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Disjointness of the KMS-States of Different Temperatures

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Abstract. Disjointness of (KMS)-states of different temperatures is proved.

Let A be a C^*-algebra with a one parameter automorphism group σ_t. A state φ of A is said to satisfy the *Kubo-Martin-Schwinger (KMS) boundary condition* for $\beta > 0$ if for every pair x, y in A there exists a function $F(z)$ holomorphic in the strip: $0 < \operatorname{Im} z < \beta$ with boundary values:

$$F(t) = \varphi(\sigma_t(x)y) \quad \text{and} \quad F(t + i\beta) = \varphi(y\sigma_t(x)). \tag{1}$$

If we assume the boundedness of the relevant function F on the whole strip: $0 \leqq \operatorname{Im} z \leqq \beta$, the condition (1) implies the σ_t-invariance of φ by Sturm's Theorem, as is shown by Winnink [11].

In quantum thermodynamics, the above β is given by $\beta = 1/kT$, where k is the Boltzmann constant and T is the absolute temperature of the system. Recently, a great deal of progress on the KMS boundary condition has been done by several physicists, for example, [1, 2, 4, 6, 7, and 11].

From the purely mathematical point of view, the author has shown recently in [9] that to every faithful normal state φ of a von Neumann algebra M there corresponds a unique one-parameter automorphism group σ_t^φ of M with respect to which φ satisfies the KMS boundary condition for $\beta = 1$. The proof is based on Tomita's theory [9, 10]. This σ_t^φ is called the *modular automorphism group* of M associated with φ.

Therefore, the following question naturally comes into consideration: *How does the modular automorphism group σ_t^φ depend on a normal faithful state φ? What changes will occur in the modular automorphism group σ_t^φ for different normal faithful states?*

In this paper, we shall show the relation between σ_t^φ and σ_t^ψ for two normal faithful states φ and ψ commuting in the sense of [9: Definition 15.1], that is, when $\varphi + i\psi$ and $\varphi - i\psi$ have the same absolute value in the sense of the polar decomposition. As an application, it is shown that if M is of type III, and ψ satisfies the KMS condition with respect to the modular automorphism σ_t^φ associated with a faithful normal state φ for some β, then $\beta = 1$ and $\sigma_t^\varphi = \sigma_t^\psi$.

The relation of σ_t^φ and σ_t^ψ for general pair φ, ψ will be discussed in a separate paper.

Before going into the discussion, the author would like to express his thanks to Professor H. A. Dye for his kind hospitality at UCLA.

Let φ be a fixed normal faithful state of a von Neumann algebra M. Considering the cyclic representation of M induced by φ, we assume that M acts on a Hilbert space $\mathscr{H}$ with a cyclic vector ξ_0 with $\varphi(x) = (x\xi_0 | \xi_0)$, $x \in M$. Put $\mathfrak{A} = M\xi_0$ and define a product and an involution in $\mathfrak{A}$ as follows:

$$(x\xi_0)(y\xi_0) = xy\xi_0, \qquad x, y \in M;$$

$$(x\xi_0)^\# = x^*\xi_0, \qquad x \in M.$$

Then, with this structure $\mathfrak{A}$ turns out to be a generalized Hilbert algebra as in [9: Theorem 12.1]. Let Δ be the modular operator of $\mathfrak{A}$. Then the modular automorphism group σ_t^φ is given by:

$$\sigma_t^\varphi(x) = \Delta^{it} x \Delta^{-it}, \qquad x \in M, \ t \in \boldsymbol{R}.$$

Let $\mathfrak{A}_0$ be the modular Hilbert algebra contained in $\mathfrak{A}$, which is constructed in [9: Theorem 10.1]. In this situation, we shall use the notations and the terminology in [9].

Let M_φ denote the set of all $x \in M$ satisfying the equality: $x\varphi = \varphi x$, that is,

$$\varphi(xy) = \varphi(yx) \quad \text{for every} \quad y \in M.$$

Then M_φ is exactly the algebra of all fixed elements of σ_t^φ by [9: Lemma 15.8].

