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Free States of the Canonical Anticommutation Relations*

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Abstract. Each gauge invariant generalized free state ω_A of the anticommutation relation algebra over a complex Hilbert space K is characterized by an operator A on K. It is shown that the cyclic representations induced by two gauge invariant generalized free states ω_A and ω_B are quasi-equivalent if and only if the operators $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class. The combination of this result with results from the theory of isomorphisms of von Neumann algebras yield necessary and sufficient conditions for the unitary equivalence of the cyclic representations induced by gauge invariant generalized free states.

Introduction

In this paper we study gauge invariant generalized free states of the canonical anticommutation relations, and in particular the question of quasi and unitary equivalence of their induced representations. If K is a separable (complex) Hilbert space $\mathfrak{A}(K)$ – the CAR-algebra of K – is the C^*-algebra generated by elements $a(f)$, where $f \to a(f)$ is a linear map of K into $\mathfrak{A}(K)$ satisfying the canonical anticommutation relations. The gauge invariant generalized free states of $\mathfrak{A}(K)$ are states ω_A whose n-point functions have the structure

$$\omega_A(a(f_n)^* \dots a(f_1)^* \dots a(g_1) \dots a(g_m)) = \delta_{nm} \det((f_i, A g_j))$$

for all $f, g \in K$, where A is a linear operator on K such that $0 \leqq A \leqq I$. These states were first defined and studied by Shale and Stinespring [26].

Since the introduction by Shale and Stinespring [26] generalized free states, which are also called quasi-free states, have been studied by several authors, see e.g. [3, 4, 8, 18, 22, and 24]. Dell'Antonio [8] and Rideau [24] have shown that gauge invariant generalized free states are factor states and have given a characterization of the types of the factors obtained from these states. It follows from their work that the gauge invariant generalized free states ω_A and ω_B are quasi-equivalent if the

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two operators $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class.

The main new result of the present paper is that this condition is also necessary for quasi-equivalence. In the proof of this result we derive some inequalities on the norm differences of states which may be of independent interest. Using these inequalities we can with little extra effort rederive the results of Dell'Antonio and Rideau mentioned above. We include these for the sake of completeness.

Then the main result in this paper (Theorem 5.1) asserts that the cyclic representations induced by gauge invariant generalized free states ω_A and ω_B are quasi-equivalent if and only if the two operators $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class. If A and B commute and have pure point spectra this result follows from the work of Kakutani [16] on equivalence of product measures. Hence our result can also be viewed as a noncommutative extension of Kakutani's theorem. For this see also the work of Segal [25]. By a result of Moore [19] on the types of product states the type of the factor obtained from ω_A can easily be characterized, hence the theory of isomorphisms of von Neumann algebras can be applied to give necessary and sufficient conditions in order that ω_A and ω_B induce unitarily equivalent cyclic representations (Theorem 5.7).

The proof of our characterization of quasi-equivalence of two states ω_A and ω_B is divided into several sections. In Section 1 we give the necessary background from the theory of the anticommutation relations. Then in Section 2 the problem is solved in the simplest case, namely when A and B are projections, in which case ω_A and ω_B are pure states. The idea of the proof is to reduce the general case to this latter situation. Note that if A is an operator on a Hilbert space K and $0 \leqq A \leqq I$, then the operator

$$E_A = \begin{pmatrix} A, & A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} \\ A^{\frac{1}{2}}(I-A)^{\frac{1}{2}}, & I-A \end{pmatrix}$$

is a projection on $K \oplus K$. Furthermore the map $f \to a(f)$ has a canonical extension to a map of $K \oplus K$ satisfying the canonical anticommutation relations, such that ω_A extends to a pure gauge invariant generalized free state ω_{E_A} on $\mathfrak{A}(K \oplus K)$. In Section 3 an analysis of the relationship between ω_A and ω_{E_A} is given when K is finite dimensional, and then an inequality relating the norm difference of ω_A and ω_B to that of ω_{E_A} and ω_{E_B} is proved. In order to complete the proof some results on Hilbert-Schmidt operators are obtained in Section 4. Then in Section 5 the developed techniques are used to show that ω_A and ω_B induce quasi-equivalent representations if and only if ω_{E_A} and ω_{E_B} induce unitarily equivalent representations. By Section 2 this is equivalent to $E_A - E_B$

being Hilbert-Schmidt. By Section 4 this holds if and only if $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (1-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators.

The reader who is only interested in understanding the main theorems and not their proofs is advised to first read Section 1 and then to proceed directly to Section 5.

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1. The Algebra of the Canonical Anticommutation Relations

Let K be a complex Hilbert space. The CAR-algebra, $\mathfrak{A}(K)$, over K is a C^*-algebra with the property, there is a linear mapping, $f \to a(f)$, of K into $\mathfrak{A}(K)$, whose range generates $\mathfrak{A}(K)$, as a C^*-algebra such that for $f, g \in K$

$$a(f)^* a(g) + a(g) a(f)^* = (f, g) I,$$

$$a(f) a(g) + a(g) a(f) = 0,$$

where $(.,.)$ is the inner product on K, and I is the unit of $\mathfrak{A}(K)$. If $\mathfrak{A}(K)$ and $\mathfrak{A}'(K)$ are both CAR-algebras over K. generated by the $a(f)$ and $a'(f)$, $f \in K$, respectively, then these C^*-algebras are $*$-isomorphic. In fact, the mapping $a(f) \to a'(f)$, $f \in K$, uniquely extends to a $*$-isomorphism of $\mathfrak{A}(K)$ into $\mathfrak{A}'(K)$. We refer to [22] or [26] for a general discussion of the CAR-algebra. Throughout this paper K will be separable, however we believe all results are valid for non-separable Hilbert spaces as well.

If K is n-dimensional then $\mathfrak{A}(K)$ is a $(2^n \times 2^n)$-matrix algebra (the algebra of all operators on a 2^n-dimensional Hilbert space). If $\mathfrak{M} \subset K$ is a linear subspace of K, we denote by $\mathfrak{A}(\mathfrak{M})$ the C^*-subalgebra of $\mathfrak{A}(K)$ generated by the $a(f)$ with $f \in \mathfrak{M}$. If K is infinite dimensional. $\mathfrak{A}(K)$ may be taken as the inductive limit, in the sense of Guichardet [13], of the subalgebras $\mathfrak{A}(\mathfrak{M})$ for all finite dimensional subspaces $\mathfrak{M} \subset K$. It follows that if K is an infinite dimensional separable Hilbert space $\mathfrak{A}(K)$ is a uniformly hyperfinite (UHF) algebra of type (2^n) as defined by Glimm [11], i.e. $\mathfrak{A}(K)$ is generated by an increasing sequence of $(2^n \times 2^n)$-matrix algebras.

A state of $\mathfrak{A}(K)$ is a positive linear functional on $\mathfrak{A}(K)$ normalized so that $\omega(I) = 1$. We denote by π_ω the cyclic representation induced by the state ω. We will use the following equivalence relations on states.

Definition 1.1. *Two states, ω_1 and ω_2, of a C^*-algebra are, respectively, unitarily equivalent and quasi-equivalent, denoted $\omega_1 \sim \omega_2$ and $\omega_1 \underset{q}{\sim} \omega_2$,* *if the induced representations, π_{ω_1} and π_{ω_2} are, respectively, unitarily equivalent and quasi-equivalent.*

We recall that two representations, π_1 and π_2, are quasi-equivalent if there is a $*$-isomorphism ϕ of the von Neumann algebras, $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$, such that $\phi(\pi_1(A)) = \pi_2(A)$ for all $A \in \mathfrak{A}$. Unitary equivalence implies quasi-equivalence, and for pure states quasi-equivalence implies unitary equivalence (see, e.g. [10]).

Since every state is norm continuous and since polynomials in the $a(f)$ and $a(g)^*$, $f, g \in K$, are dense in $\mathfrak{A}(K)$ it follows that every state of $\mathfrak{A}(K)$ is uniquely determined by its values on polynomials. In fact a state of $\mathfrak{A}(K)$ is uniquely determined by its n-point functions,

$$W_{nm}(f_1, \ldots, f_n, g_1, \ldots, g_m) = \omega(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m))$$

The n-point functions of a gauge invariant generalized free state have the following structure.

Definition 1.2. *The state ω_A is a gauge invariant generalized free state of the CAR-algebra $\mathfrak{A}(K)$, if the n-point functions of ω_A have the form*

$$\omega_A(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) = \delta_{nm} \det((f_i, A g_j)),$$

where A is a linear operator on K satisfying the condition $0 \leqq A \leqq I$.

The generalized free states were developed and studied by Shale and Stinespring [26]. The state $\omega_0 (A = 0)$ is the well-known Fock state which induces the Fock representation [7].

If A has pure point spectrum then the gauge invariant free state ω_A is a factor state, i.e., it induces a factor representation of $\mathfrak{A}(K)$. This can be seen from the fact that the state ω_A can be factorized as follows.

Since A has pure point spectrum there is an orthonormal basis $\{f_n\}$ of K such that $A f_n = \lambda_n f_n$ for $n = 1, 2, \ldots$. We define a sequence of mutually commutative (2×2)-matrix units as follows.

Let

$$V_0 = 1, \quad V_n = \prod_{i=1}^{n} (I - 2a(f_i)^* a(f_i)), \qquad n \geqq 1,$$

and

$$e_{11}^{(n)} = a(f_n)\, a(f_n)^*, \qquad e_{12}^{(n)} = a(f_n)\, V_{n-1},$$
$$e_{21}^{(n)} = a(f_n)^*\, V_{n-1}, \qquad e_{22}^{(n)} = a(f_n)^*\, a(f_n).$$

Using the anticommutation relations one can show that the $(e_{ij}^{(n)}; i,j = 1,2)$ form a set of (2×2)-matrix units, and for distinct n and m, $e_{ij}^{(n)} e_{rs}^{(m)} = e_{rs}^{(m)} e_{ij}^{(n)}$. From the definition of ω_A and the construction of these matrix units, it follows that

$$\omega_A(e_{i_1 j_1}^{(n_1)} \ldots e_{i_r j_r}^{(n_r)}) = \alpha_{i_1}^{(n)} \delta_{i_1 j_1} \ldots \alpha_{i_r}^{(n_r)} \delta_{i_r j_r},$$

for all $n_1 < n_2 < \cdots < n_r$, where $\alpha_1^{(n)} = 1 - \lambda_n$ and $\alpha_2^{(n)} = \lambda_n$.

Therefore, ω_A factorizes with respect to the (2×2)-matrix algebra N_n, generated by the $\{e_{ij}^{(n)}\}$. Since $\mathfrak{A}(K)$ is generated by the $\{N_n; n=1,2,\dots\}$ it follows that ω_A is a factor state (see e.g. [23, Lemma 4.1] or [13, Corollaire 2.1]).

We summarize this result in the form of a lemma.

Lemma 1.3. *If $0\leqq A\leqq I$ is an operator on a Hilbert space with pure point spectrum, then the generalized free state, ω_A of $\mathfrak{A}(K)$ is a factor state.*

We remark that the generalized free state ω_A is pure if and only if A is a projection (i.e. $A^2=A$) (see [3, 18 or 22]). Furthermore, if ω is a state of $\mathfrak{A}(K)$ and $\omega(a(f)^*a(g))=(f,Eg)$ where E is a projection, then $\omega=\omega_E$. This follows from the fact that the requirements $\omega(a(f)^*a(f))=0$ for all $f\in\text{Range}\,(1-E)$ and $\omega(a(f)a(f)^*)=0$ for all $f\in\text{Range}\ E$, uniquely determines all the n-point functions of ω. We will make frequent use of these facts in the following sections.

In the next section we will frequently make use of the projections $\chi_+(E), \chi_-(E)\in\mathfrak{A}(K)$, defined for all finite dimentional hermitian projections in K. The projections are defined as follows.

$$\chi_+(E)=a(f_1)^*a(f_1)a(f_2)^*a(f_2)\dots a(f_n)^*a(f_n)$$

and

$$\chi_-(E)=a(f_1)a(f_1)^*\dots a(f_n)a(f_n)^*$$

where $\{f_1,\dots,f_n\}$ is a complete orthonormal basis for EK. Using the anticommutation relations one can show that $\chi_+(E)$ and $\chi_-(E)$ depend only on E and not on the particular basis $\{f_1,\dots,f_n\}$ used to define them. Using the anticommutation relations one can show, if E and F are finite hermitian projections and $EF=FE$, then

(i) $\chi_+(E)^*=\chi_+(E)$, $\chi_-(E)^*=\chi_-(E)$,

(ii) $\chi_+(E)^2=\chi_+(E)$, $\chi_-(E)^2=\chi_-(E)$,

(iii) $\chi_+(E)\chi_+(F)=\chi_+(E+F-EF)$ and
$\chi_-(E)\chi_-(F)=\chi_-(E+F-EF)$,

(iv) $\chi_+(E)\chi_-(F)=\chi_-(F)\chi_+(E)$ and
$\chi_+(E)\chi_-(F)=0$ if $EF\neq 0$.

It follows from these relations that $\chi_+(E)\leqq\chi_+(F)$ and $\chi_-(E)\leqq\chi_-(F)$ if $E\geqq F$.

If ω_A is a gauge invariant generalized free state of $\mathfrak{A}(K)$, then, by a straightforward computation one can show that

$$\omega_A(\chi_+(E))=\det(I-E(I-A)E)\quad\text{and}\quad\omega_A(\chi_-(E))=\det(I-EAE),$$

where these determinates are defined as follows.

If $0\leqq A\leqq I$ is an operator on K, we define

$$\det(I-A)=\inf(\det((1-A)\,|\,\mathfrak{M});\quad\text{all finite subspaces}\quad\mathfrak{M}\subset K),$$

where $(I-A)|\mathfrak{M}$ is the operator $I-A$ restricted to $\mathfrak{M}$. If A is of finite rank, we have $\det(I-A)=\prod_{i=1}^{n}(1-\lambda_i)$, where $\{\lambda_1, \ldots, \lambda_n\}$ are the non-zero eigenvalues of A repeated as often as their multiplicity dictates.

In the following sections we will have occasion to compute $\det(I-A)$ when A is not of finite rank. We state here some inequalities which can be easily verified from the definition of the determinant.

Let A be an operator on K such that $0 \leqq A \leqq I$. Then, the following inequalities hold.

(i) $\det(I-A) \leqq 1-\|A\|$,

(ii) $\det(I-A) \geqq (1-\|A\|)^m$, where m is the smallest integer such that $m \geqq \|A\|^{-1}\operatorname{Tr}(A)$,

(iii) $\det(I-A) \leqq \exp(-\operatorname{Tr}(A))$,

where $\operatorname{Tr}(A)=\sum_{i=1}^{\infty}(f_i, A f_i)$ and (f_i) is an orthonormal basis of K. Condition (i) is obvious. Condition (ii) may be proved as follows. First suppose A is of finite rank. Then $\det(I-A)=\prod_{i=1}^{n}(1-\lambda_i)$ where (λ_i) are the non-zero eigenvalues of A. A minimization of this product subject to the conditions $\sum_i \tilde{\lambda}_i = \sum \lambda_i$ and $\tilde{\lambda} \leqq \|A\|$ is achieved by setting $\tilde{\lambda}_i = \|A\|$ for $i=1, \ldots, m-1$, $\tilde{\lambda}_m = \sum \lambda_i - (m-1)\|A\|$ and $\tilde{\lambda}_i = 0$ for $i > m$. Then, we have $\det(I-A)=\prod_{i=1}^{n}(1-\lambda_i) \geqq (1-\|A\|)^{m-1}(1-\tilde{\lambda}_m) \geqq (1-\|A\|)^m$. Since condition (ii) is valid for all finite rank operators it follows from the definition of the determinant that it is true for all $0 \leqq A \leqq I$. Condition (iii) follows from the fact that $\det(I-A)=\exp(\operatorname{Tr}(\log(I-A))) \leqq \exp(-\operatorname{Tr}(A))$.

