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On Limits of Separable Potentials and Operator Extensions

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Abstract. A family of self-adjoint Hamiltonians with a separable potential leading towards a contact potential (zero range) is analyzed by tools of functional analysis. It is shown that the family of time evolution operators e^{-iHt} converges strongly (for all t) though the family of Hamiltonians does not converge even weakly. In the case of three dimensions a renormalization procedure is discussed and a correspondence between the renormalized coupling constant and the self-adjoint extensions of the free Hamiltonian is established.

Introduction

The object which we are going to analyze is a one parameter family of Hamiltonians $\{H(\mu)\}_{\mu\in[0,1]} = \{T + V(\mu)\}_{\mu\in[0,1]}$ where $V(\mu)$ denotes a separable potential. We want to give a precise mathematical meaning to the statement, that for $\mu\to 0$ $H(\mu)$ "converges" towards a Hamilton with a contact potential. Furthermore the process of renormalization is a very delicate one from the mathematical point of view and should therefore be studied carefully in the simplest possible case. The detailed study is also necessary as it serves as preparation for the treatment of the problem in the framework of second quantization (to be published in a subsequent paper). We enclose our system in a box of finite length L and assume periodic boundary conditions; the spectra of $H(\mu)$ will therefore be purely discrete for all μ. At first we treat the one-dimensional case which already shows some of the relevant features and study then the case of three dimensions. Our units are $\hbar = 2m = 1$; for simplicity we assume $2\pi/L = 1$, our momenta are therefore $k = n$, n integer or $k = (n_1, n_2, n_3)$ n_i integer, $i = 1, 2, 3$ resp. We use the notation $\rightharpoonup$ and $\to$ for weak and strong convergence resp.

1. The One-Dimensional Case

We consider a one-parameter family of separable potentials $\{V(\mu)\}_{\mu\in[0,1]}$ which approach (in an intuitive sense to be specified) a contact potentials as $\mu\to 0$. In momentum space this is expressed by

$$V(\mu)_{nm} = \lambda \sigma_n^*(\mu)\,\sigma_m(\mu) \quad \text{with} \quad \sigma_n(\mu)\to 1\ \forall n \text{ as } \mu\to 0\,.$$

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We assume that the $\sigma_n(\mu)$ are continuously differentiable in $\mu \in [0, 1]$, that they are bounded by one, $|\sigma_n(\mu)| \leqq 1 \,\forall n, \forall \mu$ and that

$$\sigma(\mu) \equiv \{\sigma_n(\mu)\}_{n \in z} \in \ell_2 \,\forall \mu \neq 0 .$$

$\left(\text{One may always have in mind the particular form } \sigma_n(\mu) = \dfrac{1}{1+n^2\mu^2} .\right)$

Putting $g = n^2$ and introducing the two-dimensional vectors $\tilde{\sigma}_g(\mu) = (\sigma_n(\mu), \sigma_{-n}(\mu))$ we obtain a very convenient basis (especially in the three-dimensional case), $|g, \mu\rangle$, in which the Schroedinger-equation has two classes of solutions: those which are orthogonal to the $\tilde{\sigma}_g(\mu) \forall g (\mathfrak{H}^A)$ and those which point in the direction of $\tilde{\sigma}_g(\mu)$[1] (for each g separately). In the following we shall concentrate only on the latter ($\mathfrak{H}^S$). Defining

$$\begin{aligned} \sigma_g(\mu) \equiv |\tilde{\sigma}_g(\mu)| &= \sqrt{\sum_{n^2=g} |\sigma_n(\mu)|^2} \qquad \forall g = \text{square of an integer} \\ &= 0 \qquad\qquad\qquad\qquad\quad\; \forall g \neq \text{square} \end{aligned} \tag{1}$$

with

$$\lim_{\mu \to 0} \sigma_g(\mu) \equiv \sqrt{Ng} = \begin{matrix} \sqrt{2} & g = \text{square} \neq 0 \\ 1 & g = 0 \\ 0 & g \neq \text{square} \end{matrix} \tag{2}$$

the Schroedinger-equation reads as

$$(g - E)\,\phi_E(g, \mu) + \lambda \sigma_g(\mu) \sum_{g'=0}^{\infty} \sigma_{g'}(\mu)\,\phi_E(g', \mu) = 0; \tag{3}$$

its solutions (belonging to the second class) have the explicit form

$$\langle E | g, \mu\rangle \equiv \phi_E(g, \mu) = \sqrt{\frac{\lambda}{D_{,E}(E, \mu)}}\,\frac{\sigma_g(\mu)}{g - E(\mu)} \tag{4}$$

where E is a solution of

$$D(E, \mu) \equiv 1 + \lambda \sum_{g=0}^{\infty} \frac{\sigma_g^2(\mu)}{g - E(\mu)} = 0 . \tag{5}$$