Lemma 1. *If $h \in M_\varphi$ is positive and invertible and ψ is defined by $\psi(x) = \varphi(xh)$, $x \in M$, then σ_t^ψ is given by:*

$$\sigma_t^\psi(x) = \sigma_t^\varphi(h^{it} x h^{-it}), \qquad x \in M, \ t \in \boldsymbol{R}.$$

Proof. Since h and Δ commute, h leaves $\mathfrak{A}_0$ invariant; in particular $h\xi_0$ is in $\mathfrak{A}_0$. Take an $x \in M$ and an η in $\mathfrak{A}_0$ with $y = \pi(\eta) \in M$. Define a function $F(\alpha)$ by:

$$F(\alpha) = (\pi(\Delta^{-i\alpha}\eta) h\xi_0 | h^{i\bar{\alpha}} x^* h^{-i\bar{\alpha}} \xi_0), \qquad \alpha \in \boldsymbol{C}.$$

Then $F(\alpha)$ is analytic on the whole plane $\boldsymbol{C}$. For each $t \in \boldsymbol{R}$, we have

$$\begin{aligned}
F(t) &= (\pi(\Delta^{-it}\eta)h\xi_0 \mid h^{it}x^*h^{-it}\xi_0) \\
&= (h^{it}xh^{-it}\Delta^{-it}y\Delta^{it}h\xi_0 \mid \xi_0) \\
&= (h^{it}xh^{-it}\Delta^{-it}yh\xi_0 \mid \Delta^{-it}\xi_0) \\
&= (\Delta^{it}h^{it}xh^{-it}\Delta^{-it}yh\xi_0 \mid \xi_0);
\end{aligned}$$

$$\begin{aligned}
F(t+i) &= (\pi(\Delta^{-i(t+i)}\eta)h\xi_0 \mid h^{i(t-i)}x^*h^{-i(t-i)}\xi_0) \\
&= (\pi(\Delta^{-it+1}\eta)h\xi_0 \mid h^{it+1}x^*h^{-it-1}\xi_0) \\
&= (\Delta^{-it+1}y\Delta^{it-1}h\xi_0 \mid h^{it+1}x^*h^{-it-1}\xi_0) \\
&= (\Delta^{-it+1}yh\xi_0 \mid h^{it+1}x^*h^{-it-1}\xi_0) \\
&= (Jh^{it+1}x^*h^{-it-1}\xi_0 \mid J\Delta^{-it+1}yh\xi_0) \\
&= (\Delta^{\frac{1}{2}}Sh^{it+1}x^*h^{-it-1}\xi_0 \mid \Delta^{-it-1}Jyh\xi_0) \\
&= (\Delta^{\frac{1}{2}}h^{it-1}xh^{-it+1}\xi_0 \mid \Delta^{-it-1}\Delta^{\frac{1}{2}}hy^*\xi_0) \\
&= (h^{it-1}xh^{-it+1}\xi_0 \mid \Delta^{-it}hy^*\xi_0) \\
&= (yh\Delta^{it}h^{it-1}xh^{-it}h\xi_0 \mid \xi_0) \\
&= (y\Delta^{it}h^{it}xh^{-it}\Delta^{-it}h\xi_0 \mid \xi_0).
\end{aligned}$$

Hence we have

$$F(t) = \psi(\sigma_t^{\varphi}(h^{it}xh^{-it})y);$$

$$F(t+i) = \psi(y\sigma_t^{\varphi}(h^{it}xh^{it})).$$

For an arbitrary element $y \in M$, there exists a sequence $\{\eta_n\}$ in $\mathfrak{A}_0$ such that

$$\begin{aligned}
y\xi_0 &= \lim \pi(\eta_n)\xi_0 = \lim \eta_n; \\
y^*\xi_0 &= \lim \pi(\eta_n)^*\xi_0 = \lim \eta_n^{\#}; \\
yh\xi_0 &= \lim \pi(\eta_n)h\xi_0; \\
y^*h\xi_0 &= \lim \pi(\eta_n)^*h\xi_0.
\end{aligned}$$

