$ is a generating functional on (E_t, σ_t) (the corresponding state is the Fock state; see also [2], § 1.5); then for every $\alpha_1, \ldots \alpha_m$ in $\mathbb{C}$, the matrix with coefficients

$$\exp(i\alpha_n \bar{\alpha}_p) \cdot \psi_t(\alpha_n - \alpha_p) = \exp(i\alpha_n \bar{\alpha}_p - \tfrac{1}{2}|\alpha_n - \alpha_p|^2) \cdot \exp(F^0(\alpha_n - \alpha_p))$$

is positive since the coefficientwise product of two positive matrices is positive. Finally the same arguments prove that the function

$$\begin{aligned}x \mapsto \psi(x) &= \exp[\int F(x(t))\, dt] \\ &= \exp(-\tfrac{1}{2}\|x\|^2 + \int F^0(x(t))\, dt)\end{aligned}$$

is a generating functional on (E, σ); we can thus construct many continuous tensor product states which are translation invariant.

In particular we can take F^0 of the following form:

$$F^0(\alpha) = -u^2|\alpha|^2 + iv \cdot \alpha + \int_{\mathbb{C}} \left(e^{iw \cdot \alpha} - 1 - \frac{iw \cdot \alpha}{1 + |w|^2}\right) \frac{1 + |w|^2}{|w|^2}\, d\mu(w); \quad (1)$$

here u is real, v is complex, μ is a finite positive measure on $\mathbb{C}-O$, and

$$v\cdot\alpha=\operatorname{Re}v\operatorname{Re}\alpha+\operatorname{Im}v\operatorname{Im}\alpha$$

and similarly for $w\cdot\alpha$; conversely if E is sufficiently large, for instance if it contains all infinitely differentiable functions with compact support, every F^0 satisfying a), b), c) is of the form (1) (see for instance [1], Ch. III).

§ 4. Quasi-Free States

Definitions. Given two real vector spaces V and W denote by $\mathscr{L}(V,W)$ the vector space of all linear mappings $V\to W$; if W is a topological vector space we endow $\mathscr{L}(V,W)$ with the topology of the simple convergence; we say that a mapping $f:V\to W$ is *differentiable* if for each x in V there exists a linear mapping $f'(x;.):V\to W$ such that for every y in V:

$$h^{-1}(f(x+hy)-f(x))\to f'(x;y)\qquad\text{when } h\text{, real, tends to } O.$$

By the above procedure we can define inductively topologies on $\mathscr{L}(V,\mathbb{C})$, $\mathscr{L}(V,\mathscr{L}(V,\mathbb{C}))$, etc.; as usual $\mathscr{L}(V,\mathscr{L}(V,\mathbb{C}))$ shall be identified with the set of all bilinear mappings $V\times V\to\mathbb{C}$ and so on; we thus can speak of a mapping $f:V\to\mathbb{C}$ which is infinitely differentiable, and we have

$$\begin{aligned}&f^{(n)}(x;y_1,\dots y_n)\\&=\lim_{h=0}h^{-1}[f^{(n-1)}(x+hy_n;y_1,\dots y_{n-1})-f^{(n-1)}(x;y_1,\dots y_{n-1})];\end{aligned}$$

moreover for every $x,y_1,\dots y_n$ the function

$$(h_1,\dots h_n)\mapsto f(x+h_1y_1+\dots+h_ny_n)$$

is infinitely differentiable and we have

$$\begin{aligned}&\frac{\partial^{p_1+\cdots p_n}}{\partial h_1^{p_1}\cdots\partial h_n^{p_n}}f(x+h_1y_1+\cdots+h_ny_n)|_{h_1=\cdots=h_n=0}\\&\qquad=f^{(p_1+\dots+p_n)}(x;\underbrace{y_1,\dots y_1}_{p_1\text{-times}},\dots\underbrace{y_n,\dots y_n}_{p_n\text{-times}}).\end{aligned}\tag{2}$$

Returning to our (E,σ) we denote by E^0 the set of all real functions in E; let ψ be a generating functional such that $\psi|E^0$ is infinitely differentiable; denote by U and ξ the representation of (E,σ) and cyclic vector determined by ψ such that $\psi(x)=(U(x)\,\xi|\xi)\ \forall x\in E$; let $A(x)$ be the self-adjoint generator of the one-parameter group $h\mapsto U(hx)$.