For the matrix elements of $H(\mu)$ we obtain in this basis (g = square)

$$H(\mu)_{gg'} = g\delta_{gg'} + \lambda \sigma_g(\mu)\,\sigma_{g'}(\mu) \xrightarrow{\mu \to 0} g\delta_{gg'} + \lambda \sqrt{N_g N_{g'}} \tag{6}$$

and, by complex integration, for the resolvent

$$\operatorname{Res}(z, \mu)_{gg'} = \frac{\delta_{gg'}}{z - g} + \frac{\lambda \sigma_g(\mu)\,\sigma_{g'}(\mu)}{D(z, \mu)\,(z - g)\,(z - g')} . \tag{7}$$

[1] These directions depend on μ in general; it is only for the symmetric potential, i.e. $\sigma_n(\mu) = \sigma_{-n}(\mu) \,\forall n$, that the direction is independent of $\mu \,\forall g$. Since each element $|g, \mu\rangle$ of the basis is an element of a finite dimensional subspace, $|g, \mu\rangle$ is continuous in the strong ℓ_1 topology for all $\mu \in [0, 1]$ and $\forall g$.

As we shall see $H(0)$ is not defined *on* $|g\rangle$; by $H(0)_{gg'}$ we rather denote $\sum_{E\in\operatorname{spec}H(0)} \langle g'|E\rangle E\langle E|g\rangle$ and with this definition we have: $\langle g|H\psi\rangle = \sum_{g'} H(0)_{gg'}\langle g'|\psi\rangle \;\forall\psi\in\mathfrak{D}(H(0))$.

We now state some properties concerning the eigenvalues and eigenfunctions.

Proposition 1.1. As functions of μ the eigenvalues $E=E(\mu)$ are continuously differentiable in $[0,1]$.

Proof. Since $D_{,E}(E,\mu)=\lambda\sum_{g=0}^{\infty}\frac{\sigma_g(\mu)^2}{(g-\mathrm{E})^2}\neq 0\,\forall\mu$ and since $D_{,\mu}(E,\mu)$ exists we can apply the implicite function theorem.

The structure of $D(E,\mu)=0$ shows that there is exactly one solution $E(\mu)$ between two succeeding n^2 and $(n+1)^2$ for all n and all $\mu\in[0,1]$ and that $E(\mu)$ never coincides with a pole g of D. If $\lambda<0$ there also exists exactly one solution $E<0$. The eigenvalues belonging to eigenfunctions of the second class are therefore non degenerate. They are furthermore orthogonal and complete (μ fixed). As a consequence we see explicitly that $H(\mu)$ is self-adjoint for all $\mu\in[0,1]$.

Proposition 1.2. $|\phi_E(\mu)\rangle \xleftrightarrow{\ell_1} |\phi_E(0)\rangle$ for all ϕ_E of class $\mathfrak{H}^S$.

Proof. We have

$$|\phi_E(g,\mu)|=\sqrt{\frac{\lambda}{D_{,E}(E,\mu)}}\frac{\sigma_g(\mu)}{|g-E(\mu)|}\leqq K_E\frac{\sqrt{N_g}}{|g-E(0)|} \tag{8}$$

where

$$K_E=\operatorname*{Max}_{\mu\in[0,1]}\sqrt{\frac{\lambda}{D_{,E}(E,\mu)}}\left|\frac{g-E(0)}{g-E(\mu)}\right|;$$

this maximum exists, since everything is continuous in μ and $D_{,E}(E,\mu)$ as well as $g-E(\mu)$ are bounded against zero. Since