It follows from condition (i), (ii) and (iii) that for $0 \leqq A \leqq I$, $\det(I-A)>0$ if and only if $\|A\|<1$ and $\operatorname{Tr}(A)<\infty$. Another property of the determinant we will use in the following sections is the property. If $0 \leqq A \leqq I$ and E is a hermitian projection, the $\det(I-EAE)=\inf(\det(1-PAP); P$ finite projection, $P \leqq E)$.

This last property is a consequence of the following lemma, when the projection E is chosen to be finite dimensional.

Lemma 1.4. *Let A be an operator on the Hilbert space K such that $0 \leqq A \leqq I$. Let E be a hermitian projection on K and F a hermitian projection of finite rank such that $F \leqq E$. Then*

$$\det(I-EAE) \leqq \det(I-(E-F)A(E-F)).$$

Proof. We first assume E is finite dimensional. If P is a projection on K we denote by $\det_P(A)$ the determinant of PAP considered as an operator on PK. Then the lemma states

$$\begin{aligned}\det_E(I-A) &= \det((I-E)+(I-A)E) = \det(I-EAE)\\ &\leqq \det(I-(E-F)A(E-F)) = \det_{E-F}(I-A).\end{aligned}$$

If this inequality holds for F of rank 1 then by iteration it holds for all $F \leqq E$. We may therefore assume F has rank 1. In order to prove the inequality in this case we prove the following more general inequality. Let B be a positive $(n\times n)$-matrix, and F a hermitian projection of rank one on the n-dimensional Hilbert space $\mathbb{C}^n$. Then

$$\det B \leqq \det(FBF+(I-F)B(I-F)).$$

In fact, let $P=I-F$. We may assume F is the matrix (f_{ij}) with $f_{11}=1$, $f_{ij}=0$ if $(i,j)\neq(1,1)$. Let $B=(b_{ij})$. Let V be a unitary operator on the space $P\mathbb{C}^n$ such that $V^*(b_{ij})_{i,j\geqq 2}V$ is diagonal with entries c_{ii}, $i=2,3,\dots,n$. Let U be the unitary $n\times n$-matrix defined by $U=F+V$, where V is extended to be 0 on the complement of P. Then

$$U^*BU=\begin{pmatrix} b_{11} & c_{12} & \dots & c_{1n}\\ \bar{c}_{12} & c_{22} & \dots & 0\\ \vdots & & \ddots & \\ \bar{c}_{1n} & 0 & \dots & c_{nn}\end{pmatrix}$$

Now

$$\det(FBF+PBP) = \det FU^*BUF+PU^*BUP) = b_{11}\prod_{i=2}^{n} c_{ii},$$

and

$$\det(B)=\det(U^*BU)=b_{11}\prod_{i=2}^{n}c_{ii}-\sum_{j=2}^{n}|c_{1j}|^2\prod_{i\neq j}c_{ii}.$$

Since b_{11} and c_{ii} are all positive, $\det(B)\leqq\det(FBF+(I-F)B(I-F))$. In particular, if $0\leqq B\leqq I$, $\det B\leqq det_{I-F}(B)$. Hence the inequality $\det_E(I-A)\leqq\det_{E-F}(I-A)$ follows, and the lemma follows for E finite dimensional.

In particular, the assertion, stated before the lemma, follows, and we have for E infinite dimensional,

$$\begin{aligned}\det(I-EAE) &= \inf\{\det(I-PAP)\colon P \text{ finite dimensional, } P\leqq E\}\\ &= \inf\{\det(I-PAP)\colon P \text{ finite dimensional, } F\leqq P\leqq E\}\\ &\leqq \inf\{\det(I-(P-F)A(P-F))\colon P \text{ finite dimensional,}\\ &\qquad\qquad F\leqq P\leqq E\}\\ &= \inf\{\det(I-QAQ)\colon Q \text{ finite dimensional, } Q\leqq E-F\}\\ &= \det(I-(E-F)A(E-F)).\end{aligned}$$

2. Pure Generalized Free States

In this section we show that the pure generalized free states, ω_E and ω_F, are unitarily equivalent if and only if $E-F$ is of Hilbert-Schmidt class. We make use of the following lemma in the proof of this result.

Lemma 2.1. *Let K be a Hilbert space and ω_1, ω_2 two factor states of $\mathfrak{A}(K)$. Let $\mathfrak{M}_1 \subset \mathfrak{M}_2 \ldots$ be an increasing sequence of finite dimensional subspaces of K, the closures of whose union is K. We denote by $\mathfrak{A}(\mathfrak{M}_n)^c$ the commutant of $\mathfrak{A}(\mathfrak{M}_n)$ in $\mathfrak{A}(K)$. Then the following statements are equivalent.*

(*i*) $\omega_1 \underset{q}{\sim} \omega_2$.

(ii) *For every $\varepsilon > 0$ there is an integer n such that*

$$\|\omega_1 \,|\, \mathfrak{A}(\mathfrak{M}_n)^c - \omega_2 \,|\, \mathfrak{A}(\mathfrak{M}_n)^c\| < \varepsilon .$$

(iii) *There is a finite subspace $\mathfrak{N} \subset K$ such that*

$$\|\omega_1 \,|\, \mathfrak{A}(\mathfrak{N})^c - \omega_2 \,|\, \mathfrak{A}(\mathfrak{N})^c\| < 2 .$$

For the proof of this lemma we refer to [23, Theorem 2.7].

A difficulty in the application of this lemma lies with the unwieldy form of $\mathfrak{A}(\mathfrak{M})^c$. However, we will show that for even states, $\|\omega_1 \,|\, \mathfrak{A}(\mathfrak{M})^c - \omega_2 \,|\, \mathfrak{A}(\mathfrak{M})^c\| = \|\omega_1 \,|\, \mathfrak{A}(\mathfrak{M}^\perp) - \omega_2 \,|\, \mathfrak{A}(\mathfrak{M}^\perp)\|$, where $\mathfrak{M}^\perp$ is the orthogonal complement of $\mathfrak{M}$.

A state ω of $\mathfrak{A}(K)$ is even if it is invariant under the $*$-automorphism γ, i.e. $\omega(A) = \omega(\gamma(A))$ for all $A \in \mathfrak{A}(K)$, where γ is the unique $*$-automorphism satisfying the condition $\gamma(a(f)) = -a(f)$ for all $f \in K$. We begin by characterizing $\mathfrak{A}(\mathfrak{M})^c$.

Lemma 2.2. *Let $\mathfrak{M}$ be a finite subspace of a Hilbert space K and $\{f_i; i=1, \ldots, n\}$ an orthonormal basis of $\mathfrak{M}$. Let $V = \prod_{i=1}^{n} (I - 2a(f_i)^* a(f_i))$. Then $\mathfrak{A}(\mathfrak{M})^c$, the commutant of $\mathfrak{A}(\mathfrak{M})$ in $\mathfrak{A}(K)$, is generated by the elements $a(f)V$ for all $f \in \mathfrak{M}^\perp$.*

Proof. Let $\mathfrak{B}$ be the C^*-subalgebra of $\mathfrak{A}(K)$ generated by the elements $a(f)\, V$ for all $f \in \mathfrak{M}^\perp$. Since the generators of $\mathfrak{A}(\mathfrak{M})$ commute with the generators of $\mathfrak{B}$ (i.e. $a(f)\, Va(g) = a(g)\, a(f)\, V$ and $a(f)\, Va(g)^* = a(g)^*\, a(f)\, V$ for all $f \in \mathfrak{M}^\perp$ and $g \in \mathfrak{M}$) it follows that $\mathfrak{B} \subset \mathfrak{A}(\mathfrak{M})^c$.

Next we remark that $\mathfrak{A}(K)$ is generated by $\mathfrak{A}(\mathfrak{M})$ and $\mathfrak{B}$. To see that this is true, let $\mathfrak{A}_1$ be the algebra generated by $\mathfrak{A}(\mathfrak{M})$ and $\mathfrak{B}$. Now, we have that $V \in \mathfrak{A}_1$ since $V \in \mathfrak{A}(\mathfrak{M})$. Since $V^2 = I$, $a(f) \in \mathfrak{A}_1$ for all $f \in \mathfrak{M}^\perp$ and all $f \in \mathfrak{M}$, Hence $\mathfrak{A}_1 = \mathfrak{A}(K)$.

Suppose $\mathfrak{N}$ is a finite dimensional subspace of $\mathfrak{M}^\perp$. Let $\mathfrak{B}(\mathfrak{N})$ be the C^*-algebra generated by the $a(f)\, V$ for $f \in \mathfrak{N}$. Clearly, we have $\mathfrak{B}(\mathfrak{N}) \subset \mathfrak{B}$. Now, $\mathfrak{B}(\mathfrak{N})$ and $\mathfrak{A}(\mathfrak{M})$ are finite matrix algebras which generate $\mathfrak{A}(\mathfrak{M} \oplus \mathfrak{N})$.

Hence, $\{\mathfrak{A}(\mathfrak{M}), \mathfrak{B}(\mathfrak{N})\}$ is a factorization of $\mathfrak{A}(\mathfrak{M}\oplus\mathfrak{N})$. Since $\mathfrak{A}(\mathfrak{M})$ and $\mathfrak{B}(\mathfrak{N})$ are type I_n factors it follows that this factorization must be paired [20] i.e. $\mathfrak{A}(\mathfrak{M})$ and $\mathfrak{B}(\mathfrak{N})$ are each others commutants in $\mathfrak{A}(\mathfrak{M}\oplus\mathfrak{N})$. Hence, we have $\mathfrak{B}(\mathfrak{N}) = \mathfrak{A}(\mathfrak{M})^c \cap \mathfrak{A}(\mathfrak{M}\oplus\mathfrak{N})$. From the proof of Lemma 3.2 in [23] it follows that $\mathfrak{A}(\mathfrak{M})^c$ is generated by the algebras $\mathfrak{A}(\mathfrak{M})^c \cap \mathfrak{A}(\mathfrak{M}\oplus\mathfrak{N})$ for all finite $\mathfrak{N} \subset \mathfrak{M}^\perp$. Hence, we have $\mathfrak{A}(\mathfrak{M})^c \subset \mathfrak{B}$. This completes the proof of the lemma

Lemma 2.3. *Suppose ω_1 and ω_2 are even states of $\mathfrak{A}(K)$ and $\mathfrak{M}$ is a finite dimensional subspace of K. Then,*

$$\|\omega_1 | \mathfrak{A}(\mathfrak{M})^c - \omega_2 | \mathfrak{A}(\mathfrak{M})^c\| = \|\omega_1 | \mathfrak{A}(\mathfrak{M}^\perp) - \omega_2 | \mathfrak{A}(\mathfrak{M}^\perp)\| .$$

Proof. Let

$$\alpha^c = \|\omega_1 | \mathfrak{A}(\mathfrak{M})^c - \omega_2 | \mathfrak{A}(\mathfrak{M})^c\|$$

and

$$\alpha^\perp = \|\omega_1 | \mathfrak{A}(\mathfrak{M}^\perp) - \omega_2 | \mathfrak{A}(\mathfrak{M}^\perp)\| .$$

We prove $\alpha^c = \alpha^\perp$. Let $\varepsilon > 0$. Since $\mathfrak{A}(\mathfrak{M}^\perp)$ is generated by the $a(f)$ with $f \in \mathfrak{M}^\perp$ there is a polynomial p in the $a(f)$ and $a(f)^*$, $f \in \mathfrak{M}^\perp$ such that $\|p\| \leqq 1$ and $|\omega_1(p) - \omega_2(p)| \geqq \alpha^\perp - \varepsilon$. Let $z = \frac{1}{2}(p + \gamma(p))$. Since ω_1 and ω_2 are γ-invariant, we have $\omega_1(z) = \omega_1(p)$ and $\omega_2(z) = \omega_2(p)$. Hence, we have $|\omega_1(z) - \omega_2(z)| \geqq \alpha^\perp - \varepsilon$.

We have that $\|z\| \leqq \frac{1}{2}(\|p\| + \|\gamma(p)\|) \leqq 1$ and z is an even polynomial in the $a(f)$ and $a(f)^*$ with $f \in \mathfrak{M}^\perp$. Hence, z commutes with the $a(h)$ and $a(h)^*$ for $h \in \mathfrak{M}$. Therefore z is an element of $\mathfrak{A}(\mathfrak{M})^c$ with norm less than or equal to one. Hence $\alpha^c \geqq \alpha^\perp - \varepsilon$. Since ϵ is arbitrary, $\alpha^c \geqq \alpha^\perp$.

We prove the reverse inequality. From the preceding lemma, it follows that there is a polynomial p in the $Va(f)$ and $Va(f)^*$, $f \in \mathfrak{M}^\perp$ such that $\|p\| \leqq 1$ and $|\omega_1(p) - \omega_2(p)| \geqq \alpha^c - \varepsilon$. Again, we set $z = \frac{1}{2}(p + \gamma(p))$. We have, as before, $\|z\| \leqq 1$ and $|\omega_1(z) - \omega_2(z)| \geqq \alpha^c - \varepsilon$. Now z is an even polynomial in the $a(f)\,V$ and $a(f)^*\,V$. Since V commutes with the $a(f)^*$ for $f \in \mathfrak{M}^\perp$ and $V^2 = I$, z is a polynomial in the $a(f)$ and $a(f)^*$ with $f \in \mathfrak{M}^\perp$ (i.e. z contains no terms with V in them). Hence, we have $z \in \mathfrak{A}(\mathfrak{M}^\perp)$ and $\alpha^\perp \geqq \alpha^c - \varepsilon$. Since ε is arbitrary $\alpha^\perp = \alpha^c$.

It follows from this lemma that for even states ω_1 and ω_2, the expressions $\|\omega_1 | \mathfrak{A}(\mathfrak{M})^c - \omega_2 | \mathfrak{A}(\mathfrak{M})^c\|$ may be replaced by $\|\omega_1 | \mathfrak{A}(\mathfrak{M}^\perp) - \omega_2 | \mathfrak{A}(\mathfrak{M}^\perp)\|$ in Lemma 2.1.

In the following lemmas we develop techniques for estimating the norm difference between two states one of which is pure. The first two lemmas are closely related to two lemmas of Glimm ([11], Lemmas 3.2 and 3.3).