Lemma 1. *$A(x_1)\dots A(x_n)\,\xi$ exists for every $x_1,\dots x_n$ in E^0.*

Proof. a) The domain D of $A(x)$ is the set of all η in H such that the expression $h^{-1}(U(hx)-I)\,\eta$ has a strong limit when $h\to O$; but one can replace strong by weak; in fact let D' be the set of all η such that $h^{-1}(U(hx)-I)\eta$ has a weak limit; D' is a linear subspace containing D; set

$$A'\eta = w\text{-}\lim (ih)^{-1}(U(hx)-I)\eta \quad \text{for each} \quad \eta \in D';$$

A' is easily seen to be a symmetric operator which extends $A(x)$; then $A' = A(x)$ and $D' = D$.

b) We now prove that the expression

$$B = (h_1 \ldots h_n)^{-1}(U(h_1x_1)-I)\ldots(U(h_nx_n)-I)\xi$$

has a weak limit when $h_1, \ldots h_n$ tend to O. Denoting by T the canonical mapping $A_{E,\sigma} \to H$ we have $\xi = T(\delta_0)$ and

$$\begin{aligned} B &= (h_1 \ldots h_n)^{-1} \sum_{\substack{i_1 < \cdots < i_p \\ p = 0 \cdots n}} (-1)^{n-p} U(h_{i_1}x_{i_1}) \ldots U(h_{i_p}x_{i_p}) \cdot T(\delta_0) \\ &= (h_1 \ldots h_n)^{-1} \sum (-1)^{n-p} U(h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p}) \cdot T(\delta_0) \\ &= (h_1 \ldots h_n)^{-1} \sum (-1)^{n-p} T(\delta_{h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p}}) . \end{aligned}$$

Let us prove first that $(B|T(\delta_y))$ has a limit for every y in E; we have

$$\begin{aligned} (B|T(\delta_y)) &= (h_1 \ldots h_n)^{-1} \sum (-1)^{n-p} \chi(\delta_{-y}\delta_{h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p}}) \\ &= (h_1 \ldots h_n)^{-1} \sum (-1)^{n-p} \exp[i\sigma(y, h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p}) \\ &\quad \cdot \psi(-y + h_{i_1}x_{i_1} + \cdots) \\ &= (h_1 \ldots h_n)^{-1} \sum (-1)^{n-p} \varphi(O, \ldots h_{i_1}, O, \ldots h_{i_p}, \ldots O) \end{aligned} \tag{3}$$

where we have set

$$\varphi(h_1, \ldots h_n) = \exp[i\sigma(y, h_1x_1 + \cdots + h_nx_n)] \cdot \psi(-y + h_1x_1 + \cdots + h_nx_n);$$

it is known (and easily verified) that (3) converges to

$$\left. \frac{\partial^n \varphi}{\partial h_1 \ldots \partial h_n} \right|_{h_1 = \cdots = h_n = 0} .$$

Now to prove b) it is sufficient, since the $T(\delta_y)$'s are total in H, to prove that B is bounded; we have

$$\begin{aligned} \|B\|^2 &= (h_1 \ldots h_n)^{-2} \sum_{\substack{i_1 < \cdots < i_p \\ j_1 < \cdots < j_q \\ p, q = 0, \ldots n}} (-1)^{p+q} (T(\delta_{h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p}}) | T(\delta_{h_{j_1}x_{j_1} + \cdots + h_{j_q}x_{j_q}})) \\ &= (h_1 \ldots h_n)^{-2} \sum (-1)^{p+q} \psi(h_{i_1}x_{i_1} + \cdots + h_{i_p}x_{i_p} - h_{j_1}x_{j_1} - \cdots - h_{j_q}x_{j_q}); \end{aligned}$$

writing out an expansion of the $\sum$ and using (2) one can see that the only terms which really occur contain $h_1^{a_1} \ldots h_n^{a_n}$ where $a_1, \ldots a_n$ are non zero even integers; this establishes our assertion.

c) By the part b) we know that $A(x_n)\,\xi$ exists; then

$$\begin{aligned}\underset{h_{n-1}=h_n=0}{w\text{-lim}}\ & h_{n-1}^{-1}(U(h_{n-1}x_{n-1})-I)\cdot h_n^{-1}(U(h_n x_n)-I)\cdot\xi\\ &=\underset{h_{n-1}=0}{w\text{-lim}}\ \underset{h_n=0}{w\text{-lim}}\,(\text{the same expression})\\ &=\underset{h_{n-1}=0}{w\text{-lim}}\, h_{n-1}^{-1}(U(h_{n-1}x_{n-1})-I)\cdot A(x_n);\end{aligned}$$

this proves that $A(x_{n-1})\,A(x_n)\,\xi$ exists; and so on inductively. QED.

