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Operations and Measurements. II*

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Abstract. Results of a preceding paper on pure operations are generalized. The application to local field theory is discussed in some detail.

1. Operations

In a previous paper [1] we investigated state changes of a quantum system, called operations.

The state space of the system is a Hilbert space $\mathfrak{H}$, and in the Heisenberg picture used here its state is described by a fixed density operator W, as long as no operations are performed.

An operation was assumed to consist of an interaction of the system with an apparatus, and a subsequent measurement of some property Q' of the apparatus. If $\mathfrak{H}'$ is the state space of the apparatus, W' its initial state, and S the unitary "scattering" operator in $\mathfrak{H} \otimes \mathfrak{H}'$ which describes the interaction, the state W of the system is changed into

$$\tilde{W} = \mathrm{Tr}' \, W, \qquad W = \frac{\hat{W}}{\mathrm{Tr}\, \hat{W}}, \qquad \hat{W} = (1 \otimes Q') \, S (W \otimes W') \, S^* (1 \otimes Q') \,. \tag{1}$$

This state change may also be described as

$$\tilde{W} = \frac{\hat{W}}{\mathrm{Tr}\, \hat{W}}, \qquad \hat{W} = \sum_{k \in K} \sum_{i=1}^{n} c_i A_{ki} W A_{ki}^* \,, \tag{2}$$

with the following definitions [1]. Consider the spectral decomposition

$$W' = \sum_{i=1}^{n} c_i P_{\varphi'_i} \tag{3}$$

with a complete orthonormal system $\{\varphi'_i, i = 1 \dots n\}$ in $\mathfrak{H}'$ [1], $c_i \geqq 0$ and $\sum_{i=1}^{n} c_i = 1$. The subset of all i with $c_i \neq 0$ is denoted by I. Furthermore,

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[1] Our discussion applies to finite n as well as to $n = \infty$.

choose another complete orthonormal system $\{\psi'_k, k=1 \dots n\}$ in $\mathfrak{H}'$, so that with a suitable subset K of $\{1 \dots n\}$ the vectors ψ'_k, $k \in K$ span the subspace $Q'\mathfrak{H}'$ of $\mathfrak{H}'$. Then the operators A_{ki} are defined by

$$(\psi, A_{ki}\varphi) = ((\psi \otimes \psi'_k), S(\varphi \otimes \varphi'_i)) \tag{4}$$

for all $\varphi, \psi \in \mathfrak{H}$.

In Ref. [1] we investigated a particular case of Eq. (2), called pure operations. The purpose of the present note is to investigate the general case.

For the following discussion it is convenient to define the A_{ki} in a more abstract way [2]. The space $\mathfrak{H} \otimes \mathfrak{H}'$ can be canonically identified with $\sum_{i=1}^{n} \oplus \mathfrak{H}_i$, with $\mathfrak{H}_i = \mathfrak{H} \otimes \varphi'_i$ isomorphic to $\mathfrak{H}$ for all i. Therefore, there are partially isometric mappings U_i from $\mathfrak{H} \otimes \mathfrak{H}'$ onto $\mathfrak{H}$ with

$$U_i U_j^* = \delta_{ij} 1_{\mathfrak{H}}, \quad U_i^* U_i = P_{\mathfrak{H}_i}, \quad U_i(\varphi \otimes \varphi'_i) = \varphi\,. \tag{5}$$

The same consideration with ψ'_k and $\bar{\mathfrak{H}}_k = \mathfrak{H} \otimes \psi'_k$ instead of φ'_i and $\mathfrak{H}_i = \mathfrak{H} \otimes \varphi'_i$ leads to partially isometric mappings V_k with

$$V_k V_l^* = \delta_{kl} 1_{\mathfrak{H}}, \quad V_k^* V_k = P_{\bar{\mathfrak{H}}_k}, \quad V_k(\varphi \otimes \psi'_k) = \varphi\,. \tag{6}$$