Lemma 2.4. *Suppose ω_1 and ω_2 are states of a C^*-algebra $\mathfrak{A}$ and there is a $B \in \mathfrak{A}$ such that $\omega_2(A) = \omega_1(B^* A B)$ for all $A \in \mathfrak{A}$. Then the following* *inequality holds,* $\|\omega_1 - \omega_2\| \leqq 2(1 - |\omega_1(B)|^2)^{\frac{1}{2}}$. *Furthermore, if* ω_1 *is pure the equality sign holds.*

Proof. Let π be the cyclic representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ induced by ω_1 and let f_1 be a cyclic vector for π such that $\omega_1(A) = (f_1, \pi(A) f_1)$ for all $A \in \mathfrak{A}$. Let $f_2 = \pi(B) f_1$. Then we have $\omega_2(A) = (f_2, \pi(A) f_2)$ for all $A \in \mathfrak{A}$. Let χ be the operator on $\mathfrak{H}$ defined by the relation. $\chi f = (f_1, f) f_1 - (f_2, f) f_2$ for all $f \in \mathfrak{H}$. We have that

$$\begin{aligned}\omega_1(A) - \omega_2(A) &= (f_1, \pi(A) f_1) - (f_2, \pi(A) f_2)\\ &= \mathrm{Tr}(\pi(A)\chi).\end{aligned}$$

Diagonalizing χ we find χ can be expressed as follows, $\chi f = \lambda\{(g_1, f) g_1 - (g_2, f) g_2\}$ where $\lambda = (1 - |(f_1, f_2)|^2)^{\frac{1}{2}} = (1 - |\omega_1(B)|^2)^{\frac{1}{2}}$ and g_1, g_2 are two orthonormal vectors contained in the span of f_1 and f_2. Hence we have that

$$\begin{aligned}|\omega_1(A) - \omega_2(A)| &= \lambda\{(g_1, \pi_1(A) g_1) - (g_2, \pi_1(A) g_2)\}\\ &\leqq 2\lambda\,\|\pi(A)\| \leqq 2\lambda\|A\|.\end{aligned}$$

Hence, we have $\|\omega_1 - \omega_2\| \leqq 2\lambda = 2(1 - |\omega_1(B)|^2)^{\frac{1}{2}}$.

Now suppose ω_1 is pure. Then π is irreducible, and the closure of $\pi(\mathfrak{A})$ in the weak operator topology is $\mathfrak{B}(\mathfrak{H})$. Let E_1 be the projection onto g_2, i.e. $E_2 f = (g_2, f) g_2$ for all $f \in \mathfrak{H}$ and let $U = I - 2E_2$. Notice that $U f_1 = f_1$ and $U f_2 = -f_2$. Since $\pi(\mathfrak{A})$ is irreducible it is algebraically irreducible, and since U is self-adjoint there exists a self-adjoint operator $\pi(A) \in \pi(\mathfrak{A})$ such that $\pi(A) f_i = U f_i$ [10, Théorème 2.8.3]. Replacing A by $\frac{1}{2}(A + A^*)$ we may assume A is self-adjoint. Let h denote the real function defined by $h(x) = x$ for $|x| \leqq 1$, $h(x) = 1$ for $x \geqq 1$, $h(x) = -1$ for $x \leqq -1$. Let $B = h(A)$. Then $\|B\| = 1$. Since f_1 and f_2 are eigenvectors for $\pi(A)$, $\pi(B) f_i = h(\pi(A)) f_i = \pi(A) f_i = U f_i$. Thus $|\omega_1(B) - \omega_2(B)| = 2\lambda$. Hence, for pure states we have $\|\omega_1 - \omega_2\| = 2\lambda$. This completes the proof of the lemma.

Lemma 2.5. *Suppose* ω_1 *and* ω_2 *are states of a* C^**-algebra* $\mathfrak{A}$. *Suppose* $\{E_\gamma; \gamma \in I_0\}$ *is a decreasing net of projections in* $\mathfrak{A}$ *(i.e.* $E_\alpha \leqq E_\beta$ *for* $\alpha \succ \beta$*) with the property that* $\omega_1(E_\gamma) = 1$ *for all* $\gamma \in I_0$ *and if* ω *is any state of* $\mathfrak{A}$ *such that* $\omega(E_\gamma) = 1$ *for all* $\gamma \in I_0$, *then* $\omega = \omega_1$. *Let* $\alpha = \inf(\omega_2(E_\gamma); \gamma \in I_0)$. *Then, the following inequalities are valid*

$$2(1 - \alpha) \leqq \|\omega_1 - \omega_2\| \leqq 2(1 - \alpha)^{\frac{1}{2}}.$$

Furthermore if ω_2 *is pure, then*

$$\|\omega_1 - \omega_2\| = 2(1 - \alpha)^{\frac{1}{2}}.$$

Proof. Let $U_\gamma = 2E_\gamma - I$. Since $\|U_\gamma\| = 1$ we have

$$\begin{aligned}\|\omega_1 - \omega_2\| &\geqq \sup_{\gamma\in I_0} |\omega_1(U_\gamma) - \omega_2(U_\gamma)| \\ &\geqq \sup_{\gamma\in I_0} (2 - 2\omega_2(E_\gamma)) = 2(1-\alpha).\end{aligned}$$

Hence, we have $\|\omega_1 - \omega_2\| \geqq 2(1-\alpha)$.

In the proof of the other inequalities we assume $\alpha > 0$, since for $\alpha = 0$ these inequalities are trivally satisfied. Hence, we have $\omega_2(E_\gamma) \geqq \alpha > 0$ for all $\gamma \in I_0$. Let $P_\gamma = \omega_2(E_\gamma)^{-\frac{1}{2}} E_\gamma$ for all $\gamma \in I_0$.

Let $\varrho_\gamma(A) = \omega_2(P_\gamma A P_\gamma)$ for all $A \in \mathfrak{A}$ and $\gamma \in I_0$. We show that $\lim \varrho_\gamma = \omega_1$ where the convergence is in norm. Let π be a cyclic representation of $\mathfrak{A}$ on $\mathfrak{H}$ induced by ω_2 and $f \in \mathfrak{H}$ a cyclic vector such that $\omega_2(A) = (f, \pi(A) f)$ for all $A \in \mathfrak{A}$. Since $(\pi(E_\gamma), \gamma \in I_0)$ is a decreasing net of operators bounded below by 0, $\lim \pi(E_\gamma)$ exists in the sense of strong convergence ([10], Appendix II, p. 331). Let $g = \lim \pi(E_\gamma) f$. Note $g \neq 0$ since $(g, g) = \alpha > 0$.

Let $\varrho(A) = (g, \pi(A) g)\|g\|^{-2}$. Since $\pi(E_\gamma) f$ converges in norm to g it follows that ϱ_γ converges in norm to ϱ. We note that $\varrho(E_\gamma) = 1$ for all $\gamma \in I_0$ since $\varrho(E_\gamma) = \lim \varrho_\beta(E_\gamma)$, and $\varrho_\beta(E_\gamma) = 1$ for all $\beta > \gamma$. From the properties of the $\{E_\gamma\}$ it follows that $\varrho = \omega_1$. Hence, ϱ_γ converges in norm to ω_1 as $\gamma \to \infty$.

Now, we have that

$$\|\omega_1 - \omega_2\| \leqq \|\omega_1 - \varrho_\gamma\| + \|\varrho_\gamma - \omega_2\| \quad \text{for all} \quad \gamma \in I_0 .$$

From the preceding lemma, we have

$$\|\varrho_\gamma - \omega_2\| \leqq 2(1 - |\omega_2(P_\gamma)|^2)^{\frac{1}{2}} = 2(1 - \omega_2(\mathrm{E}_\gamma))^{\frac{1}{2}} .$$

Hence, we have

$$\|\omega_1 - \omega_2\| \leqq \|\omega_1 - \varrho_\gamma\| + 2(1 - \omega_2(\mathrm{E}_\gamma))^{\frac{1}{2}}$$

$$\|\omega_1 - \omega_2\| \leqq \|\omega_1 - \varrho_\gamma\| + 2(1 - \omega_2(E_\gamma))^{\frac{1}{2}}$$

for all $\gamma \in I_0$. Since $\|\omega_1 - \varrho_\gamma\|$ and $\omega_2(E_\gamma) - \alpha$ can be made arbitrarily small for sufficiently large γ, it follows that, $\|\omega_1 - \omega_2\| \leqq 2(1-\alpha)^{\frac{1}{2}}$.

Now suppose ω_2 is pure. Then from the preceding lemma $\|\varrho_\gamma - \omega_2\| = 2(1 - \omega_2(E_\gamma))$. We have that

$$\begin{aligned}\|\omega_1 - \omega_2\| &\geqq \|\varrho_\gamma - \omega_2\| - \|\omega_1 - \varrho_\gamma\| \\ &= 2(1 - \omega_2(E_\gamma))^{\frac{1}{2}} - \|\omega_1 - \varrho_\gamma\|\end{aligned}$$

for all $\gamma \in I_0$. Since $\|\omega_1 - \varrho_\gamma\|$ and $|\alpha - \omega_2(E_\gamma)|$ can be made arbitrarily small for sufficiently large γ it follows that, $\|\omega_1 - \omega_2\| \geqq 2(1-\alpha)^{\frac{1}{2}}$. Hence, if ω_2 is pure, $\|\omega_1 - \omega_2\| = 2(1-\alpha)^{\frac{1}{2}}$.

Theorem 2.6. *Suppose ω_E and ω_F are pure generalized free states of $\mathfrak{A}(K)$. Let $\alpha_1 = \det(I - E(I-F)E)$ and $\alpha_2 = \det(I - (I-E)F(I-E))$ and*

$\alpha = \min(\alpha_1, \alpha_2)$. *Then*

$$\|\omega_E - \omega_F\| = 2(1-\alpha)^{\frac{1}{2}}.$$

Proof. Consider the collection of projections $\{\chi_+(P)\,\chi_-(Q) \in \mathfrak{A}(K)\}$ defined for all finite projections $P \leqq E$ and $Q \leqq I - E$. This collection forms a decreasing net of projections, since for $P' \geqq P$ and $Q' \geqq Q$ we have.

$$\chi_+(P')\,\chi_-(Q') \leqq \chi_+(P)\,\chi(Q') \leqq \chi_+(P)\,\chi_-(Q).$$

Furthermore, we have $\omega_E(\chi_+(P)) = \omega_E(\chi_-(Q)) = 1$ for all finite $P \leqq E$ and $Q \leqq I - E$. Hence, we have $\omega_E(\chi_+(P)\,\chi_-(Q)) = 1$ for all $P \leqq E$ and $Q \leqq 1 - E$.

Next, suppose ω is any state of $\mathfrak{A}(K)$ with the property $\omega(\chi_+(P)\,\chi_-(Q)) = 1$ for all $P \leqq E$ and $Q \leqq 1 - E$. We show that $\omega = \omega_E$. Let A be the operator defined by the relation $\omega(a(f)^* a(g)) = (f, Ag)$ for $f, g \in K$. We have that $0 \leqq A \leqq I$. Furthermore we have $\omega(a(f)^* a(f)) = (f, Af) = (f, f)$ for $f \in \text{Range}\, E$ and $\omega(a(f)\, a(f)^*) = (f, (I - A) f) = (f, f)$ for $f \in \text{Range}$ $(1 - E)$. Hence, we have $A = E$ and therefore from the discussion in Section 1, $\omega = \omega_E$. Therefore, the net $\{\chi_+(P)\,\chi_-(Q)\}$ satisfies the conditions of Lemma 2.5. Hence, we have $\|\omega_E - \omega_F\| = 2(1 - \alpha')^{\frac{1}{2}}$ where

$$\alpha' = \inf\{\omega_F(\chi_+(P)\,\chi_-(Q));\ P \leqq E \text{ and } Q \leqq I - E\}.$$

We complete the proof of the theorem by showing $\alpha = \alpha'$.

Now, we have from Section 1 that $\omega_F(\chi_+(P)) = \det(I - P(1-F)P)$ $\omega_F(\chi_-(Q)) = \det(I - QFQ)$. Let $\alpha_1 = \inf\{\omega_F(\chi_+(P));\ P \leqq E\}$ and $\alpha_2 = \inf\{\omega_F(\chi_-(Q));\ Q \leqq I - E\}$. Clearly, we have $\alpha_1 = \det(I - E(I-F)E)$ and $\alpha_2 = \det(I - (I-E)\,F(I-E))$. Furthermore $\alpha_1 \geqq \alpha'$ and $\alpha_2 \geqq \alpha'$. We assume $\alpha_1 > 0$ and $\alpha_2 > 0$, for if $\alpha_1 = 0$ or $\alpha_2 = 0$, we have $\alpha' = 0$ and the conclusion of the theorem follows immediately.

Since $\alpha_1 > 0$ and $\alpha_2 > 0$ we have that

$$\|E(I-F)E\| < 1 \quad \text{and} \quad \|(I-E)\,F(I-E)\| < 1.$$

Since these operators have orthogonal ranges we have

$$\|E(I-F)E + (I-E)\,F(I-E)\| = \|(E-F)^2\| < 1.$$

Hence, $\|E - F\| < 1$. We will make use of this fact in a minute.

Let $\tau_P(A) = \omega_F(\chi_+(P))^{-1}\,\omega_F(\chi_+(P)\,A\,\chi_+(P))$ and $\omega = \lim\limits_{P \nearrow E} \tau_P$. It follows from the argument in Lemma 2.5 that τ_P converges in norm to a state ω as $P \nearrow E$. We will show that $\omega = \omega_E$. Let $0 \leqq A \leqq I$ be the operator on K defined by the relation $\omega(a(f)^* a(g)) = (f, Ag)$ for all $f, g \in K$. For $f \in \text{Range}\, E$ and $\|f\| = 1$ we have $\omega(a(f)^* a(f)) = (f, Af) = 1$, since $\tau_P(a(f)^* a(f)) = 1$ for all $P \leqq E$ such that $Pf = f$. Hence, we have $A \geqq E$. We will show $A = \mathrm{E}$ by assuming $A \neq E$ and arriving at a contradiction.

Suppose $A \neq E$. Then there is a vector $f_1 \in \text{Range}(1-E)$ with $\|f_1\|=1$ and $(f_1, Af_1)>0$. Let E_1 be the projection onto f_1; i.e. $E_1 f=(f_1,f)f_1$ for all $f \in K$. Next we show that $\inf(\omega_F(\chi_+(P)); \ P \leqq E+E_1) = \det(I-(E+E_1)(I-F)(E+E_1))=(f_1, Af_1)\alpha_1>0$.

We have that

$$\begin{aligned}\inf(\omega_F(\chi_+(P)); P \leqq E_1+E) &= \inf(\omega_F(\chi_+(P)); \mathrm{E}_1 \leqq P \leqq E_1+E)\\ &= \inf(\omega_F(\chi_+(P)\,\chi_+(E_1)\,\chi_+(P)); P \leqq E)\\ &= \inf(\omega_F(\chi_+(P))\,\tau_P(\chi_+(E_1)); P \leqq E)\\ &= \alpha_1\,\omega(\chi_+(E_1)) = (f_1, Af_1)\,\alpha_1 > 0\,.\end{aligned}$$

Hence, $\det(I-(E+E_1)\ (I-F)\ (E+E_1))>0$. Since $\alpha_2>0$ we have using Lemma 1.4, $\det(I-(I-E-E_1)\ F(I-E-E_1))>0$. Therefore $\|E+E_1-F\|<1$, by the argument used earlier in this proof. Hence, we have that $\|E-F\|<1$ and $\|E+E_1-F\|<1$. The next lemma shows that this is impossible. Hence, we have reached a contradiction. We conclude that $A=E$ and therefore $\omega=\omega_E$.