Then we have a sequence $\{F_n\}$ of analytic functions defined by:

$$\begin{aligned}
F_n(\alpha) &= (\pi(\Delta^{-i\alpha}\eta_n)h\xi_0 \mid h^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0) \\
&= (\Delta^{-i\alpha}\pi(\eta_n)h\xi_0 \mid h^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0).
\end{aligned}$$

Observing that

$$\begin{aligned}
\lim \Delta^{\frac{1}{2}}\pi(\eta_n)h\xi_0 &= \lim JS\pi(\eta_n)h\xi_0 \\
&= \lim Jh\pi(\eta_n)^*\xi_0 \\
&= Jhy^*\xi_0,
\end{aligned}$$

we have

$$\lim (1+\Delta^{\frac{1}{2}})\,\pi(\eta_n)h\xi_0 = yh\xi_0 + Jhy^*\xi_0\,.$$

Since $\|\Delta^t(1+\Delta)^{-1}\| \leqq 1$ for $0 \leqq t \leqq \frac{1}{2}$, we have

$$\lim \Delta^t\pi(\eta_n)h\xi_0 = \Delta^t yh\xi_0$$

uniformly for $0 \leqq t \leqq \frac{1}{2}$. Since $\Delta^{-i\alpha} = \Delta^{-is}\Delta^t$, where $\alpha = s + it$, $s, t \in \boldsymbol{R}$, the sequence $\{F_n(\alpha)\}$ converges uniformly to a function $F^1(\alpha)$ in the lower half strip: $0 \leqq \operatorname{Im}\alpha \leqq \frac{1}{2}$, which is defined by

$$F^1(\alpha) = (\Delta^{-i\alpha}yh\xi_0 \mid h^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0), \qquad 0 \leqq \operatorname{Im}\alpha \leqq \tfrac{1}{2}\,;$$

hence $F^1(\alpha)$ is holomorphic in and continuous on the lower half strip.

Now, we shall consider the upper half strip: $\frac{1}{2} \leqq \operatorname{Im}\alpha \leqq 1$. If $\frac{1}{2} \leqq \operatorname{Im}\alpha \leqq 1$, then we have

$$\begin{aligned} F_n(\alpha) &= (\Delta^{-i\alpha}\pi(\eta_n)h\xi_0 \mid h^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0) \\ &= (Jh^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0 \mid J\Delta^{-i\alpha}\pi(\eta_n)h\xi_0) \\ &= (Jh^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0 \mid \Delta^{-i\bar{\alpha}}J\pi(\eta_n)h\xi_0) \\ &= (\Delta^{\frac{1}{2}}Sh^{i\bar{\alpha}}x^*h^{-i\bar{\alpha}}\xi_0 \mid \Delta^{-i\bar{\alpha}}\Delta^{\frac{1}{2}}S\pi(\eta_n)h\xi_0) \\ &= (\Delta^{\frac{1}{2}}h^{i\alpha}xh^{-i\alpha}\xi_0 \mid \Delta^{\frac{1}{2}-i\bar{\alpha}}h\pi(\eta_n)^*\xi_0) \\ &= (h^{i\alpha}xh^{-i\alpha}\xi_0 \mid \Delta^{1-i\bar{\alpha}}h\pi(\eta_n)^*\xi_0)\,. \end{aligned}$$

By the same reason as for the lower half strip, $F_n(\alpha)$ converges uniformly to a function $F^2(\alpha)$ on the upper half strip: $\frac{1}{2} \leqq \operatorname{Im}\alpha \leqq 1$, which is defined by:

$$F^2(\alpha) = (h^{i\alpha}xh^{-i\alpha}\xi_0 \mid \Delta^{1-i\bar{\alpha}}hy^*\xi_0), \qquad \tfrac{1}{2} \leqq \operatorname{Im}\alpha \leqq 1\,;$$

hence $F^2(\alpha)$ is holomorphic in and continuous on the upper half strip. The functions $F^1(\alpha)$ and $F^2(\alpha)$ coincide on the line: $\operatorname{Im}\alpha = \frac{1}{2}$; so they define a function F holomorphic in and continuous on the strip: $0 \leqq \operatorname{Im}\alpha \leqq 1$.