Then, obviously,

$$A_{ki} = V_k S U_i^*\,. \tag{7}$$

Eq. (7) now allows a very simple characterization of the operators A_{ki}. With $\sum_{i=1}^{n} P_{\mathfrak{H}_i} = \sum_{k=1}^{n} P_{\bar{\mathfrak{H}}_k} = 1_{\mathfrak{H} \otimes \mathfrak{H}'}$ and the unitarity of S, Eqs. (5) to (7) lead to

$$\sum_{i=1}^{n} A_{ki} A_{li}^* = \delta_{kl} 1, \qquad \sum_{k=1}^{n} A_{ki}^* A_{kj} = \delta_{ij} 1\,. \tag{8}$$

In other words, the $n \times n$ matrix of operators A_{ki} represents a unitary operator in the direct sum of n copies of $\mathfrak{H}$. The conditions (5) and (7) of Ref. [1] are immediate consequences of (8).

However, only the operators A_{ki} with $k \in K$ and $i \in I$ actually enter Eq. (2) which describes the operation. Eq. (8) implies that the "$K \times I$" matrix of operators

$$A = (A_{ki}), \quad k \in K,\ i \in I \tag{9}$$

which maps the space $\mathscr{H} = \sum_{i \in I} \oplus \mathfrak{H}^{(i)}$, $\mathfrak{H}^{(i)} \equiv \mathfrak{H}$ into $\bar{\mathscr{H}} = \sum_{k \in K} \oplus \bar{\mathfrak{H}}^{(k)}$, $\bar{\mathfrak{H}}^{(k)} \equiv \mathfrak{H}$, is a contraction, i.e.,

$$A^* A \leqq 1_{\mathscr{H}}, \quad A A^* \leqq 1_{\bar{\mathscr{H}}}\,. \tag{10}$$

Conversely, any operator matrix (9) with (10) may be considered as a part of a unitary operator matrix. Consider the Hilbert space

$\hat{\mathscr{H}} = \bar{\mathscr{H}} \oplus \mathscr{H}$. The operator matrix

$$T = \begin{pmatrix} (1 - AA^*)^{1/2} & A \\ A^* & -(1 - A^*A)^{1/2} \end{pmatrix} \tag{11}$$

then represents a unitary operator in $\hat{\mathscr{H}}$. This follows as a straightforward generalization of a well known result [3] [2].

These results allow a complete characterization of operations. Any operation may be described by Eq. (2) with a "$K \times I$" matrix A of operators A_{ki} fulfilling (10) and numbers $c_i > 0$ with $\sum_{i \in I} c_i = 1$. Conversely, any state change described by Eq. (2) with $A = (A_{ki})$ fulfilling (10) and numbers $c_i > 0$ with $\sum_{i \in I} c_i = 1$ is an operation in the sense defined above, i.e., there exists a Hilbert space $\mathfrak{H}'$, a state W', and a property Q' of an apparatus and a unitary operator S in $\mathfrak{H} \otimes \mathfrak{H}'$ so that the state change may also be described by Eq. (1).

The last statement follows easily from Eq. (11). The Hilbert space $\hat{\mathscr{H}} = \sum_{k \in K} \oplus \bar{\mathfrak{H}}^{(k)} \oplus \sum_{i \in I} \oplus \mathfrak{H}^{(i)}$, $\bar{\mathfrak{H}}^{(k)} \equiv \mathfrak{H} \equiv \mathfrak{H}^{(i)}$ is canonically isomorphic [2] to $\mathfrak{H} \otimes \mathfrak{H}'$ with a "$K + I$"-dimensional Hilbert space $\mathfrak{H}'$ and a suitable basis $\{\chi'_k | k \in K\} \cup \{\eta'_i | i \in I\}$ in $\mathfrak{H}'$. Then $W' = \sum_{i \in I} c_i P_{\eta'_i}$, $Q' = \sum_{k \in K} P_{\chi'_k}$, and $S \equiv T$ as given by (11) have the desired properties.

