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The Even CAR-Algebra

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Abstract. It is shown that the even CAR-algebra over a separable Hilbert space is *-isomorphic to the CAR-algebra.

Let K be a separable infinite dimensional complex Hilbert space. Let $\mathfrak{A}(K)$ be the CAR-algebra over K. Then $\mathfrak{A}(K)$ is the C^*-algebra generated by elements $a(f)$, where $f \to a(f)$ is a linear map of K into $\mathfrak{A}(K)$ satisfying the canonical anticommutation relations

$$a(f)a(g)^* + a(g)^* a(f) = (g, f)I\,,$$
$$a(f)a(g) + a(g)\ a(f) = 0\,,$$

for all $f, g \in K$, I denoting the identity operator in $\mathfrak{A}(K)$. Let γ be the *-automorphism of $\mathfrak{A}(K)$ such that $\gamma(a(f)) = -a(f)$ for all $f \in K$, and let $\mathfrak{A}(K)_e$ be the C^*-algebra of even elements in $\mathfrak{A}(K)$, i.e. $x \in \mathfrak{A}(K)$ if and only if $\gamma(x) = x$. It has been shown by Doplicher and Powers [1] that $\mathfrak{A}(K)_e$ is a simple C^*-algebra. In the present note we sharpen this result by showing that $\mathfrak{A}(K)_e$ is *-isomorphic to $\mathfrak{A}(K)$. We refer the reader to the thesis of Powers [3] for an account of the general theory of the CAR-algebra.

Theorem. *$\mathfrak{A}(K)_e$ is *-isomorphic to $\mathfrak{A}(K)$.*

Proof. Let $f_1, f_2, \ldots,$ be an orthonormal basis for K. Let K_n be the linear span of $f_1, \ldots, f_n$, and $\mathfrak{A}(K_n)$ the CAR-algebra over K_n considered as a subalgebra of $\mathfrak{A}(K)$. Let $\mathfrak{A}(K_n)_e$ be the even subalgebra of $\mathfrak{A}(K_n)$. Since $\gamma(\mathfrak{A}(K_n)) = \mathfrak{A}(K_n)$ we clearly have $\mathfrak{A}(K_n)_e = \mathfrak{A}(K_n) \cap \mathfrak{A}(K)_e$. Let $U_i = I - 2a(f_i)^* a(f_i)$, $V_n = U_1 U_2 \ldots U_n$. Then for $x \in \mathfrak{A}(K_n)$, $\gamma(x) = V_n x V_n$. Indeed, it suffies to show this for each $a(f_j)$, $j = 1, \ldots, n$. But

$$V_n a(f_j) V_n = \prod_{i=1}^{n} U_i a(f_j) \prod_{i=1}^{n} U_i = U_j a(f_j) U_j = -a(f_j) = \gamma(a(f_j))\,.$$

Let P_n and Q_n be the spectral projections of V_n in $\mathfrak{A}(K_n)$, so that $V_n = P_n - Q_n$. Then P_n and Q_n are both projections of dimension 2^{n-1} in the $2^n \times 2^n$

matrix algebra $\mathfrak{A}(K_n)$. Let

$$J_1 = \{i : 1 \leqq i \leqq 2^{n-1}\}, \qquad J_2 = \{i : 2^{n-1} < i \leqq 2^n\},$$

and let $L_1 = (J_1 \times J_1) \cup (J_2 \times J_2)$, $L_2 = (J_1 \times J_2) \cup (J_2 \times J_1)$.

Let $\{e_{ij}^{(n)} : i, j \in J_1 \cup J_2\}$ be a complete set of matrix units for $\mathfrak{A}(K_n)$ such that

$$\sum_{i \in J_1} e_{ii}^{(n)} = P_n, \qquad \sum_{i \in J_2} e_{ii}^{(n)} = Q_n.$$

Then $e_{ij}^{(n)}$ is even (resp. odd) if and only if $(i,j) \in L_1$ (resp. $(i,j) \in L_2$). Let

$$b_{ij}^{(n)} = \begin{cases} I & \text{if} \quad (i,j) \in L_1 \\ a(f_{n+1}) - a(f_{n+1})^* & \text{if} \quad (i,j) \in L_2. \end{cases}$$

Let $E_{ij}^{(n)} = e_{ij}^{(n)} b_{ij}^{(n)}$. Then $E_{ij}^{(n)} \in \mathfrak{A}(K_{n+1})_e$. Furthermore a straightforward computation shows that the set $\{E_{ij}^{(n)} : i, j \in J_1 \cup J_2\}$ is a complete set of $2^n \times 2^n$ matrix units. Let $\mathfrak{B}(K_{n+1})$ be the I_{2^n} factor which they generate. Then we have $\mathfrak{A}(K_n)_e \subset \mathfrak{B}(K_{n+1}) \subset \mathfrak{A}(K_{n+1})_e$. Thus $\mathfrak{A}(K)_e$ is generated by the I_{2^n} factors $\mathfrak{B}(K_{n+1})$, hence is a UHF-algebra of type $\{2^n\}$, so it is *-isomorphic to $\mathfrak{A}(K)$, see [2].

References

1. Doplicher, S., Powers, R. T.: On the simplicity of the even CAR algebra and free field models. Commun. Math. Phys. **7**, 77 (1968).
2. Glimm, J. G.: On a certain class of operator algebras. Trans. Amer. Math. Soc. **95**, 318 (1960).
3. Powers, R. T.: Representations of the canonical anticommutation relations. Thesis Princeton Univ. (1967).

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