By the above lemma we may consider the multilinear forms on E^0

$$(x_1,\dots x_n)\mapsto(A(x_1)\dots A(x_n)\,\xi\,|\,\xi);$$

they are called *Wightman distributions* and denoted by $\mathscr{W}_n$; we have

$$\begin{aligned}\psi(h_1x_1+\cdots+h_nx_n)&=(U(h_1x_1+\cdots+h_nx_n)\,\xi\,|\,\xi)\\ &=(U(h_1x_1)\dots U(h_nx_n)\,\xi\,|\,\xi)\end{aligned}$$

whence, by (2)

$$\begin{aligned}\psi^{(n)}(0;x_1,\dots x_n)&=\frac{\partial^n}{\partial h_1\dots\partial h_n}\psi(h_1x_1+\cdots+h_nx_n)|_{h_1=\cdots=h_n=0}\\ &=i^n\,\mathscr{W}_n(x_1,\dots x_n).\end{aligned}\tag{4}$$

Then one defines the *truncated Wightman distributions* $\mathscr{W}_n^T$ by the following recurrence formulae

$$\mathscr{W}_1^T=\mathscr{W}_1,$$

$$\mathscr{W}_n(x_1,\dots x_n)=\sum_{\mathscr{P}}\mathscr{W}_{n_1}^T(x_{i_{1,1}},\dots x_{i_{1,n_1}})\dots\mathscr{W}_{n_r}^T(x_{i_{r,1}},\dots x_{i_{r,n_r}})$$

where the sum is taken for all partitions $\mathscr{P}$ of the set $\{1,2,\dots n\}$ into subsets

$$i_{1,1}<i_{1,2}<\cdots<i_{1,n_1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$i_{r,1}<i_{r,2}<\cdots<i_{r,n_r}$$

with $n_1+\cdots+n_r=n$.

The state associated with ψ is said *quasi-free* if $\mathscr{W}_n^T=0\ \forall\, n\geqq 3$ (cf. [7]).

Let us now suppose that ψ has the form $\psi(x)=e^{\omega(x)}$ where ω is an infinitely differentiable mapping $E\to\mathbb{C}$, with $\omega(0)=0$; we have

$$\begin{aligned}\psi'(x;y)&=e^{\omega(x)}\cdot\omega'(x;y),\\ \psi''(x;y_1,y_2)&=e^{\omega(x)}[\omega'(x;y_1)\,\omega'(x;y_2)+\omega''(x;y_1,y_2)]\end{aligned}$$

and by induction

$$\psi^{(n)}(x;y_1,\dots y_n)=e^{\omega(x)}\sum_{\mathscr{P}}\omega^{(n_1)}(x;y_{i_{1,1}},\dots y_{i_{1,n_1}})\dots\omega^{(n_r)}(x;y_{i_{r,1}},\dots);$$

it follows that, by (4)

$$\mathscr{W}_n^T(x_1, \dots x_n) = i^n \omega^{(n)}(O; x_1, \dots x_n). \tag{5}$$

Assume now that ω has the form $\omega(x) = \int F((x(t))\,dt$ where F is a complex function on $\mathbb{C}$ whose restriction to $\mathbb{R}$ is infinitely differentiable; then, for $x, y_1, \dots y_n \in E^0$ we have

$$\omega(x + h y_1) = \int F(x(t) + h y_1(t))\,dt$$

and by derivation under $\int$:

$$\omega'(x; y_1) = \frac{d}{dh}\,\omega(x + h y_1)|_{h=0} = \int y_1(t)\,F'(x(t))\,dt;$$

then by induction

$$\omega^{(n)}(x; y_1, \dots y_n) = \int F^{(n)}(x(t)) \cdot y_1(t) \dots y_n(t) \cdot dt;$$

by (5)

$$\mathscr{W}_n^T(x_1, \dots x_n) = i^{-n} \cdot F^{(n)}(O) \cdot \int x_1(t) \dots x_n(t) \cdot dt.$$

We have thus proved the following:

Proposition 2. *The state associated with a generating functional ψ of the form $\psi(x) = \exp[\int F(x(t))\,dt]$ with $F|\mathbb{R}$ infinitely differentiable, is quasi-free if and only if $F^{(n)}(O) = O$ $\forall n \geqq 3$.*

Examples. We take $F(\alpha) = -\frac{1}{2}|\alpha|^2 + F^0(\alpha)$ where F^0 is given by (1), and suppose that

$$\int |w|^n d\mu(w) < +\infty \qquad \forall n = 1, 2, \dots;$$

if α is real we have, by setting $v_1 = \operatorname{Re} v$, $w_1 = \operatorname{Re} w$:

$$F(\alpha) = -\frac{1}{2}\alpha^2 - u^2\alpha^2 + i v_1 \alpha + \int \left(e^{i w_1 \alpha} - 1 - \frac{i w_1 \alpha}{1 + |w|^2}\right) \frac{1 + |w|^2}{|w|^2}\,d\mu(w);$$

whence, for $n \geqq 3$:

$$F^{(n)}(O) = i^n \int w_1^n \frac{1 + |w|^2}{|w|^2}\,d\mu(w);$$

we see that the corresponding state is not quasi-free unless μ is null.

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