From this result, it follows that $\alpha'=\alpha_1$, since

$$\begin{aligned}\alpha' &= \inf(\omega_F(\chi_+(P)\,\chi_-(Q)); P \leqq E, Q \leqq I-E)\\ &= \inf(\omega_F(\chi_+(P))\,\tau_P(\chi_-(Q)); P \leqq E, Q \leqq I-E)\\ &= \inf(\alpha_1\,\omega(\chi_-(Q)); Q \leqq I-E) = \alpha_1\end{aligned}$$

where the last equality follows from the fact that $\omega(\chi_-(Q))=\omega_E(\chi_-(Q))=1$ for all $Q \leqq 1-E$. By interchanging the roles of α_1 and $\alpha_2, \chi_+(P)$ and $\chi_-(Q)$, we could have equally well argued that $\alpha'=\alpha_2$. Hence we have $\alpha'=\alpha_1=\alpha_2$, when $\alpha_1>0$ and $\alpha_2>0$ (i.e. $\alpha_1 \neq \alpha_2$ only if α_1 or α_2 vanishes). Hence, we have $\alpha'=\alpha=\min(\alpha_1,\alpha_2)$. This completes the proof of the theorem.

Lemma 2.7. *Suppose E and F are hermitian projections on a Hilbert space K and $\|E-F\|<1$. Let E_1 be a non zero hermitian projection orthogonal to E, i.e. $E_1 E=0$. Then $\|E_1+E-F\|=1$.*

Proof. Let $\mathfrak{M}$ be the range of F. We begin by showing that the range of FE is $\mathfrak{M}$. Let $\delta=1-\|E-F\|^2$.

$$I-(E-F)^2 \geqq (1-\|E-F\|^2)\,I=\delta I>0\,.$$

Hence, we have

$$FEF=F(I-(E-F)^2)\,F \geqq \delta F$$

Hence, FEF is strictly positive on $\mathfrak{M}$ and therefore FEF has an inverse on $\mathfrak{M}$. Therefore, the range of FEF and FE is $\mathfrak{M}$.

We complete the proof as follows. Let $h \neq 0$ be a vector in the range of E_1, i.e. $E_1 h=h$. Since $Fh \in \mathfrak{M}$ there is a $g \in K$ and that $FEg=Fh$.

Let $f=h-Eg$. Note $f \neq 0$ since $\|f\|^2=\|h\|^2+\|Eg\|^2$. We have that $(E_1+E)f=E_1h-E^2g=f$ and $Ff=Fh-FEg=0$. Hence, we have $(E_1+E-F)f=f$ and therefore $\|E_1+E-F\| \geqq 1$. Since $\|E_1+E-F\| \leqq 1$ the conclusion of the lemma follows.

Theorem 2.8. *Let ω_E and ω_F be pure generalized free states of $\mathfrak{A}(K)$. Then, ω_E and ω_F are unitarily equivalent if and only if $E-F$ is a Hilbert-Schmidt class operator.*

Proof. Suppose $\omega_E \sim \omega_F$. Then, we have $\omega_E \underset{q}{\sim} \omega_F$. From Lemma 2.1 it follows there is a finite dimensional subspace $\mathfrak{M} \subset K$ such that

$$\|\omega_E \,|\, \mathfrak{A}(\mathfrak{M})^c - \omega_F \,|\, \mathfrak{A}(\mathfrak{M})^c\| < 1 .$$

Let $\mathfrak{N}$ be the subspace of K spanned by $\{E\mathfrak{M}\}$ and $\{(I-E)\mathfrak{M}\}$ and let P_0 be the projection onto $\mathfrak{N}$. We note that $\mathfrak{N} \supset \mathfrak{M}$ and $P_0E=EP_0$. Since $\mathfrak{N} \supset \mathfrak{M}$ we have $\|\omega_E|\mathfrak{A}(\mathfrak{N})^c-\omega_F|\mathfrak{A}(\mathfrak{N})^c\|<1$. Since ω_E and ω_F are even states of $\mathfrak{A}(K)$ we have from Lemma 2.3 that

$$\|\omega_E \,|\, \mathfrak{A}(\mathfrak{N}^\perp) - \omega_F \,|\, \mathfrak{A}(\mathfrak{N}^\perp)\| < 1 .$$

Since for all finite projections $P \leqq (1-P_0)E$ we have

$$2\chi_+(P) - I \in \mathfrak{A}(\mathfrak{N}^\perp) \quad \text{and} \quad \|2\chi_+(P)-I\| = 1 ,$$

it follows that

$$\begin{aligned} 1 &> \|\omega_E \,|\, \mathfrak{A}(\mathfrak{N}^\perp) - \omega_F \,|\, \mathfrak{A}(\mathfrak{N}^\perp)\| \\ &\geqq \sup(|\omega_E(2\chi_+(P)-I)-\omega_F(2\chi_+(P)-I)|;\, P \leqq (I-P_0)E) \\ &= 2-2\inf(\omega_F(\chi_+(P));\; P \leqq (I-P_0)E) \\ &= 2-2\det(I-(I-P_0)\,E(I-F)\,(I-P_0)E) . \end{aligned}$$

Hence, we have $\det(I-(I-P_0)\,E(I-F)\,E(I-P_0)) > \frac{1}{2}$, and therefore
a) $\operatorname{Tr}((I-P_0)\,E(I-F)\,E(I-P_0)) < \infty$.

Similarly $2\chi_-(Q)-I \in \mathfrak{A}(\mathfrak{N}^\perp)$, and $\|2\chi_-(Q)-I\|=1$ for all finite projections $Q \leqq (I-P_0)\,(I-E)$. Therefore

$$\begin{aligned} 1 &> \sup(|\omega_E(2\chi_-(Q)-I)-\omega_F(2\chi_-(Q)-I)|;\; Q \leqq (I-P_0)\,(I-E)) \\ &= 2-2\inf(\omega_F(\chi_-(Q));\; Q \leqq (I-P_0)\,(I-E)) \\ &= 2-2\det(I-(I-P_0)\,(I-E)\,F(I-E)\,(I-P_0)) . \end{aligned}$$

Hence, $\det(I-(I-P_0)\,(I-E)\,F(I-E)\,(I-P_0)) > \frac{1}{2}$, and therefore we have
b) $\operatorname{Tr}((I-P_0)\,(I-E)\,F(I-E)\,(I-P_0)) < \infty$.

Adding inequalities (a) and (b), we find $\operatorname{Tr}((I-P_0)(E-F)^2(I-P_0)) < \infty$. Since P_0 is finite dimensional it follows that $\operatorname{Tr}((E-F)^2) < \infty$. Hence, $E-F$ is of Hilbert-Schmidt class.

Next we suppose $E-F$ is of Hilbert-Schmidt class. We show $\omega_E \sim \omega_F$. Let $\mathfrak{M}_+$ be the subspace of vectors $f \in K$ such that $(E-F)f=f$ and $\mathfrak{M}_-$ be the subspace of vectors $g \in K$ such that $(E-F)g=-g$. Let P_+ and P_- be the hermitian projections onto $\mathfrak{M}_+$ and $\mathfrak{M}_-$ respectively. Since $E-F$ is of Hilbert-Schmidt class, $\mathfrak{M}_+$ and $\mathfrak{M}_-$ are both finite dimensional. Let $E_1=E-P_+$ and $F_1=F-P_-$. Note E_1 and F_1 are projections and $\|E_1-F_1\|<1$; this last inequality follows from the fact that E_1-F_1 is compact and the spectrum of E_1-F_1 is contained in the open interval $(-1,1)$. We will show that $\omega_E \sim \omega_{E_1}$, $\omega_F \sim \omega_{F_1}$ and $\omega_{E_1} \sim \omega_{F_1}$.

From Lemmas 2.1 and 2.3 it follows that $\omega_{E_1} \underset{q}{\sim} \omega_E$ and $\omega_{F_1} \underset{q}{\sim} \omega_F$, since $\omega_{E_1} | \mathfrak{A}(\mathfrak{M}_+^\perp) = \omega_E | \mathfrak{A}(\mathfrak{M}_+^\perp)$ and $\omega_{F_1} | \mathfrak{A}(\mathfrak{M}_-^\perp) = \omega_F | \mathfrak{A}(\mathfrak{M}_-^\perp)$. Since these states are pure we have $\omega_{E_1} \sim \omega_E$ and $\omega_{F_1} \sim \omega_F$.

Since $\mathrm{Tr}((E_1-F_1)^2) \leqq \mathrm{Tr}((E-F)^2) < \infty$, it follows that $\mathrm{Tr}(E_1(I-F_1)E_1) + \mathrm{Tr}((I-E_1)F_1(I-E_1)) = \mathrm{Tr}((E_1-F_1)^2) < \infty$. Since, $\|E_1-F_1\|^2<1$ we have that $\|E_1(I-F_1)\,E_1\| \leqq \|(E_1-F_1)^2\| < 1$ and $\|(I-E_1)\,F_1(I-E_1)\| \leqq \|(E_1-F_1)^2\| < 1$. Hence we have that $\alpha_1 = \det(I-E_1(I-F_1)\,E_1) > 0$ and $\alpha_2 = \det(I-(I-E_1)\;F_1(I-E_1)) > 0$. Hence, $\alpha = \min(\alpha_1, \alpha_2) > 0$ (in fact $\alpha = \alpha_1 = \alpha_2$). Then, by Theorem 2.6 we have that $\|\omega_{E_1} - \omega_{F_1}\| = 2(1-\alpha)^{\frac{1}{2}} < 2$. Since ω_{E_1} and ω_{F_1} are pure, we have by Lemma 2.1, or by [12], $\omega_{E_1} \sim \omega_{F_1}$. Since $\omega_E \sim \omega_{E_1}$ and $\omega_F \sim \omega_{F_1}$, we have $\omega_E \sim \omega_F$. This completes the proof of the theorem.

3. States of Matrix Algebras

Every gauge invariant generalized free state ω_A of $\mathfrak{A}(K)$ can be extended to a pure generalized free state of $\mathfrak{A}(K \oplus K)$. We consider $\mathfrak{A}(K)$ as the subalgebra of $\mathfrak{A}(K \oplus K)$ generated by the $a(f)$ with $f=\{f_1, f_2\}$ and $f_2=0$. Given an operator $0 \leqq A \leqq I$ on I we define a projection E_A on $K \oplus K$ defined by the matrix of operators,

$$\begin{pmatrix} A & A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} \\ A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} & I-A \end{pmatrix},$$

i.e. $E_A\{f_1, f_2\} = \{A f_1 + A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} f_2,\ A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} f_1 + (I-A)\,f_2\}$. The mapping $\omega_A \to \omega_{E_A}$ carries generalized free states of $\mathfrak{A}(K)$ into pure generalized free state of $\mathfrak{A}(K \oplus K)$. The main result of this section is the estimate

$$\tfrac{1}{4}\|\omega_{E_A} - \omega_{E_B}\|^2 \leqq \|\omega_A - \omega_B\| \leqq \|\omega_{E_A} - \omega_{E_B}\|.$$

We begin by first considering the case where K is of finite dimension. Since for K of finite dimension $\mathfrak{A}(K)$ is a finite matrix algebra, we begin with a discussion of states on matrix algebras.

Suppose M is an $(n\times n)$-matrix algebra and ω is a state of M. Then ω may be represented by a density matrix, $\Omega\in M$, by the relation.

$$\omega(A)=\operatorname{Tr}(A\Omega) \quad \text{for all} \quad A\in M,$$

where $\Omega\geqq 0$ and $\operatorname{Tr}(\Omega)=1$. If ω_1 and ω_2 are states of M represented by Ω_1 and Ω_2 then $\|\omega_1-\omega_2\|=\|\Omega_1-\Omega_2\|_{\mathrm{Tr}}$, where $\|A\|_{\mathrm{Tr}}=\operatorname{Tr}(A^*A)^{\frac{1}{2}})$ $=\sup\{|\operatorname{Tr}(BA)|;\ \|B\|\leqq 1\}$.

In general it is difficult to compute $\|\Omega_1-\Omega_2\|_{\mathrm{Tr}}$, while it is not as difficult to compute $\|\Omega_1^{\frac{1}{2}}-\Omega_2^{\frac{1}{2}}\|^2_{\mathrm{H.S.}}=2-2\operatorname{Tr}(\Omega_1^{\frac{1}{2}}\,\Omega_2^{\frac{1}{2}})$. In the next section we show that

$$\|\Omega_1^{\frac{1}{2}}-\Omega_2^{\frac{1}{2}}\|^2_{\mathrm{H.S.}}\leqq\|\Omega_1-\Omega_2\|_{\mathrm{Tr}},$$

where Ω_1,Ω_2 are positive operators of trace one. We will use this result to obtain estimates on $\|\omega_A-\omega_B\|$.

We define a mapping ϕ from the states of M to the pure states of $M\otimes M$.

Let M act irreducibly on a Hilbert space $\mathfrak{H}$, *i.e.* we identify M as $\mathfrak{B}(\mathfrak{H})$. We define a linear mapping F of M onto $\mathfrak{H}\otimes\mathfrak{H}$, the tensor product of $\mathfrak{H}$ with itself. Let V be an antiunitary operator on $\mathfrak{H}$, i.e. V is conjugate linear, $V(h_1+h_2)=Vh_1+Vh_2$, $V\alpha h=\bar{\alpha}Vh$, all h, $h_1,h_2\in\mathfrak{H}$, and V is isometric, $\|Vh\|=\|h\|$ all $h\in\mathfrak{H}$. If A is expressed as a linear combination of rank one operators, i.e. $Ah=\sum_{i=1}^{n}(g_i,h)f_i$ for $h\in\mathfrak{H}$, we define $F(A)=\sum_{i=1}^{n}f_i\otimes Vg_i$. The fact that $F(A)$ depends only on A and not on the decomposition of A into rank one operators, follows from the relation, $\|F(A)\|^2=\operatorname{Tr}(A^*A)$. If $A,B\in M$ we have $(F(A),F(B))=\operatorname{Tr}(A^*B)$. We now define the mapping ϕ from states ω of M to pure states $\phi(\omega)$ of $M\otimes M$ by the relation,

$$\phi(\omega)(A\otimes B)=(F(\Omega^{\frac{1}{2}}),\ A\otimes B\,F(\Omega^{\frac{1}{2}})),$$

where

$$\omega(A)=\operatorname{Tr}(A\Omega).$$

We note that $\phi(\omega)|M\otimes I=\omega$, i.e. $\phi(\omega)(A\otimes I)=\omega(A)$ for $A\in M$. This can be seen as follows. Let Ω be the density matrix representing ω, i.e. $\omega(A)=\operatorname{Tr}(A\Omega)$. Diagonalizing Ω, we express Ω in the form, $\Omega h=\sum_{i=1}^{n}\alpha_i(f_i,h)f_i$, where (f_i) is an orthonormal basis of $\mathfrak{H}$. Then, we have $F(\Omega^{\frac{1}{2}})=\sum_{i=1}^{n}\alpha_i^{\frac{1}{2}}f_i\otimes Vf_i$ and $\phi(\omega)\,(A\otimes I)=\sum_{ij=1}^{n}\alpha_i^{\frac{1}{2}}\,\alpha_j^{\frac{1}{2}}(f_i,Af_j)\,(Vf_i,Vf_j)$ $=\sum_{i=1}^{n}\alpha_i(f_i,Af_i)=\omega(A)$.

The mapping ϕ depends only on V. Such a mapping, constructed from an antiunitary operator V in the manner described above will be called a *purification map*.

Lemma 3.1. *Let ϕ be a purification map from the states of an $(n \times n)$-matrix algebra M to the pure states of $M \otimes M$. Then, it follows that* $\frac{1}{4}\|\phi(\omega_1) - \phi(\omega_2)\|^2 \leqq \|\omega_1 - \omega_2\| \leqq \|\phi(\omega_1) - \phi(\omega_2)\|$, *for* ω_1, ω_2 *states of* M.