For each $t \in \boldsymbol{R}$, we have

$$\begin{aligned} F(t) &= (\Delta^{-it}yh\xi_0 \mid h^{it}x^*h^{-it}\xi_0) \\ &= (h^{it}xh^{-it}\Delta^{-it}yh\xi_0 \mid \Delta^{-it}\xi_0) \\ &= (\Delta^{it}h^{it}xh^{-it}\Delta^{-it}yh\xi_0 \mid \xi_0) \\ &= \psi(\sigma_t^{\varphi}(h^{it}xh^{-it})y)\,; \\ F(t+i) &= (h^{i(t+i)}xh^{-i(t+i)}\xi_0 \mid \Delta^{1-i(t-i)}hy^*\xi_0) \\ &= (y\Delta^{it}h^{it}xh^{-it}\Delta^{-it}h\xi_0 \mid \xi_0) \\ &= \psi(y\sigma_t^{\varphi}(h^{it}xh^{-it}))\,. \end{aligned}$$

Thus, the one parameter automorphism group: $x \in M \to \sigma_t^\varphi(h^{it} x h^{-it})$, $t \in \boldsymbol{R}$, is actually the modular automorphism group associated with ψ. This completes the proof.

Remark. If σ_t is the modular automorphism group associated with a normal faithful state φ, then for each $x, y \in M$, the function $F(\alpha)$ on the strip: $0 \leqq \operatorname{Im} \alpha \leqq 1$ satisfying condition (1) is bounded.

In fact, as seen above, $F(\alpha)$ is given by:

$$\begin{aligned} F(\alpha) &= (\Delta^{-i\alpha} y \xi_0 \mid x^* \xi_0) \quad \text{if} \quad 0 \leqq \operatorname{Im} \alpha \leqq \tfrac{1}{2}; \\ &= (x \xi_0 \mid \Delta^{1-i\bar{\alpha}} y^* \xi_0) \quad \text{if} \quad \tfrac{1}{2} \leqq \operatorname{Im} \alpha \leqq 1. \end{aligned}$$

Hence we have, for $s \in \boldsymbol{R}$ and $0 \leqq t \leqq \frac{1}{2}$,

$$|F(s+it)| \leqq \|\Delta^t y \xi_0\| \, \|x^* \xi_0\|;$$

for $s \in \boldsymbol{R}$ and $\frac{1}{2} \leqq t \leqq 1$, we have

$$|F(s+it)| \leqq \|x \xi_0\| \, \|\Delta^{1-t} y^* \xi_0\|.$$

Since $y\xi_0$ and $y^*\xi_0$ are both in $\mathscr{D}(\Delta^{\frac{1}{2}})$, we have

$$\sup\{\|\Delta^t y \xi_0\| : 0 \leqq t \leqq \tfrac{1}{2}\} < +\infty;$$

$$\sup\{\|\Delta^{1-t} y^* \xi_0\| : \tfrac{1}{2} \leqq t \leqq 1\} < +\infty,$$

so that $F(\alpha)$ is bounded.

Therefore, we can estimate the behavior of $F(\alpha)$ in the strip: $0 \leqq \operatorname{Im} \alpha \leqq 1$ by Phragmen-Lindelöf theorem.