To every operation there belongs a Hermitean operator

$$F = \sum_{k \in K} \sum_{i \in I} c_i A^*_{ki} A_{ki} \tag{12}$$

with $0 \leqq F \leqq 1$ (Ref. [1], Eq. (7)), called effect. The physical meaning of F is explained in Ref. [1]. The transition probability from the state W to the new state $\tilde{W}$ is $\mathrm{Tr}\,\hat{W} = \mathrm{Tr}(FW)$ [1]. Therefore, we speak of an operation to act selectively, or non-selectively, on the state W, if $\mathrm{Tr}(FW) < 1$ or $= 1$, respectively, and an operation with $F < 1$, or $F = 1$, is called selective, or non-selective, respectively. $\mathrm{Tr}(FW) = 1$ implies $FW = W$ since, with $W = \sum_i \alpha_i P_{\varphi_i}$, $\mathrm{Tr}(FW) = \sum_i \alpha_i (\varphi_i, F\varphi_i) = 1$, $\alpha_i > 0$ and $\sum_i \alpha_i = 1$ yield $(\varphi_i, F\varphi_i) = 1$, and since $F \leqq 1$, $F\varphi_i = \varphi_i$ for all i.

2. Local Operations

In field theory with local von Neumann algebras $\mathfrak{R}_C$, the natural requirement

$$S \in \mathfrak{R}_C \otimes \mathfrak{L}(\mathfrak{H}') \tag{13}$$

for operations performed in the space-time region C implies $A_{ki} \in \mathfrak{R}_C$ [1]. Conversely, with $A_{ki} \in \mathfrak{R}_C$, $k \in K$, $i \in I$ fulfilling (10), the operator

[2] We take this occasion to point to a missing minus sign in front of $(1 - A^*A)^{1/2}$ in Eq. (11) of Ref. [1].

T given by (11) belongs to $\mathfrak{R}_C \otimes \mathfrak{L}(\mathfrak{H}')$ [2], and therefore such operators A_{ki} describe a local operation.

Quantum theory predicts the statistics of experimental results for many repetitions of the same experiment. In field theory, "the same" means: identical except the location in space-time. It is then almost inevitable to assume that, prior to any experiment, the field is in a state W which is invariant with respect to space-time displacements. Otherwise, the statistics of experimental results would depend on the space-time location of the trial experiments. The only candidate for this state is, in the usual framework, $W = P_\omega$ with the unique vacuum vector ω.

A local operation in the space-time region C transforms the original field state W into $\tilde{W}$ (Eq. (2)) in the future and side cone of C. This is explained in detail in a forthcoming paper [4]. (Compare also Schlieder [5].) Sequences of local operations may be described with the formalism proposed there.

Some propositions about local operations may now be proved easily.

Proposition 1. *A local operation is non-selective if and only if it acts non-selectively on the vacuum state* $W = P_\omega$.

Proof. "Only if" is trivial. Vice versa, $\mathrm{Tr}(F P_\omega) = 1$ implies $F\omega = \omega$. As a consequence of the Reeh-Schlieder theorem[3], ω is a separating vector for $\mathfrak{R}_C$. Thus $F\omega = \omega$ implies $F = 1$.

Proposition 2. *A local operation in C which acts non-selectively on the field state W leaves invariant expectation values in the side cone C′ of C.*

Proof. $\mathrm{Tr}(FW) = 1$ implies $\tilde{W} = \hat{W}$ and $FW = W$. Take $B \in \mathfrak{R}_{C'}$. By locality, $[B, A^*_{ki}] = 0$, and thus

$$\mathrm{Tr}(B\tilde{W}) = \mathrm{Tr}(B\hat{W}) = \mathrm{Tr}\left(B \sum_{k\in K} \sum_{i=1}^{n} c_i A^*_{ki} A_{ki} W\right) = \mathrm{Tr}(BFW) = \mathrm{Tr}(BW).$$

Proposition 2 expresses the causal behavior of non-selective local operations.