Proof. To prove second inequality we note that

$$\begin{aligned}\|\phi(\omega_1) - \phi(\omega_2)\| &\geqq \|\phi(\omega_1) | M \otimes I - \phi(\omega_2) | M \otimes I\| \\ &= \|\omega_1 - \omega_2\|.\end{aligned}$$

To prove the first inequality we identify M as $\mathfrak{B}(\mathfrak{H})$ and $M \otimes M$ as $\mathfrak{B}(\mathfrak{H} \otimes \mathfrak{H})$ and $\phi(\omega)\,(A \otimes B) = (F(\Omega^{\frac{1}{2}}), A \otimes B\, F(\Omega^{\frac{1}{2}}))$, where $\omega(A) = \mathrm{Tr}(A\,\Omega)$ for all $A \in M$. Since $M \otimes M$ acts irreducibly on $\mathfrak{H} \otimes \mathfrak{H}$ it follows from Lemma 2.4 that $\|\phi(\omega_1) - \varphi(\omega_2)\| = 2(1 - |(F(\Omega_1^{\frac{1}{2}}), F(\Omega_2^{\frac{1}{2}}))|^2)^{\frac{1}{2}}$. Since $(F(\Omega_1^{\frac{1}{2}}), F(\Omega_2^{\frac{1}{2}})) = \mathrm{Tr}(\Omega_1^{\frac{1}{2}}\,\Omega_2^{\frac{1}{2}})$ it follows from simple algebra that

$$2 - 2(1 - \tfrac{1}{4}\|\phi(\omega_1) - \phi(\omega_2)\|^2)^{\frac{1}{2}} = \|\Omega_1^{\frac{1}{2}} - \Omega_2^{\frac{1}{2}}\|_{\mathrm{H.S.}}^2.$$

From Lemma 4.1 it follows that $\|\Omega_1^{\frac{1}{2}} - \Omega_2^{\frac{1}{2}}\|_{\mathrm{H.S.}}^2 \leqq \|\Omega_1 - \Omega_2\|_{\mathrm{Tr}} = \|\omega_1 - \omega_2\|$. Hence by simple algebra we have

$$\tfrac{1}{4}\|\phi(\omega_1) - \phi(\omega_2)\|^2 \leqq \|\omega_1 - \omega_2\|\,(1 - \tfrac{1}{4}\|\omega_1 - \omega_2\|).$$

Hence, we have $\|\omega_1 - \omega_2\| \geqq \frac{1}{4}\|\phi(\omega_1) - \phi(\omega_2)\|^2$.

If K is an n-dimensional Hilbert space $M = \mathfrak{A}(K)$ is a $(2^n \times 2^n)$-matrix algebra. Since $\mathfrak{A}(K \oplus K)$ is a $(2^{2n} \times 2^{2n})$-matrix algebra we may identify $M \otimes M$ and $\mathfrak{A}(K \oplus K)$. We will construct a purification map ϕ from the states of $\mathfrak{A}(K)$ to the pure states of $\mathfrak{A}(K \oplus K)$ such that for generalized free states ω_A of $\mathfrak{A}(K)$ $\phi(\omega_A) = \omega_{E_A}$.

We identify K as the subspace of $K \oplus K$, consisting of all vectors $\{f_1, f_2\} \in K \oplus K$ where $f_2 = 0$. $K^\perp$ consists of the vectors $\{f_1, f_2\}$ where $f_1 = 0$. We identify $\mathfrak{A}(K)$ as the subalgebra of $\mathfrak{A}(K \oplus K)$ generated by the $a(f)$ with $f \in K$. Let U be the isometry of K into $K^\perp$ defined by the relation $U\{f_1, 0\} = \{0, f_1\}$. From Lemma 2.2 it follows that the commutant $\mathfrak{A}(K)^c$ of $\mathfrak{A}(K)$ in $\mathfrak{A}(K \oplus K)$ is generated by the elements $b(f) = a(Uf)\,V$ for $f \in K$, and $V = \prod_{i=1}^{n} (1 - 2a(h_i)^*\,a(h_i))$ where $\{h_1, \ldots, h_n\}$ is an orthonormal basis of K. By the argument of Lemma 2.2 we have $\mathfrak{A}(K \oplus K) = \mathfrak{A}(K) \otimes \mathfrak{A}(K)^c$.

Let ω_0 and ω_0^c be the Fock states of $\mathfrak{A}(K)$ and $\mathfrak{A}(K)^c$, i.e. $\omega_0(a(f)^*\,a(f)) = 0$ all $f \in K$ and $\omega_0^c(b(f)^*\,b(f)) = 0$ all $f \in K$. Let π and π^c be the Fock representations of $\mathfrak{A}(K)$ and $\mathfrak{A}(K)^c$ induced by ω_0

and ω_0^c on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}^c$. And let F_0 and F_0^c be cyclic vectors in $\mathfrak{H}$ and $\mathfrak{H}^c$ such that $\omega_0(A) = (F_0, \pi(A) F_0)$ and $\omega_0^c(B) = (F_0^c, \pi^c(B) F_0^c)$ for all $A \in \mathfrak{A}(K)$ and $B \in \mathfrak{A}(K)^c$. We define an antiunitary operator V_1 from $\mathfrak{H}$ into $\mathfrak{H}^c$ by the relations,

$$V_1 \pi(a(f_1)^* \ldots a(f_r)^*) F_0 = (\pi^c(b(f_1)^* \ldots b(f_r)^*))^* F_1^c$$

for $f_1, \ldots, f_r \in K$, where $F_1^c = \pi^c(b(h_1)^* \ldots b(h_n)^*) F_0^c$. V_1 may be defined for all $F \in \mathfrak{H}$ since $\mathfrak{H}$ is spanned by vectors of the above form. By straightforward computation one can show that V_1 is a conjugate linear isomorphism of $\mathfrak{H}$ onto $\mathfrak{H}^c$.

We will construct the purification map ϕ using V_1. Let ω_{00} be the Fock state of $\mathfrak{A}(K \oplus K)$, π_0 the induced Fock representation of $\mathfrak{A}(K \oplus K)$ on a Hilbert space $\mathfrak{H}_0$ with $F_{00} \in \mathfrak{H}_0$ such that $\omega_{00}(A) = (F_{00}, \pi_0(A) F_{00})$ for all $A \in \mathfrak{A}(K \oplus K)$. We note the following identifications, $\mathfrak{H}_0 = \mathfrak{H} \otimes \mathfrak{H}^c$, $F_{00} = F_0 \otimes F_0^c$, $\omega_{00} = \omega_0 \otimes \omega_0^c$ and $\pi_0(A \otimes B) = \pi(A) \otimes \pi^c(B)$ for $A \in \mathfrak{A}(K)$ and $B \in \mathfrak{A}(K)^c$.

In accordance with the previous discussion of purification maps, ϕ is constructed as follows. Suppose ω is a state of $\mathfrak{A}(K)$, and $\omega(A) = \sum_{i=1}^{s} \alpha_i (F_i, \pi(A) F_i)$ for $A \in \mathfrak{A}(K)$, $(F_i, F_j) = \delta_{ij}$, $\alpha_i \geqq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$. Let $G = \sum_{i=1}^{s} \alpha_i^{\frac{1}{2}} F_i \otimes V_1 F_i$. Then, we define $\phi(\omega)$ by the relation

$$\phi(\omega)(A \otimes B) = (G, \pi(A) \otimes \pi^c(B) G)$$

for all $A \in \mathfrak{A}(K)$ and $B \in \mathfrak{A}(K)^c$. We show that if ω_A is a generalized free state of $\mathfrak{A}(K)$ then $\phi(\omega_A) = \omega_{E_A}$.

In the calculations below we will make use of the following results. Suppose $\mathfrak{M}$ is a subspace of $K \oplus K$ and $(h_1, \ldots, h_n)$ and $(k_1, \ldots, k_n)$ are orthonormal bases for $\mathfrak{M}$. Then, if

$$A = a(h_1)^* \ldots a(h_n)^* \quad \text{and} \quad B = a(k_1)^* \ldots a(k_n)^*$$

we have $A = \alpha B$ were α is a complex number of modulus one. The pure generalized free state ω_E, where E is the projection onto $\mathfrak{M}$, is related to the Fock state ω_{00} by the relation $\omega_E(D) = \omega_{00}(A^* D A)$ for all $D \in \mathfrak{A}(K \oplus K)$.

We proceed to show $\phi(\omega_A) = \omega_{E_A}$. Suppose ω_A is a generalized free state of $\mathfrak{A}(K)$ and $\{f_1, \ldots, f_n\}$ is an orthonormal basis of K such that $A f_i = \lambda_i f_i$ for $i = 1, \ldots, n$. Let S be the set of subsets σ of the integers $(1, \ldots, n)$. Note S has 2^n elements. By a rather laborious calculation one can show

$$\omega_A(B) = \sum_{\sigma \in S} \alpha(\sigma) (F_\sigma, \pi(B) F_\sigma) \quad \text{for} \quad B \in \mathfrak{A}(K),$$

where

$$\alpha(\sigma)=\left(\prod_{i\in\sigma}\lambda_i\right)\left(\prod_{i\notin\sigma}(1-\lambda_i)\right),$$

$$F_\sigma=\pi(A_\sigma)\,F_0\,,\qquad A_\sigma=\prod_{i\in\sigma}a(f_i)^*\,.$$

Let $G=\sum_{\sigma\in S}\alpha(\sigma)^{\frac{1}{2}}F_\sigma\otimes V_1F_\sigma$. We have that $\phi(\omega_A)(A\otimes B)=(G,\pi(A)\otimes\pi^c(B)\,G)$ for $A\in\mathfrak{A}(K)$ and $B\in\mathfrak{A}(K)^c$.

We begin by computing $F_\sigma\otimes V_1\,F_\sigma$. We have

$$F_\sigma\otimes V_1\,F_\sigma=(\pi(A_\sigma)\otimes\pi^c(B_\sigma^*))\,(F_0\otimes F_1^c)$$

where $B_\sigma=\prod_{i\in\sigma}b(f_i)^*$ and $F_1^c=\pi^c(b(h_1)^*\dots b(h_n)^*)\,F_0^c$. We have that $F_1^c=z\,\pi^c(b(f_1)^*\dots b(f_n)^*)\,F_0^c$ where $|z|=1$, since $(f_1\dots f_n)$ and $(h_1\dots h_n)$ are both orthonormal basis of K. Since $\pi_0(A\otimes B)=\pi(A)\otimes\pi^c(B)$ for $A\in\mathfrak{A}(K)$ and $B\in\mathfrak{A}(K)^c$, it follows that

$$F_\sigma\otimes V_1\,F_\sigma=z\,\pi_0(A'_\sigma)\,F_{00}\,,$$

where $A'_\sigma=\prod_{i\in\sigma}a(f_i)^*\left(\prod_{i\in\sigma}b(f_i)^*\right)^*\prod_{i=1}^n b(f_i)^*$. Since $b(f_i)^*=a(U\,f_i)^*\,V$ and $\pi_0(V)\,F_{00}=F_{00}$, it follows that

$$F_\sigma\otimes V_1\,F_\sigma=z\,\pi_0(C_\sigma)\,F_{00}\,,$$

where $C_\sigma=\prod_{i\in\sigma}a(f_i)^*\left(\prod_{i\in\sigma}a(U\,f_i)^*\right)^*\prod_{i=1}^n a(U\,f_i)^*$. We can replace C_σ by the matrix

$$C'_\sigma=\prod_{i=1}^n q_{i\sigma}\,,$$

where

$$q_{i\sigma}=\begin{cases}a(f_i)^* & \text{for}\quad i\in\sigma\\ a(U\,f_i)^* & \text{for}\quad i\notin\sigma\,.\end{cases}$$

Hence, we have

$$G=\sum_{\sigma\in S}\alpha(\sigma)^{\frac{1}{2}}\,F_\sigma\otimes V_1\,F_\sigma=z\sum_{\sigma\in S}\pi_0(D_\sigma)\,F_{00}\,,$$

where $D_\sigma=\prod_{i=1}^n S_{i\sigma}$ and $S_{i\sigma}=\lambda_i^{\frac{1}{2}}\,a(f_i)^*$ for $i\in\sigma$ and $S_{i\sigma}=(1-\lambda_i)^{\frac{1}{2}}\,a(U\,f_i)^*$ for $i\notin\sigma$. Summing over all $\sigma\in S$, we find

$$G=z\,\pi_0(D_0)\,F_{00}\,,$$

where

$$D_0=\prod_{i=1}^n a(\lambda_i^{\frac{1}{2}}\,f_i+(1-\lambda_i)^{\frac{1}{2}}\,U\,f_i)^*\,.$$

Note that the vectors $g_i = \lambda_i^{\frac{1}{2}} f_i + (1-\lambda_i)^{\frac{1}{2}}\, U f_i$ from an orthonormal set which span the range of E_A. Since $\phi(\omega_A)(B) = (G, \pi_0(B)\, G)$ for all $B \in \mathfrak{A}(K \oplus K)$ it follows that $\phi(\omega_A) = \omega_{E_A}$.

Lemma 3.2. *Let ω_A and ω_B be generalized free states of $\mathfrak{A}(K)$. Let E_A and E_B be the projections on $K \oplus K$ given by the matrices of operators,*

$$E_A = \begin{pmatrix} A & A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} \\ A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} & I-A \end{pmatrix}, \quad E_B = \begin{pmatrix} B & B^{\frac{1}{2}}(I-B)^{\frac{1}{2}} \\ B^{\frac{1}{2}}(I-B)^{\frac{1}{2}} & I-B \end{pmatrix},$$

and let ω_{E_A} and ω_{E_B} be the pure generalized free states of $\mathfrak{A}(K \oplus K)$ determined by these projections. Then it follows that

$$\tfrac{1}{4}\, \|\omega_{E_A} - \omega_{E_B}\|^2 \leqq \|\omega_A - \omega_B\| \leqq \|\omega_{E_A} - \omega_{E_B}^{2}\|\,.$$

Proof. We identify K as a subspace of $K \oplus K$ and $\mathfrak{A}(K)$ as a subalgebra of $\mathfrak{A}(K \oplus K)$. If $\mathfrak{N}$ is a subspace of K, we will write $\mathfrak{N} \oplus 0$ to denote $\mathfrak{N}$ as a subspace of $K \oplus K$ (e.g. $\mathfrak{N} \oplus 0 \subset K \oplus 0 \subset K \oplus K$ and $\mathfrak{A}(\mathfrak{N} \oplus 0) \subset \mathfrak{A}(K \oplus K)$). From the construction of E_A and E_B and from the definition of generalized free states (Def. 1.2) it follows that $\omega_{E_A} | \mathfrak{A}(\mathfrak{N} \oplus 0) = \omega_A | \mathfrak{A}(\mathfrak{N})$ and $\omega_{E_B} | \mathfrak{A}(\mathfrak{N} \oplus 0) = \omega_B | \mathfrak{A}(\mathfrak{N})$ for all subspaces $\mathfrak{N} \subset K$. Since $\mathfrak{A}(\mathfrak{N} \oplus 0) \subset \mathfrak{A}(\mathfrak{N} \oplus \mathfrak{N})$ we have

$$\|(\omega_A - \omega_B) | \mathfrak{A}(\mathfrak{N})\| \leqq \|(\omega_{E_A} - \omega_{E_B}) | \mathfrak{A}(\mathfrak{N} \oplus \mathfrak{N})\|\,. \tag{a}$$

Setting $\mathfrak{N} = K$, we have $\|\omega_A - \omega_B\| \leqq \|\omega_{E_A} - \omega_{E_B}\|$.