Theorem 2. *If ψ is a σ_t^φ-invariant, normal, faithful state of M, then there exists a non-singular positive self-adjoint operator h affiliated with M_φ such that*

$$\sigma_t^\psi(x) = \sigma_t(h^{it} x h^{-it}).$$

Proof. By [9: Theorem 15.2], there exists a positive self-adjoint operator k affiliated with M_φ such that

$$\psi(x) = (x k \xi_0 \mid k \xi_0), \qquad x \in M.$$

Since the range projection of k is the support projection of ψ, k has dense range; hence it is non-singular. Let

$$k = \int_0^\infty \lambda \, de(\lambda)$$

be the spectral decomposition of k. Then all projections $\{e(\lambda)\}$ are in M_φ. Put

$$k_n = e(1/n) + \int_{1/n}^n \lambda \, de(\lambda) + (1 - e(n));$$

$$\psi_n(x) = \varphi(k_n x k_n) = \varphi(x k_n^2), \qquad n = 1, 2, \ldots .$$

Since $k_n\xi_0$ converges strongly to $k\xi_0$, ψ_n converges to ψ with respect to the norm topology in M_*. Put $h_n = k_n^2$. Then by Lemma 1, the modular automorphism group σ_t^n of ψ_n is given by:

$$\sigma_t^n(x) = \sigma_t^\varphi(h_n^{it} x h_n^{-it}), \qquad x \in M,\ t \in \boldsymbol{R}.$$

Put $h = k^2$ and $\sigma_t'(x) = \sigma_t^\varphi(h^{it} x h^{-it})$, $x \in M$, $t \in \boldsymbol{R}$. Take a pair x, y in M. For each n, there exists a function $F_n(z)$ holomorphic in the strip: $0 \leqq \operatorname{Im} z \leqq 1$ with boundary values:

$$F_n(t) = \psi_n(\sigma_t^n(x) y);$$

$$F_n(t+i) = \psi_n(y \sigma_t^n(x)).$$

Consider functions f and g on $\boldsymbol{R}$ defined by:

$$f(t) = \psi(\sigma_t'(x) y) = (\sigma_t^\varphi(h^{it} x h^{-it}) y k\xi_0 | k\xi_0);$$

$$g(t) = \psi(y \sigma_t'(x)) = (y \sigma_t(h^{it} x h^{-it}) k\xi_0 | k\xi_0).$$

Since h_n^{it} converges strongly to h^{it} as $n \to \infty$ and the product operation is strongly continuous on the bounded part of M as a function of two variables, $h_n^{it} x h_n^{-it}$ converges strongly to $h^{it} x h^{-it}$ as $n \to \infty$. Then we have

$$\begin{aligned}
|F_n(t) - f(t)| &= |(\Delta^{it} h_n^{it} x h_n^{-it} \Delta^{-it} y k_n \xi_0 | k_n \xi_0) - (\Delta^{it} h^{it} x h^{-it} \Delta^{-it} y k \xi_0 | k \xi_0)| \\
&= |(h_n^{it} x h_n^{-it} \Delta^{-it} y k_n \xi_0 | k_n \xi_0) - (h^{it} x h^{-it} \Delta^{-it} y k \xi_0 | k \xi_0)| \\
&\leqq |(h_n^{it} x h_n^{-it} \Delta^{-it} y k_n \xi_0 | (k_n - k)\xi_0)| \\
&\quad + |(\{h_n^{it} x h_n^{-it} \Delta^{-it} y k_n - h^{it} x h^{-it} \Delta^{-it} k\} \xi_0 | k \xi_0)| \\
&\leqq \|x\|\,\|y\|\,\|k_n \xi_0\|\,\|(k_n - k)\xi_0\| + \|h_n^{it} x h_n^{-it} \Delta^{-it} y (k_n - k)\xi_0\|\,\|k\xi_0\| \\
&\quad + \|(h_n^{it} x h_n^{-it} - h^{it} x h^{-it}) \Delta^{-it} y k \xi_0\|\,\|k\xi_0\|;
\end{aligned}$$

hence $F_n(t)$ converges to $f(t)$ for each $t \in \boldsymbol{R}$. Similarly $F_n(t+i)$ converges to $g(t)$ for each $t \in \boldsymbol{R}$. The sequence $\{F_n(z)\}$ is uniformly bounded on the boundary of the strip: $0 \leqq \operatorname{Im} z \leqq 1$, so that it is uniformly bounded on the strip. Let Φ be a C^∞-function on $\boldsymbol{R}$ with compact support. Then it's Fourier transform $\hat{\Phi}$:

$$\hat{\Phi}(t) = \int_{-\infty}^{\infty} \exp(-ist)\, \Phi(s)\, ds$$

is a C^∞-function of rapidly decreasing, which is extended to a entire function on the whole plane $\boldsymbol{C}$. Then we have

$$\int_{-\infty}^{\infty} \hat{\Phi}(t)\, F_n(t)\, dt = \int_{-\infty}^{\infty} \hat{\Phi}(t+i)\, F_n(t+i)\, dt$$

for $n=1,2,\ldots$. Hence by Lebesgue's convergence theorem, we have

$$\int_{-\infty}^{\infty} \hat{\Phi}(t)\,\psi(\sigma_t'(x)y)\,dt = \int_{-\infty}^{\infty} \hat{\Phi}(t+i)\,\psi(y\sigma_t'(x))\,dt\,,$$

which is equivalent to the KMS-boundary condition (1) for $\beta=1$, see for example [1]. Therefore, σ_t' is the modular automorphism group associated with ψ.

Corollary 3. *If M is of type III, then there is no normal state of M satisfying the KMS-boundary condition with respect to σ_t^φ for $\beta \neq 1$.*

Proof. Suppose ψ is a normal state of M satisfying the KMS-boundary condition with respect to σ_t^φ for $\beta \neq 1$. By [9: Theorem 13.3], the support projection e of ψ is central. Considering eM, we may assume that ψ is faithful. Since ψ is σ_t^φ-invariant, we can apply Theorem 2 to ψ. Namely, there exists a positive self-adjoint operator h affiliated with M_φ such that the modular automorphism group σ_t^ψ associated with ψ is given by: $\sigma_t^\psi(x)=\sigma_t^\varphi(h^{it}xh^{-it})$. On the other hand, by the assumption for ψ, $\sigma_{\beta t}^\varphi$ is the modular automorphism group associated with ψ. Therefore, by the unicity of the modular automorphism group [9: Theorem 13.2] we have

$$\sigma_t^\varphi(h^{it}xh^{-it}) = \sigma_{\beta t}^\varphi(x), \qquad x\in M,\ t\in \boldsymbol{R}\,;$$

$$h^{it}xh^{-it} = \sigma_{(\beta-1)t}^\varphi(x), \qquad x\in M,\ t\in \boldsymbol{R}\,.$$

Therefore, we have

$$\sigma_t^\varphi(x) = h^{it/(\beta-1)}xh^{-it/(\beta-1)}$$

hence the modular automorphism group σ_t^φ is inner, which means by [9: Theorem 14.1] that M is semi-finite. This is a contradiction.

Corollary 4. *If a normal state ψ satisfies the KMS-boundary condition with respect to σ_t^φ for $\beta=1$, then there exists a positive self-adjoint operator k affiliated with the center Z of M such that*

$$\psi(x) = (xk\xi_0\,|\,k\xi_0), \qquad x\in M\,.$$

In particular, if M is a factor, then $\varphi=\psi$.

Proof. As in Corollary 3, we may assume that ψ is faithful. As in the proof of Theorem 2, ψ has the form: $\psi(x)=(xk\xi_0\,|\,k\xi_0)$, $x\in M$; and the modular automorphism group σ_t^ψ associated with ψ is given by $\sigma_t^\psi(x)=\sigma_t^\varphi(k^{2it}xk^{-2it})$. On the other hand, by the assumption on ψ and by the unicity of the modular automorphism group [9: Theorem 13.2] we have $\sigma_t^\varphi(x)=\sigma_t^\varphi(k^{2it}xk^{-2it})$; hence k^{2it} belongs to Z for every $t\in\boldsymbol{R}$, which completes the proof.