According to Licht [7], a state W is called strictly localized outside C if $\mathrm{Tr}(BW) = (\omega, B\omega)$ for all $B \in \mathfrak{R}_C$. Proposition 2 then leads to:

Corollary. *A non-selective local operation in C changes the vacuum state* P_ω *into a state* $\tilde{W}$ *strictly localized outside C′.*

Proposition 3. *Any state* $\tilde{W}$ *strictly localized outside C′ has the form* $\tilde{W} = \sum_{k=1}^{n} B_k P_\omega B_k^*$ *(including the possibility* $n = \infty$*) with* $B_k \in \mathfrak{R}'_{C'}$, $\sum_{k=1}^{n} B_k^* B_k = 1$, *and* $(\omega, B_k^* B_l \omega) = 0$ *if* $k \neq l$.

[3] This theorem is used here in the form proved by Araki [6].

This has been proved by Licht [7].

Corollary. *Assume the duality theorem*[4] $\mathfrak{R}_{C'} = \mathfrak{R}'_C$ *for the region* C. *Any state* $\tilde{W}$ *strictly localized outside* C' *may then be produced from the vacuum state* P_ω *by a non-selective local operation in* C.

Proof. Take $K = \{1 \ldots n\}$, $I = \{1\}$, $c_1 = 1$, and $A_{k1} = B_k \in \mathfrak{R}'_{C'} = \mathfrak{R}_C$. This choice satisfies (10), Eq. (12) yields $F = \sum_{k=1}^{n} B_k^* B_k = 1$, and Eq. (2) with $W = P_\omega$ leads to $\tilde{W} = \sum_{k=1}^{n} B_k P_\omega B_k^*$.

We conclude with a remark on the Reeh-Schlieder theorem [6], according to which vectors of the form $A\omega$ with $A \in \mathfrak{R}_C$ are dense in $\mathfrak{H}$. Any unit vector $\psi \in \mathfrak{H}$ or, in other words, any pure state of the field, may then be approximated in norm by vectors of the form $\varphi = \frac{A\omega}{\|A\omega\|}$ with $A \in \mathfrak{R}_C$, $\|A\| \leqq 1$ or, in other words, by pure states which are generated from the vacuum state by a local pure [1] operation in C.

At first sight this looks very paradoxical, for instance if we think of a field state ψ which is very different from the vacuum state ω at a large space-like distance from C [7]. However, the local pure operation $\omega \to \varphi = \frac{A\omega}{\|A\omega\|}$ which approximates ψ is in general a selective one. (It is non-selective if and only if the transition probability $\mathrm{Tr}(F P_\omega) = (\omega, F\omega) = \|A\omega\|^2$ is equal to one.) Therefore Proposition 2 does not apply, and φ may be different from the vacuum in C'.

Consider a field property measurable in C', i.e., a projection operator $P \in \mathfrak{R}_{C'}$, and a pure state $\varphi = \frac{A\omega}{\|A\omega\|}$ as above. Then

$$(\omega, P\omega) \geqq (\varphi, P\varphi)\, \|A\omega\|^2 \,, \tag{14}$$

in words: the probability for P in the vacuum state ω is greater than or equal to the probability for P in the state φ times the transition probability from ω to φ. Indeed, from $[A, P] = 0$ and $\|A\| \leqq 1$ follows

$$(\varphi, P\varphi)\, \|A\omega\|^2 = (A\omega, PA\omega) = \|AP\omega\|^2 \leqq \|P\omega\|^2 = (\omega, P\omega) \,.$$

The same consideration applies if P is replaced by a local effect [1] $F \in \mathfrak{R}_{C'}$.

The estimate (14) indicates that any deviation of φ from ω in C' is produced solely by the selection performed in C. One may imagine that the observer exploits some vacuum fluctuations occuring simultaneously

[4] Haag and Schroer [8], Araki [6].

in C and C' with a suitable correlation, and thereby selects those fields, which have the desired properties in C'[5].

If he wants a state φ very different from the vacuum in C', i.e., $(\varphi, P\varphi) \gg (\omega, P\omega)$ for some $P \in \mathfrak{R}_{C'}$, (14) implies that the transition probability $\|A\omega\|^2$ is very small, and therefore the preparation of the field state φ may be practically impossible. We hope this remark solves the apparent paradox mentioned above.

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[5] Note that $(\omega, P\omega) \neq 0$ unless $P = 0$ since ω is a separating vector. Therefore, the vacuum state has "virtually" any desired property.