Next we show $\frac{1}{4}\|\omega_{E_A} - \omega_{E_B}\|^2 \leqq \|\omega_A - \omega_B\|$.

If K is finite dimensional, then there is a purification map ϕ from the states of $\mathfrak{A}(K)$ to the pure states of $\mathfrak{A}(K \oplus K)$ such that $\phi(\omega_A) = \omega_{E_A}$ and $\phi(\omega_B) = \omega_{E_B}$. Then, it follows from Lemma 3.1 that $\frac{1}{4}\|\omega_{E_A} - \omega_{E_B}\|^2 \leqq \|\omega_A - \omega_B\|$.

Now suppose K is infinite dimensional and $\varepsilon > 0$. Since polynomials in the $a(f)$ and $a(f)^*$, $f \in K \oplus K$ are dense in $\mathfrak{A}(K \oplus K)$ it follows that there is a polynomial p such that $\|p\| \leqq 1$ and $|\omega_{E_A}(p) - \omega_{E_B}(p)| > \|\omega_{E_A} - \omega_{E_B}\| - \varepsilon$.

Since p is a polynomial there is a finite dimensional subspace $\mathfrak{M} \subset K \oplus K$ such that $p \in \mathfrak{A}(\mathfrak{M})$. Since $\mathfrak{M}$ is finite dimensional there is a finite dimensional subspace $\mathfrak{N} \subset K$ such that $\mathfrak{M} \subset \mathfrak{N} \oplus \mathfrak{N}$. Let $\{\mathfrak{N}_n; n = 1, 2, \ldots\}$ be an increasing sequence of finite dimensional subspaces of K, each of which contains $\mathfrak{N}$ and the closure of whose union is K. Let F_n be the hermitian projection onto $\mathfrak{N}_n$, and let $A_n = F_n A F_n$, $B_n = F_n B F_n$ for $n = 1, 2, \ldots$ Since $F_n \to I$ as $n \to \infty$ in the strong operator topology it follows from the work of Kaplansky [17], that $A_n \to A$, $B_n \to B$, $A_n^{\frac{1}{2}}(I - A_n)^{\frac{1}{2}} \to A^{\frac{1}{2}}(I-A)^{\frac{1}{2}}$ and $B_n^{\frac{1}{2}}(I - B_n)^{\frac{1}{2}} \to B^{\frac{1}{2}}(I-B)^{\frac{1}{2}}$ as $n \to \infty$ in the sense of strong convergence. Let P_n and Q_n be the projections on $K \oplus K$ defined by the matrices,

$$P_n = \begin{pmatrix} A_n & A_n^{\frac{1}{2}}(I-A_n)^{\frac{1}{2}} \\ A_n^{\frac{1}{2}}(I-A_n)^{\frac{1}{2}} & I-A_n \end{pmatrix}, \quad Q_n = \begin{pmatrix} B_n & B_n^{\frac{1}{2}}(I-B_n)^{\frac{1}{2}} \\ B_n^{\frac{1}{2}}(I-B_n)^{\frac{1}{2}} & I-B_n \end{pmatrix}.$$

Clearly, we have $P_n \to E_A$ and $Q_n \to E_B$ as $n \to \infty$ in the sense of strong convergence. Let ω_{P_n} and ω_{Q_n} be the generalized free states of $\mathfrak{A}(K \oplus K)$ determined by P_n and Q_n. Since p is a polynomial in the $a(f)$ and $a(f)^*$ and ω_{P_n} is a generalized free state, $\omega_{P_n}(p)$ can be expressed as a polynomial in matrix elements, $(f, P_n g)$ of P_n. Since $P_n \to E_A$, we have $\omega_{P_n}(p) \to \omega_{E_A}(p)$ as $n \to \infty$. Similarly we have $\omega_{Q_n}(p) \to \omega_{E_B}(p)$ as $n \to \infty$. Since $\mathfrak{N}_n$ is finite dimensional, we have

$$\begin{aligned} \|(\omega_{P_n} - \omega_{Q_n}) \,|\, \mathfrak{A}(\mathfrak{N}_n \oplus \mathfrak{N}_n)\| &\leqq 2\,\|(\omega_{A_n} - \omega_{B_n}) \,|\, \mathfrak{A}(\mathfrak{N}_n)\|^{\frac{1}{2}} \\ &\leqq 2\,|\omega_A - \omega_B\|^{\frac{1}{2}}. \end{aligned}$$

Since $p \in \mathfrak{A}(\mathfrak{N}_n \oplus \mathfrak{N}_n)$ for all $n = 1, 2, \ldots$ and $\|p\| \leqq 1$ we have

$$|\omega_{P_n}(p) - \omega_{Q_n}(p)| \leqq 2\,\|\omega_A - \omega_B\|^{\frac{1}{2}}.$$

Taking the limit as $n \to \infty$ we have $|\omega_{E_A}(p) - \omega_{E_B}(p)| \leqq 2\,\|\omega_A - \omega_B\|^{\frac{1}{2}}$. Hence, we have $\|\omega_{E_A} - \omega_{E_B}\| \leqq 2\,\|\omega_A - \omega_B\|^{\frac{1}{2}} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary we have $\frac{1}{4}\,\|\omega_{E_A} - \omega_{E_B}\|^2 \leqq \|\omega_A - \omega_B\|$. This completes the proof of the Lemma.

4. Hilbert-Schmidt Operators

In the present section we collect some results on Hilbert-Schmidt operators which will be needed later. The first two lemmas compare $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $A - B$ in Hilbert-Schmidt norms and trace norms. We denote by $\|S\|_{\mathrm{H.S.}}$ and $\|S\|_{\mathrm{Tr}}$ the Hilbert-Schmidt and the trace norm respectively of an operator S, i.e. $\|S\|_{\mathrm{H.S.}} = \mathrm{Tr}(S^* S)^{\frac{1}{2}}$ and $\|S\|_{\mathrm{Tr}} = \mathrm{Tr}(|S|)$ where $|S| = (S^* S)^{\frac{1}{2}}$.

Lemma 4.1. *Let A and B be positive operators on a Hilbert space K. Then*

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_{\mathrm{H.S.}}^2 \leqq \|A - B\|_{\mathrm{Tr}}.$$

Proof. If $A - B$ is not of trace class its trace norm is infinite, and the lemma is trivial. We therefore assume $A - B$ is of trace class, hence it is in particular compact. Let $S = A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $T = A^{\frac{1}{2}} + B^{\frac{1}{2}}$. Let π be a representation of $\mathfrak{B}(K)$ annihilating the compact operators. Then $\pi(A) = \pi(B)$, hence $\pi(A^{\frac{1}{2}}) = \pi(A)^{\frac{1}{2}} = \pi(B)^{\frac{1}{2}} = \pi(B^{\frac{1}{2}})$ by the uniqueness of the positive square root of a positive operator. Thus $\pi(S) = 0$, so S is compact. In particular S has pure point spectrum. Let $(f_i)_{i \geqq 1}$ be an orthonormal basis for K consisting of eigenvectors for S with eigenvalues λ_i. Then, since $T \geqq \pm S$ and $\frac{1}{2}(ST+TS)=A-B$, we have

$$\begin{aligned}\operatorname{Tr}(|A-B|) &= \sum_i \tfrac{1}{2}(f_i, |ST+TS|\, f_i)\\ &\geqq \sum_i |\tfrac{1}{2}(f_i, (ST+TS)\, f_i)|\\ &= \sum_i |\lambda_i (f_i, T f_i)|\\ &\geqq \sum_i \lambda_i^2\\ &= \sum_i (f_i, S^2 f_i)\\ &= \operatorname{Tr}\big((A^{\frac{1}{2}}-B^{\frac{1}{2}})^2\big).\end{aligned}$$

Lemma 4.2. *Let A and B be positive operators on a Hilbert space K. Suppose B has pure point spectrum and that there exists $\varepsilon > 0$ such that either $A \geqq \varepsilon I$ or $B \geqq \varepsilon I$. Then*

$$\sqrt{\varepsilon}\, \|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_{\mathrm{H.S.}} \leqq \|A-B\|_{\mathrm{H.S.}}\,.$$

Proof. If $A-B$ is not of Hilbert-Schmidt class its Hilbert-Schmidt norm is infinite, and the lemma is trivial. We therefore assume $A-B$ is a Hilbert-Schmidt operator. Let $(f_i)_{i \geqq 1}$ be an orthonormal basis for K consisting of eigenvectors for B with eigenvalues λ_i. Then

$$\begin{aligned}\operatorname{Tr}((A-B)^2) &= \sum_i (f_i, (A-B)^2 f_i)\\ &= \sum_i (f_i, (A-\lambda_i)^2 f_i)\\ &= \sum_i (f_i, (A^{\frac{1}{2}}-\lambda_i^{\frac{1}{2}})^2 (A^{\frac{1}{2}}+\lambda_i^{\frac{1}{2}})^2 f_i)\\ &\geqq \sum_i \varepsilon (f_i, (A^{\frac{1}{2}}-\lambda_i^{\frac{1}{2}})^2 f_i)\\ &= \varepsilon \operatorname{Tr}\big((A^{\frac{1}{2}}-B^{\frac{1}{2}})^2\big).\end{aligned}$$

Von Neumann [21] (see also [8]) has showed that every self-adjoint operator A on a separable Hilbert space can be written in the form $A=B+H$, where B is a self-adjoint operator with pure point spectrum and H is a self-adjoint Hilbert-Schmidt operator of arbitrarily small Hilbert-Schmidt norm. Moreover, the eigenvalues of B are dense in the spectrum of A. We first modify this result for our purposes.

Lemma 4.3. *Let A be an operator on a Hilbert space K such that $0 \leqq A \leqq I$, and let $\varepsilon > 0$. Then there exists an operator B, $0 \leqq B \leqq I$, with pure point spectrum such that $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class with Hilbert-Schmidt norms less than ε. Furthermore, the eigenvalues of B are dense in the spectrum of A.*

Proof. Let E be the spectral projection of A such that $EA \leqq \frac{1}{2}E$. If we consider the operators EA and $(I-E)A$ separately and use symmetry arguments we may assume $0 \leqq A \leqq \frac{1}{2}I$. Furthermore, if F is the projection on the null space of A, and B is defined to be 0 on FK, we may restrict attention to $(I-F)A$, hence we may assume A has no null space. Finally we may assume $\varepsilon \leqq 1$. Let E_n be the spectral projection of A such that

$$2^{-n-1}E_n < E_n A \leqq 2^{-n} E_n,$$

$n = 1, 2, \ldots$, and let $A_n = E_n A$. By von Neumann's theorem [21] there exist a self-adjoint operator B_n with pure point spectrum on the Hilbert space $E_n K$, such that the eigenvalues of B_n are dense in the spectrum of A_n, and a self-adjoint Hilbert-Schmidt operator H_n acting on $E_n K$ with $\|H_n\|_{\text{H.S.}} < \varepsilon 2^{-2n}$, such that $A_n = B_n + H_n$. Since $\|H_n\| \leqq \|H_n\|_{\text{H.S.}}$, $B_n = A_n - H_n \geqq 0$. Hence by Lemma 4.2

$$\|A_n^{\frac{1}{2}} - B_n^{\frac{1}{2}}\|_{\text{H.S.}} \leqq 2^{\frac{1}{2}(n+1)} \|H_n\|_{\text{H.S.}} < \varepsilon 2^{-n}.$$

Let $B = \sum^{\infty} B_n$. Then B has pure point spectrum with eigenvalues dense in the spectrum of A, and

$$\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_{\text{H.S.}} \leqq \sum_{n=1}^{\infty} \|A_n^{\frac{1}{2}} - B_n^{\frac{1}{2}}\|_{\text{H.S.}} < \varepsilon \sum_{n=1}^{\infty} 2^{-n} = \varepsilon.$$

Note that $E_n - A_n \geqq (1 - 2^{-n}) E_n \geqq \frac{1}{2} E_n$ and $E_n - B_n \geqq 0$. Hence by Lemma 4.2

$$\begin{aligned}\|(E_n - A_n)^{\frac{1}{2}} - (E_n - B_n)^{\frac{1}{2}}\|_{\text{H.S.}} &\leqq \sqrt{2}\,\|(E_n - A_n) - (E_n - B_n)\|_{\text{H.S.}} \\ &= \sqrt{2}\|H_n\|_{\text{H.S.}} < \varepsilon 2^{-n}.\end{aligned}$$

Since $E_n(I-A)^{\frac{1}{2}} = (E_n - A_n)^{\frac{1}{2}}$ and similarly for B,

$$\|(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}\|_{\text{H.S.}} \leqq \sum_{n=1}^{\infty} \|(E_n - A_n)^{\frac{1}{2}} - (E_n - B_n)^{\frac{1}{2}}\|_{\text{H.S.}} < \varepsilon.$$

The proof is complete.

Recall from the previous section that if A is an operator on a Hilbert space K, and $0 \leqq A \leqq I$, then E_A denotes the projection on the Hilbert space $K \oplus K$ defined by the matrix,

$$E_A = \begin{pmatrix} A & A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} \\ A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} & I - A \end{pmatrix}.$$

If B is another operator between 0 and I we next give a criterion in order that $E_A - E_B$ be Hilbert-Schmidt.

Lemma 4.4. *Let A and B be operators on a Hilbert space K such that $0 \leqq A \leqq I$ and $0 \leqq B \leqq I$. Then $E_A - E_B$ is a Hilbert-Schmidt operator if and only if the operator $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class.*

Proof. Let $H = A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $G = (I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$. Then

$$E_A - E_B = \begin{pmatrix} X & Y \\ Y^* & -X \end{pmatrix},$$

where

(a) $X = A^{\frac{1}{2}} H + H A^{\frac{1}{2}} - H^2$,

(b) $Y = A^{\frac{1}{2}} G + H(I-A)^{\frac{1}{2}} - HG$.

Now $E_A - E_B$ is Hilbert-Schmidt if and only if X and Y are both Hilbert-Schmidt operators on K. In particular, if G and H are Hilbert-Schmidt we have by (a) and (b) that $E_A - E_B$ is Hilbert-Schmidt. Conversely assume $E_A - E_B$ is Hilbert-Schmidt. By Lemma 4.3 there exists an operator C with pure point spectrum such that $0 \leqq C \leqq I$ and such that $A^{\frac{1}{2}} - C^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-C)^{\frac{1}{2}}$ are Hilbert-Schmidt operators. By the first part of the proof $E_A - E_C$ is Hilbert-Schmidt. If we can show that $B^{\frac{1}{2}} - C^{\frac{1}{2}}$ and $(I-B)^{\frac{1}{2}} - (I-C)^{\frac{1}{2}}$ are Hilbert-Schmidt, it follows that $A^{\frac{1}{2}} - B^{\frac{1}{2}} = (A^{\frac{1}{2}} - C^{\frac{1}{2}}) + (C^{\frac{1}{2}} - B^{\frac{1}{2}})$ is Hilbert-Schmidt, and similarly $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ is Hilbert-Schmidt. Hence in order to prove the lemma we may assume B has pure point spectrum.