Now, let A be a C^*-algebra with a one parameter automorphism group σ_t, $t \in \boldsymbol{R}$. In the following σ_t will be fixed and let a β-(KMS)-state of A be a state of A satisfying the KMS-boundary condition with respect to σ_t for β. Let K_β denote the set of all β-(KMS)-states of A. Put $K = \bigcup_{\beta > 0} K_\beta$. Clearly each K_β is convex. If we assume the continuity of the map: $t \to \sigma_t(x)$, $x \in A$, in the norm topology in A, then it is easily seen that K_β is compact. Therefore, by Corollary 4, K_β is a Choquet simplex in the sense of [8]. But if we do not assume the continuity for σ_t, then we can not expect compactness for K_β. In fact, K_β has no extremal point in many cases (see [5]).

Theorem 5. *In the above situation, let φ and ψ be a β-(KMS) state and a γ-(KMS) state of A respectively. Suppose one of the cyclic representations π_φ and π_ψ induced by φ and ψ is of type III. Then if $\beta \neq \gamma$, then π_φ and π_ψ are disjoint.*

Proof. Let $\mathscr{H}$ and $\mathscr{K}$ be the representation spaces of π_φ and π_ψ respectively. Let M and N be the von Neumann algebras generated by $\pi_\varphi(A)$ and $\pi_\psi(A)$ respectively. Suppose π_φ and π_ψ are not disjoint. Then there exist a central projection p in M and a central projection q in N and an isomorphism π of Mp onto Nq such that $\pi(\pi_\varphi(x)p) = \pi_\psi(x)q$, $x \in A$. Let $\xi_\varphi \in \mathscr{H}$ and $\xi_\psi \in \mathscr{H}$ denote the cyclic vectors corresponding to φ and ψ respectively. Then it is not too hard to see that the states of A defined by

$$\varphi_1(x) = \frac{1}{\|p\xi_\varphi\|^2}(\pi_\varphi(x)p\xi_\varphi | p\xi_\varphi);$$

$$\psi_1(x) = \frac{1}{\|q\xi_\psi\|^2}(\pi_\psi(x)q\xi_\psi | q\xi_\psi), \qquad x \in A,$$

are β-(KMS) and γ-(KMS) respectively and that the cyclic representations π_{φ_1} and π_{ψ_1} induced by φ_1 and ψ_1 are quasi-equivalent. Therefore, we may assume that π_φ and π_ψ are quasi-equivalent. Let π be an isomorphism of M onto N such that $\pi \circ \pi_\varphi = \pi_\psi$. By [9: Theorem 13.3], there exist one parameter automorphism groups σ_t^M of M and σ_t^N of N such that

$$\begin{aligned} \pi_\varphi(\sigma_t(x)) &= \sigma_t^M \pi_\varphi(x); \\ \pi_\psi(\sigma_t(x)) &= \sigma_t^N \pi_\varphi(x), \qquad x \in A. \end{aligned} \tag{$*$}$$

Furthermore, the normal states $\tilde{\varphi}$ of M and $\tilde{\psi}$ of N defined by

$$\tilde{\varphi}(x) = (x\xi_\varphi | \xi_\varphi), \ x \in M; \qquad \tilde{\psi}(x) = (x\xi_\psi | \xi_\psi), \ x \in N,$$

are both β-(KMS) and γ-(KMS) with respect to σ_t^M and σ_t^N respectively. Define a normal state $\tilde{\psi}_1$ of M by $\tilde{\psi}_1(x) = (\pi(x)\xi_\psi | \xi_\psi)$, $x \in M$. Then $\tilde{\psi}_1$

is a γ-(KMS) state of M with respect to the one parameter automorphism group $\pi^{-1}\sigma_t^N\pi$. But equality (*) shows that $\pi^{-1}\sigma_t^N\pi = \sigma_t^M$. Hence the one parameter automorphism group σ_t^M of M admits a β-(KMS) state $\tilde{\varphi}$ and a γ-(KMS) state $\tilde{\varphi}_1$ simultaneously for different β and γ. Then by Corollary 3 M is not of type III. This completes the proof.

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