$$\sum_{g=0}^{\infty}\frac{N_g}{|g-E(0)|}$$

converges, also

$$\|\phi_E(\mu)\|_{\ell_1}=\sum_{g=0}^{\infty}\sqrt{N_g}|\phi_E(g,\mu)|\leqq K_E\sum_{g=0}^{\infty}\frac{N_g}{|g-E(0)|}$$

exists and thus

$$|\phi_E(\mu)\rangle\in\ell_1\;\forall\mu\in[0,1].$$

We can now draw two important conclusions:

a) the family of ℓ_1-norms $\{\|\phi_E(\mu)\|_{\ell_1}\}_{\mu\in[0,1]}$ is (uniformly) bounded and

b) the series $\sum_{g=0}^{\infty} \sqrt{N_g}|\phi_E(g,\mu)|$ converge uniformly in μ, i.e. $\forall\varepsilon>0\,\exists N(\varepsilon)$ (independently of μ) with

$$\sum_{g\geq N(\varepsilon)}^{\infty} \sqrt{N_g}|\phi_E(g,\mu)|<\varepsilon \qquad \forall\mu\in[0,1].$$

We have

$$\begin{aligned}\|\phi_E(\mu)-\phi_E(0)\|_{\ell_1} &= \sum_{g=0}^{N} \sqrt{N_g}|\phi_E(g,\mu)-\phi_E(g,0)| \\ &\quad + \sum_{g=N+1}^{\infty} \sqrt{N_g}|\phi_E(g,\mu)-\phi_E(g,0)| \\ &\leqq \sum_{g=0}^{N} \sqrt{N_g}|\phi_E(g,\mu)-\phi_E(g,0)| \\ &\quad +2\sum_{g=N+1}^{\infty} \sqrt{N_g}|\phi_E(g,0)|;\end{aligned} \tag{9}$$

the second term on the right hand side can be made arbitrarily small by choosing N large enough (independently of μ), while the first term can be made arbitrarily small by choosing μ close enough to zero. Thus

$$|\phi_E(\mu)\rangle \xrightarrow{\ell_1} |\phi_E(0)\rangle .$$

Since the strong ℓ_1-topology is finer than the strong ℓ_2-topology, ℓ_1-convergence implies ℓ_2-convergence. We remark that according to a general theorem [1] conditions a) and b) are equivalent to saying that the family of vectors

$$\{|\phi_E(\mu)\rangle\}_{\mu\in[0,1]}$$

forms a relatively compact set in the strong ℓ_1-topology. We shall have to use this fact lateron. As a trivial byproduct of Proposition 3 we find that the eigenfunctions $\psi_E(x,\mu)$ in x-space are uniformly convergent in x.

Considering the respective domains of definition $\mathfrak{D}(T)$ and $\mathfrak{D}(\mu)$ of the Hamiltonians $H_{\text{free}}\equiv T$ and $H(\mu)$, $\mu\neq 0$, we see that they are identical for all $\mu\in(0,1]$ since the potential $V(\mu)$ is a bounded operator $\forall\mu\neq 0$. On the other hand – disregarding the functions of class $\mathfrak{H}^A$, since these are not effected by the potential and thus appear trivially in $\mathfrak{D}(0)$ – we have

$$\mathfrak{D}(T)\cap\mathfrak{D}(0)\equiv\mathfrak{D}=\left\{\phi\in\text{class}\,\mathfrak{H}^S/\sum_{g=0}^{\infty}\sqrt{N_g}\,\phi(g)=0,\ \sum_{g=0}^{\infty} g^2|\phi(g)|^2<\infty\right\}.$$

This follows from the conditions $\|(H(0)-T)\phi\|_{\ell_2}<\infty$ and $\|T\phi\|_{\ell_2}<\infty$ for any ϕ of $\mathfrak{D}$ (remember that $\phi(g)=0$ by definition if $g\neq$ square of an integer). Due to the first condition we have $H(0)/\mathfrak{D} = T/\mathfrak{D}$, since for any $\phi \in \mathfrak{D}$

$$\sum_{g'} H(0)_{gg'}\, \phi(g') = g\phi(g) + \lambda \sum_{g'} \sqrt{N_{g'}}\, \phi(g') = g\phi(g)\,.$$

Thus our limit operator $H(0)$ is an extension of T restricted to $\mathfrak{D}$, which is dense in $\mathfrak{H}^S$. We shall come back to this question in the three-dimensional case.