Let $R_A = A^{\frac{1}{2}} + (I-A)^{\frac{1}{2}}$ and $R_B = B^{\frac{1}{2}} + (I-B)^{\frac{1}{2}}$. Then $R_A \geqq I$ and $R_B \geqq I$. Furthermore R_B, and hence R_B^2, has pure point spectrum. Thus by Lemma 4.2 we have

$$\begin{aligned} \|G+H\|_{\text{H.S.}} &= \|R_A - R_B\|_{\text{H.S.}} \\ &\leqq \|R_A^2 - R_B^2\|_{\text{H.S.}} \\ &= 2\|A^{\frac{1}{2}}(I-A)^{\frac{1}{2}} - B^{\frac{1}{2}}(I-B)^{\frac{1}{2}}\|_{\text{H.S.}} \\ &= 2\|Y\|_{\text{H.S.}} < \infty, \end{aligned}$$

so $G+H$ is Hilbert-Schmidt. Then by (b) $-A^{\frac{1}{2}} H + H(I-A)^{\frac{1}{2}} + H^2$ is a Hilbert-Schmidt operator. Adding this operator to X we obtain from (a) that HR_A is Hilbert-Schmidt. Since R_A is invertible H is Hilbert-Schmidt, hence so is G. The proof is complete.

If we combine this lemma with Lemma 4.1 we have the following corollary.

Corollary 4.5. *Let A and B be operators on a Hilbert space K such that $0 \leqq A \leqq I$, $0 \leqq B \leqq I$, and $A - B$ is of trace class. Then $E_A - E_B$ is a Hilbert-Schmidt operator.*

Lemma 4.6. *Let ω_A and ω_B be gauge invariant generalized free states of $\mathfrak{A}(K)$. Suppose $0 < \varepsilon < 2$. Then if $\|A^{\frac{1}{2}} - B^{\frac{1}{2}}\|_{\text{H.S.}} < \varepsilon/12$, and $\|(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}\|_{\text{H.S.}} < \varepsilon/12$, then $\|\omega_A - \omega_B\| < \varepsilon$.*

Proof. By Eqs. (a) and (b) in the proof of Lemma 4.4

$$\|X\|_{\text{H.S.}} \leqq \|A^{\frac{1}{2}} H\|_{\text{H.S.}} + \|H A^{\frac{1}{2}}\|_{\text{H.S.}} + \|H^2\|_{\text{H.S.}} < \varepsilon/4$$

and similarly $\|Y\|_{\text{H.S.}} < \varepsilon/4$. Therefore we have

$$\begin{aligned} \|E_A - E_B\|^2_{\text{H.S.}} &= \|X^2 + Y Y^*\|_{\text{Tr}} + \|Y^* Y + X^2\|_{\text{Tr}} \\ &\leqq 2\|X\|^2_{\text{H.S.}} + 2\|Y\|^2_{\text{H.S.}} \\ &< 4(\varepsilon/4)^2 = \varepsilon^2/4 . \end{aligned}$$

Now $\|E_A - E_B\|^2_{\text{H.S.}} = \text{Tr}((I - E_A) E_B (I - E_A)) + \text{Tr}(E_A (I - E_B) E_A)$. Hence $(I - E_A) E_B (I - E_A)$ and $E_A (I - E_B) E_A$ are of trace class. Let $(f_i)_{i \geqq 1}$ and $(g_i)_{i \geqq 1}$ be orthonormal bases for $K \oplus K$ consisting of eigenvectors for $(I - E_A) E_B (I - E_A)$ and $E_A (I - E_B) E_A$ respectively with respective eigenvalues λ_i and μ_i. Then $\lambda_i \geqq 0$, $\mu_i \geqq 0$, $\sum_{i=1}^{\infty} \lambda_i < \varepsilon^2/4$, and $\sum_{i=1}^{\infty} \mu_i < \varepsilon^2/4$. Hence

$$\det(I - (I - E_A) E_B (I - E_A)) = \prod_{i=1}^{\infty} (1 - \lambda_i) \geqq 1 - \sum_{i=1}^{\infty} \lambda_i > 1 - \varepsilon^2/4 ,$$

and similarly $\det(I - E_A (I - E_B) E_A) > 1 - \varepsilon^2/4$. Since $\varepsilon < 2$ it follows from the proof of Theorem 2.6 that the two determinants are equal, and

$$\|\omega_{E_A} - \omega_{E_B}\| < 2(1 - (1 - \varepsilon^2/4))^{\frac{1}{2}} = \varepsilon .$$

Since $\|\omega_A - \omega_B\| \leqq \|\omega_{E_A} - \omega_{E_B}\|$ by Lemma 3.2, the proof is complete.

5. The Main Theorems

In this section we shall give necessary and sufficient conditions for quasi and unitary equivalence of two gauge invariant generalized free states of the CAR-algebra $\mathfrak{A}(K)$, K being, as before, a separable Hilbert space. Recall that if ω is a state of a C^*-algebra $\mathfrak{A}$, and π_ω is its cyclic representation, we say ω is a factor state if $\pi_\omega(\mathfrak{A})''$ is a factor.

Theorem 5.1. *Every gauge invariant generalized free state of the CAR-algebra $\mathfrak{A}(K)$ is a factor state. Two gauge invariant generalized free state ω_A and ω_B are quasi-equivalent if and only if the operators $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I - A)^{\frac{1}{2}} - (I - B)^{\frac{1}{2}}$ are of Hilbert-Schmidt class.*

Proof. Let ω_A be a gauge invariant generalized free state of $\mathfrak{A}(K)$. By Lemma 4.3 we can choose a sequence of operators A_n on K with pure point spectra with eigenvalues dense in the spectrum of A, such that $0 \leqq A_n \leqq I$, $\|A^{\frac{1}{2}} - A_n^{\frac{1}{2}}\|_{\text{H.S.}} < 1/12n$, and $\|(I - A)^{\frac{1}{2}} - (I - A_n)^{\frac{1}{2}}\|_{\text{H.S.}} < 1/12n$. By Lemma 4.6 $\lim_n \|\omega_A - \omega_{A_n}\| \leqq \lim_n 1/n = 0$. By Lemma 1.3 ω_{A_n} is a factor state. Since the factor states of a C^*-algebra form a norm closed set by a theorem of Combes [6], ω_A is a factor state.

Suppose ω_B is another gauge invariant generalized free state. Suppose $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators. By Lemma 4.4 $E_A - E_B$ is a Hilbert-Schmidt operator, hence by Theorem 2.8 $\omega_{E_A} \sim \omega_{E_B}$. By Lemmas 2.1 and 2.3 there exists a finite dimensional subspace $\mathfrak{M}$ of $K \oplus K$ such that

$$\|(\omega_{E_A} - \omega_{E_B}) \,|\, \mathfrak{A}(\mathfrak{M}^\perp)\| < 2 .$$

Now there exists a finite dimensional subspace $\mathfrak{N}$ of K such that $\mathfrak{M} \subset \mathfrak{N} \oplus \mathfrak{N}$. Since $\mathfrak{N}^\perp \oplus \mathfrak{N}^\perp \subset (\mathfrak{N} \oplus \mathfrak{N})^\perp \subset \mathfrak{M}^\perp$, it follows by inequality (a) in the proof of Lemma 3.2 that

$$\begin{aligned} \|(\omega_A - \omega_B) \,|\, \mathfrak{A}(\mathfrak{N}^\perp)\| &\leqq \|(\omega_{E_A} - \omega_{E_B}) \,|\, \mathfrak{A}(\mathfrak{N}^\perp \oplus \mathfrak{N}^\perp)\| \\ &\leqq \|(\omega_{E_A} - \omega_{E_B}) \,|\, \mathfrak{A}(\mathfrak{M}^\perp)\| < 2 . \end{aligned}$$

By Lemmas 2.1 and 2.3 $\omega_A \underset{q}{\sim} \omega_B$.

Conversely, suppose $\omega_A \underset{q}{\sim} \omega_B$. Again by Lemmas 2.1 and 2.3 there exists a finite dimensional subspace $\mathfrak{M}$ of K such that

$$\|(\omega_A - \omega_B) \,|\, \mathfrak{A}(\mathfrak{M}^\perp)\| < 1 .$$

Let E denote the hermitian projection on $\mathfrak{M}$. Let $A_1 = EAE + (I-E) \cdot A(I-E)$ and $B_1 = EAE + (I-E)B(I-E)$. Then $A - A_1$ and $B - B_1$ have finite rank. Hence $E_A - E_{A_1}$ and $E_B - E_{B_1}$ are Hilbert-Schmidt operators by Corollary 4.5. Therefore by Theorem 2.8 $\omega_{E_A} \sim \omega_{E_{A_1}}$ and $\omega_{E_B} \sim \omega_{E_{B_1}}$. Now $\omega_{A_1} | \mathfrak{A}(\mathfrak{M}^\perp) = \omega_A | \mathfrak{A}(\mathfrak{M}^\perp)$ and similarly for B and B_1. Therefore

$$\|(\omega_{A_1} - \omega_{B_1}) \,|\, \mathfrak{A}(\mathfrak{M}^\perp)\| < 1 .$$

But A_1 and B_1 coincide on $\mathfrak{M}$, and both ω_{A_1} and ω_{B_1} are product states in the sense that if $S \in \mathfrak{A}(\mathfrak{M})$ and $T \in \mathfrak{A}(\mathfrak{M}^\perp)$ then $\omega_{A_1}(ST) = \omega_{A_1}(S)\,\omega_{A_1}(T)$ and similarly for B_1. Therefore $\|\omega_{A_1} - \omega_{B_1}\| < 1$, so by Lemma 3.2 we have

$$\|\omega_{E_{A_1}} - \omega_{E_{B_1}}\| \leqq 2\|\omega_{A_1} - \omega_{B_1}\|^{\frac{1}{2}} < 2 .$$

By Lemma 2.1, or by [12], $\omega_{E_{A_1}} \sim \omega_{E_{B_1}}$. By transitivity of unitary equivalence $\omega_{E_A} \sim \omega_{E_B}$. By Theorem 2.8 $E_A - E_B$ is a Hilbert-Schmidt operator. Hence by Lemma 4.4 $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators. The proof is complete.

In the course of the proof we showed

Corollary 5.2. *Let ω_A and ω_B be two gauge invariant generalized free states of $\mathfrak{A}(K)$. Then $\omega_A \underset{q}{\sim} \omega_B$ if and only if $\omega_{E_A} \sim \omega_{E_B}$.*

Our next objective is to study unitary equivalence of two states ω_A and ω_B. For this a more detailed knowledge of the factors obtained from

ω_A and ω_B is necessary. Let π_A denote the cyclic representation of $\mathfrak{A}(K)$ induced by ω_A. Let $R_A = \pi_A(\mathfrak{A}(K))''$. By Theorem 5.1 R_A is a factor. We say ω_A is of type X if R_A is of type X, $X = I, II_1, II_\infty, III$. Our next lemma gives characterizations of the type of ω_A in terms of A. An equivalent result has been obtained by Rideau [24], see also the paper of Dell'Antonio [8].

Lemma 5.3. *Let ω_A be a gauge invariant generalized free state of the CAR-algebra $\mathfrak{A}(K)$. Then*

(i) *ω_A is of type I if and only if there exist a spectral projection E of A and a trace class operator T such that $A = E + T$.*

(ii) *ω_A is of type II_1 if and only if there exists a Hilbert-Schmidt operator H such that $A = \frac{1}{2}I + H$.*

(iii) *ω_A is of type II_∞ if and only if there exist two orthogonal spectral projections P and Q of A with Q and $I - Q$ of infinite dimension, a self-adjoint trace class operator T such that $TQ = 0$, and a self-adjoint Hilbert-Schmidt operator H such that $HQ = H$, such that $A = P + T + \frac{1}{2}Q + H$.*

(iv) *ω_A is of type III otherwise.*

Proof. Suppose first A has pure point spectrum. Let $(f_i)_{i \geqq 1}$, $i = 1, 2, \ldots$, be an orthonormal basis for K consisting of eigenvectors for A. Let $A f_i = \lambda_i f_i$. Let ω_j be a state on the complex 2×2;matrices $\mathfrak{A}_j$ defined by

$$\omega_j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (1 - \lambda_j)a + \lambda_j d\,.$$

Use the notation in [13]. Let $\omega = \overset{*}{\bigotimes}_{i \geqq 1} \omega_i$ be the corresponding product state on $\mathfrak{A} = \overset{*}{\bigotimes}_{i \geqq 1} \mathfrak{A}_i$. Then $\mathfrak{A} \cong \mathfrak{A}(K)$ and ω can be identified with ω_A, see Lemma 1.3. Let E denote the spectral projection of A such that $AE \geqq \frac{1}{2}E$. For P any spectral projection of A we write $i \in P$ if $P f_i = f_i$. Then by a result of Araki [1] and Bures [5], see also [19], ω_A is of type I if and only if

a) $\sum_{i \in E} (1 - \lambda_i) + \sum_{i \in I - E} \lambda_i < \infty$,

hence if and only if $E(I - A) + (I - E)A$ is of trace class. Since this is equivalent to $A = E + T$ with T of trace class (i) as follows.

By a result of Moore [19] ω_A is of type II_1 if and only if

$$\sum_{i \geqq 1} (\lambda_i^{\frac{1}{2}} - 2^{-\frac{1}{2}})^2 + ((1 - \lambda_i)^{\frac{1}{2}} - 2^{-\frac{1}{2}})^2 < \infty$$

hence if and only if $A^{\frac{1}{2}} - 2^{-\frac{1}{2}} I$ and $(I - A)^{\frac{1}{2}} - 2^{-\frac{1}{2}} I$ are Hilbert-Schmidt. An easy argument using Lemma 4.2 shows that this is equivalent to $A - \frac{1}{2}I$ being a Hilbert-Schmidt operator. Thus (ii) follows.

If $c>0$ and x is a real number let $|x|_c = \inf(|x|, c)$. Then by a result of Moore [19] ω_A is not of type III if and only if for some, and hence all, $c>0$

$$\text{b)}\quad \sum_{i\in E}(1-\lambda_i)\left|\frac{\lambda_i}{1-\lambda_i}-1\right|_c^2 + \sum_{i\in I-E}\lambda_i\left|\frac{1-\lambda_i}{\lambda_i}-1\right|_c^2 < \infty .$$

Let F and G be spectral projections of A such that $AF \leqq \frac{1}{3}F$ and $AG \geqq \frac{2}{3}G$. Assume ω_A is of type II_∞ and put $c=1$. Then by b) we have

$$\begin{aligned}
\infty > & \sum_{i\in E-G}(1-\lambda_i)\left(\frac{\lambda_i}{1-\lambda_i}-1\right)^2 + \sum_{i\in G}(1-\lambda_i) \\
& + \sum_{i\in I-E-F}\lambda_i\left(\frac{1-\lambda_i}{\lambda_i}-1\right)^2 + \sum_{i\in F}\lambda_i \\
\geqq & \sum_{i\in E-G}(\lambda_i^{\frac{1}{2}}-(1-\lambda_i)^{\frac{1}{2}})^2 + \sum_{i\in G}(1-\lambda_i) + \sum_{i\in I-E-F}((1-\lambda_i)^{\frac{1}{2}}-\lambda_i^{\frac{1}{2}})^2 + \sum_{i\in F}\lambda_i \\
\geqq & \tfrac{1}{2}\sum_{i\in I-F-G}\{(\lambda_i^{\frac{1}{2}}-2^{-\frac{1}{2}})^2 + ((1-\lambda_i)^{\frac{1}{2}}-2^{-\frac{1}{2}})^2\} + \sum_{i\in F}\lambda_i + \sum_{i\in G}(1-\lambda_i),
\end{aligned}$$

since $2(\lambda_i^{\frac{1}{2}}-(1-\lambda_i)^{\frac{1}{2}})^2 \geqq (\lambda_i^{\frac{1}{2}}-2^{-\frac{1}{2}})^2 + ((1-\lambda_i)^{\frac{1}{2}}-2^{-\frac{1}{2}})^2$. Therefore, if ω_A is of type II_∞ then $\dim(F+G)=\dim(I-F-G)=\infty$, and by using the arguments employed in the proofs of (i) and (ii),

$$A = G + T + \tfrac{1}{2}(I-F-G) + H,$$

where T is a self-adjoint trace class operator such that $T(I-F-G)=0$, and H is a self-adjoint Hilbert-Schmidt operator such that $H = H(I-F-G)$. Letting $P=G$ and $Q=I-F-G$, A has the form in (iii). Conversely, by (i) and (ii), if A has the form in (iii) then ω_A is of type II_∞.

Finally, if A does not have pure point spectrum then from the proof of Theorem 5.1 ω_A can be approximated in norm by states ω_{A_n}, where A_n has pure point spectrum, and its eigenvalues are dense in the spectrum of A. By cases (i), (ii), and (iii) ω_{A_n} is of type III for all n, hence ω_A is of type III. The proof is complete.

Remark 5.4. From the proof of Theorem 5.1 if A does not have pure point spectrum then ω_A is quasi-equivalent to a state ω_B, where B has pure point spectrum and its eigenvalues are dense in the spectrum of A. By the work of Araki and Woods [2] the factor R_B obtained from ω_B belongs to the class S_∞ defined in [2]. Since all factors in the class S_∞ are isomorphic [2, Theorem 7.6] it follows that if A and A' do not have pure point spectra then the factors R_A *and* $R_{A'}$ are isomorphic, i.e. ω_A and $\omega_{A'}$ are algebraically equivalent.

In order to study unitary equivalence of two states ω_A and ω_B one needs information on the commutants of R_A and R_B. For this the following definition is convenient.

Definition 5.5. *Let A be an operator on the Hilbert space K such that $0 \leqq A \leqq I$. Let N be the hermitian projection on the null space of $A(I-A)$. We say A is elementary if $A = P + \frac{1}{2}(I-N) + H$, where P is a hermitian projection, $P \leqq N$, and H is a self-adjoint Hilbert-Schmidt operator such that $HN = 0$.*

Lemma 5.6. *Let ω_A be a gauge invariant generalized free state of $\mathfrak{A}(K)$. Let π_A denote the cyclic representation of $\mathfrak{A}(K)$ induced by ω_A, and let $R_A = \pi_A(\mathfrak{A}(K))''$. Then R'_A is a finite factor if and only if A is elementary.*

Proof. Let N denote the null space of $A(I-A)$. Let $P = AN$. Then P is a hermitian projection. By Lemma 5.3 R_A is semi-finite if and only if $A = P + P_1 + T + \frac{1}{2}Q + H'$, where P_1 and Q are orthogonal spectral projections of $A(I-N)$, T is a self-adjoint trace class operator such that $TQ = 0$, and H' is a self-adjoint Hilbert-Schmidt operator such that $H'Q = H'$. Use the notation introduced in the proof of Lemma 5.3. For $A_j \in \mathfrak{A}_j$, let $\omega_j(A_j) = (f_j, \pi_j(A_j) f_j)$, where π_j is a cyclic representation of the 2×2-matrices $\mathfrak{A}_j$ with f_j as a cyclic vector. Let $R = I - N - Q$. Let $f_N = \bigotimes_{j \in N} f_j$, $f_R = \bigotimes_{j \in R} f_j$, and $f_Q = \bigotimes_{j \in Q} f_j$. Put

$$R_1 = \left(\overset{*\,f_N}{\bigotimes_{i \in N}} \pi_i(\mathfrak{A}_i) \right)'', \quad R_2 = \left(\overset{*\,f_R}{\bigotimes_{i \in R}} \pi_i(\mathfrak{A}_i) \right)'', \quad \text{and} \quad R_3 = \left(\overset{*\,f_Q}{\bigotimes_{i \in Q}} \pi_i(\mathfrak{A}_i) \right)''.$$

The $R_A = R_1 \otimes R_2 \otimes R_3$, where the tensor product is that of von Neumann algebras. By Lemma 5.3 R_3 is a finite factor. Since the eigenvalues of $\frac{1}{2}Q + H'$ are all different from 0 and 1, R_3 has a separating and cyclic vector (see e.g. [2, Lemma 2.10]). Then it follows from [9, Théorème 5, p. 235] that R'_3 is finite. For $i \in N$, $\pi_i(\mathfrak{A}_i)$ equals all bounded operators on the two dimensional Hilbert space. Therefore R_1 is all bounded operators on a Hilbert space ([13], Corollaire 2.1). In particular R'_1 is finite. Now, by a result of Araki and Woods [2, Lemma 6.10]

$$R'_2 = \left(\overset{*\,f_R}{\bigotimes_{i \in R}} \pi_i(\mathfrak{A}_i)' \right)''.$$

Let $k = \dim R$. Since all the eigenvalues of $P_1 + T$ are different from 0 and 1, R_2 has a separating and cyclic vector by [2], Lemma 2.10. By Lemma 5.3 R_2 is of type I. In fact it is of type I_{2^k} by construction. Since $R'_2 \cong R_2$, R'_2 is of type I_{2^k}. By [9], Proposition 14, p. 102, $R'_A = R'_1 \otimes R'_2 \otimes R'_3$. Therefore R'_A is finite if and only if $k < \infty$. But if $k < \infty$ we can replace Q by $I - N$ and replace $T + \frac{1}{2}Q + H'$ by $\frac{1}{2}(I-N) + H$, where H is a Hilbert-Schmidt operator. This completes the proof of the lemma.

We are now in the position to give characterizations in terms of A and B for unitary equivalence of the states ω_A and ω_B. Due to the highly complicated situation which occurs when R_A and R_B are of type II_∞ with finite commutants, we cannot give a satisfactory characterization in that case. We are indebted to E. Effros for pointing out the relevance of Ref. [14].

Theorem 5.7. *Let ω_A and ω_B be gauge invariant generalized free states of $\mathfrak{A}(K)$, K a Hilbert space. Let N and N' denote the projections on the null spaces of $A(I-A)$ and $B(I-B)$ respectively. There are two cases.*

(1) *If A is not elementary* (see Def. 5.5) *then ω_A is unitarily equivalent to ω_B if and only if $A^{\frac{1}{2}}-B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}}-(I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators and B is not elementary.*

(2) *If A is elementary then ω_A is unitarily equivalent to ω_B if and only if $A^{\frac{1}{2}}-B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}}-(I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators, B is elementary, and*

(i) *if* $\dim N<\infty$ *then* $\dim N'=\dim N$,

(ii) *if* $\dim(I-N)<\infty$ then $\dim(I-N')=\dim(I-N)$,

(iii) *if* $\dim N=\dim(I-N)=\infty$, then $\dim N'=\dim(I-N')=\infty$, *and*

$$\mathrm{Tr}(\alpha([R'_A f_A]))=\mathrm{Tr}([R'_B f_B]),$$

where π_A is a cyclic representation of $\mathfrak{A}(K)$ and f_A a cyclic vector such that $\omega_A(S)=(f_A,\pi_A(S)f_A)$, $R_A=\pi_A(\mathfrak{A}(K))''$, and similarly for B, α is the isomorphism of R_A onto R_B such that $\pi_B=\alpha\circ\pi_A$, and Tr *is a normal trace on R_B.*

Proof. If R_1 and R_2 are factors on separable Hilbert spaces with infinite commutants then every $*$-isomorphism of R_1 onto R_2 is unitarily implemented ([9, Corollaire 7, p. 321]). Since in our case R_A and R_B act on separable Hilbert spaces, since $\mathfrak{A}(K)$ is norm separable and f_A and f_B are cyclic vectors (see [11, Theorem 3.5]), case (1) follows from Theorem 5.1 and Lemma 5.6.

Suppose A is elementary. Let $a=\dim N$, $b=\dim(I-N)$. From the proof of Lemma 5.6 $R_A=\mathfrak{M}_A\otimes\mathfrak{N}_A$, where $\mathfrak{M}_A$ is all bounded operators on a Hilbert space of dimension 2^a, and $\mathfrak{N}_A$ is a finite factor of coupling 1, having a separating and cyclic vector. If $b<\infty$, $\mathfrak{N}_A$ is of type I_{2^b}, otherwise it is of type II_1. Since unitary equivalence is a stricter property than quasi-equivalence we may by Theorem 5.1 assume $A^{\frac{1}{2}}-B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}}-(I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt operators. Furthermore by Lemma 5.6 R'_A is finite, hence if $\omega_A\sim\omega_B$, R'_B is finite, hence by Lemma 5.6 B is elementary. We may therefore assume B is elementary. Let α denote the isomorphism of R_A onto R_B such that $\alpha\circ\pi_A=\pi_B$, found by Theorem 5.1. Consider the three cases separately.

(i) Suppose $a<\infty$. T hen $b=\infty$, so R_A is of type II_1 with coupling 2^a. By [9], Proposition 10, p. 286, α is unitarily implemented if and only if R_B has coupling 2^a, hence if and only if $\dim N' = \dim N$.

(ii) Suppose $b<\infty$. Then $R'_A \cong \mathfrak{R}'_A$ is of type I_{2^b}. By [9, Proposition 3, p. 253], α is unitarily implemented if and only if R'_B is of type I_{2^b}, hence if and only if $\dim(I-N') = \dim(I-N)$.

(iii) Suppose $a=b=\infty$. Then R_A is of type II_∞ with finite commutant. As shown by Kadison [14] this case is much more complicated, indeed by [14, Theorem 2], α is unitarily implemented if and only if the conditions in (iii) hold. The proof is complete.

It should be noted that if A and B are bounded away from 0 and 1 then the study of unitary equivalence of ω_A and ω_B is greatly simplified. Indeed, we have the following corollary.

Corollary 5.8. *Let ω_A and ω_B be gauge invariant generalized free states of $\mathfrak{A}(K)$. Suppose 0 and 1 do not belong to the spectra of A and B. Then ω_A and ω_B are unitarily equivalent if and only if $A-B$ is a Hilbert-Schmidt operator.*

Proof. By assumption it follows from the compactness of the spectra of A and B that there exists $\varepsilon>0$ such that $\varepsilon I \leqq A \leqq (1-\varepsilon)I$ and $\varepsilon I \leqq B \leqq (1-\varepsilon)I$. This the corollary is an immediate consequence of Lemma 4.2 and Theorem 5.7.

Remark 5.9. In applications of Theorem 5.7 the case 2 (iii) in the theorem is fortunately a very special case. The condition we have given, essentially requires that one must know the isomorphism α in detail in order to be able to conclude whether ω_A and ω_B are unitarily equivalent. We give a simple example in which α is not unitarily implemented, thus pointing out that no simple criterion seems available. For a detailed analysis of this problem the reader is referred to the work of Kadison [15].

Let K be as in Theorem 5.7. Let E be a projection on K such that $\dim E = \dim(I-E) = \infty$. Let F be a one-dimensional projection orthogonal to E, and let $G = I-E-F$. Let $A=\frac{1}{2}E$, $B=\frac{1}{2}(E+F)$. By Theorem 5.1 ω_A and ω_B are quasi-equivalent. Let (f_i) be a basis for K consisting of eigenvectors for A and B. As before we say $i\in P$ if $P f_i = f_i$ for a spectral projection P of A and B. Use the notation introduced in the proof of Lemma 5.3 with the addition that we put primes on the corresponding representations for B. Then we have

$$\pi_A = \left(\overset{*}{\bigotimes_{i\in E}}{}^{f_E}\,\pi_i\right) \otimes \left(\bigotimes_{i\in F}{}^{f_F}\,\pi_i\right) \otimes \left(\overset{*}{\bigotimes_{i\in G}}{}^{f_G}\,\pi_i\right)$$

$$\pi_B = \left(\overset{*}{\bigotimes_{i\in E}}{}^{f'_E}\,\pi'_i\right) \otimes \left(\overset{*}{\bigotimes_{i\in F}}{}^{f'_F}\,\pi'_i\right) \otimes \left(\overset{*}{\bigotimes_{i\in G}}{}^{f'_G}\,\pi'_i\right),$$

where for example $f_E = \bigotimes_{i \in E} f_i$, where $\omega_i(A) = (f_i, \pi_i(A) f_i)$ for $i \in E$, and $A \in \mathfrak{A}_i$. If $i \in E$ then π_i and π_i' are both of the form $A \to A \otimes I$ with I the identity on the 2-dimensional Hilbert space. If $i \in G$ π_i and π_i' are both irreducible; hence if $i \in E + G$ then π_i and π_i' are unitarily equivalent. If $i \in F$ then π_i is irreducible while π_i' has multiplicity two. Let $R_A = \pi_A(\mathfrak{A}(K)''$ and $R_B = \pi_B(\mathfrak{A}(K))''$. Since π_A and π_B are quasi-equivalent there exists an isomorphism α of R_A onto R_B such that $\alpha \circ \pi_A = \pi_B$. If $A_i \in \mathfrak{A}_i$ and $A_i = I$ for all but a finite number of i's then $\alpha(\otimes \pi_i(A_i)) = \otimes \pi_i'(A_i)$, hence an easy argument shows the existence of isomorphisms α_i of $\pi_i(\mathfrak{A}_i)$ onto $\pi_i'(\mathfrak{A}_i)$ such that $\alpha_i \circ \pi_i = \pi_i'$ and such that $\alpha \mid \pi_A(\mathfrak{A}(K)) = \overset{*}{\bigotimes} \alpha_i$. Let P (resp. Q) denote the one-dimensional projection in

$$\left(\overset{*}{\bigotimes_{i \in E}}{}^{f_E} \pi_i(\mathfrak{A}_i) \right)'' \left(\text{resp.} \left(\overset{*}{\bigotimes_{i \in E}}{}^{f_E'} \pi_i'(\mathfrak{A}_i) \right)'' \right)$$

onto f_E (resp. f_E'), and let $[y]$ denote the one-dimensional projection in $\pi_i(\mathfrak{A}_i)$, $i \in F$, on $y = f_i$. Then we have

$$[R_A' f_A] = P \otimes [y] \otimes I ,$$
$$[R_B' f_B] = Q \otimes I \otimes I ,$$

where the identity I to the right is the identity in the algebra obtained from the portion in G. Let Tr denote the trace on R_B. Then, if $k \in F$,

$$\begin{aligned} \operatorname{Tr}(\alpha([R_A' f_A])) &= \operatorname{Tr}(Q \otimes \alpha_k([y]) \otimes I) \\ &= \tfrac{1}{2} \operatorname{Tr}(Q \otimes I \otimes I) \\ &= \tfrac{1}{2} \operatorname{Tr}([R_B' f_B]) . \end{aligned}$$

By [14, Theorem 2], α is not unitarily implemented.

It should be remarked that in this example there exists a unitary operator U on K such that $B = U A U^{-1}$. In all other cases than 2(iii) in Theorem 5.7 it follows from that theorem that if $B = U A U^{-1}$, and $A^{\frac{1}{2}} - B^{\frac{1}{2}}$ and $(I-A)^{\frac{1}{2}} - (I-B)^{\frac{1}{2}}$ are Hilbert-Schmidt, then ω_A and ω_B are unitarily equivalent.

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