We are now going to give a precise mathematical meaning to the intuitive statement that "the operators $V(\mu)$ converge towards the δ-potential". One can certainly not speak of a weak convergence of $V(\mu) \to \lambda\delta$ since on the common domain of definition $\lambda\delta$ is zero. The same trouble appears with $H(\mu)$: neither $H(\mu)$, $\mu \neq 0$, nor $H(0)$ are essentially self-adjoint on their common dense domain $\mathfrak{D}$ and it makes no sense to speak of the convergence of a family of operators towards a limit if this limit does not possess a unique self-adjoint extension. We therefore define the "convergence" in a different way by proving the

Theorem A. *Let $\{A(\mu)\}_{\mu \in [0,1]}$ be a family of uniformly bounded normal operators on ℓ_2 with a point-spectrum (i.e. they can be represented as $A(\mu) = \text{strong} \sum_i a_i(\mu)\, P_i(\mu)$, $\sum_i P_i(\mu) = 1$, where the eigenvalues $a_i(\mu)$ are uniformly bounded in i and μ and the $P_i(\mu)$ are the projection-operators). We assume that the $a_i(\mu)$ are continuous in $\mu \forall i$ and that the projectors $P_i(\mu)$ are strongly continuous in $\mu \forall i$. (For our purpose we may replace the assumption on the P_i's by assuming strong continuity of the eigenfunctions $|i, \mu\rangle$ of $A(\mu)$, since $P_i(\mu) = |i, \mu\rangle \langle i, \mu|$.) Then the map $A : [0, 1] \to \{A(\mu)\}_{\mu \in [0,1]}$ is strongly continuous in μ. (Here "strongly" always means the strong topology of bounded operators on the ℓ_2.)*

Proof. Since the $A(\mu)$ are bounded operators it is sufficient to show the strong convergence on a dense set which may be generated by the finite linear combinations of some basis vectors. Taking the eigenfunctions $|i, \mu\rangle$ of $A(\mu)$ as complete and orthonormal basis we obtain by explicit calculation

$$\begin{aligned}
\|(A(\mu') - A(\mu))\,|i, \mu\rangle\|^2 &= \sum_k |\langle k, \mu' |\, A(\mu') - A(\mu) \,| i, \mu\rangle|^2 \\
&= \sum_k |a_k(\mu') - a_i(\mu)|^2\, |\langle k, \mu' | i, \mu\rangle|^2 \\
&= |a_i(\mu') - a_i(\mu)|^2\, |\langle i, \mu' | i, \mu\rangle|^2 \\
&\quad + \sum_{k \neq i} b_{ki}(\mu', \mu)\, |\langle k, \mu' | i, \mu\rangle|^2
\end{aligned}$$

with

$$b_{ki}(\mu', \mu) = |a_k(\mu') - a_i(\mu)|^2\,.$$

The first term goes to zero since $a_i(\mu)$ is continuous and

$$|\langle i, \mu' | i, \mu\rangle|^2 \leqq \||i, \mu'\rangle\|^2 \||i, \mu\rangle\|^2$$

is bounded. In the sum we have $|b_{ik}(\mu', \mu)| \leqq K \forall \mu, \mu', \forall i, k$ and the sum itself converges to zero due to the following argument: as continuous image of the compact set $[0, 1]$ the set $\{|i, \mu\rangle\}_{\mu \in [0,1]}$ is also compact and thus

$$\sum_{k \neq i} |\langle k, \mu' | i, \mu\rangle|^2 = \||i, \mu\rangle\|^2 - |\langle i, \mu' | i, \mu\rangle|^2$$

converges uniformly in μ according to the theorem mentioned in the proof of Proposition 1.2; thus the rest of the series becomes arbitrarily small (independently of μ), while each of the first finitely many terms tends to zero since $\langle k, \mu' | i, \mu\rangle \to 0$ for $k \neq i$. Hence $A(\mu') \to A(\mu)$ for $\mu' \to \mu \in [0, 1]$. Applying this result to our problem we have the

Corollary. For each continuous and bounded function f, $f(H(\mu)) \to f(H(0))$. This holds especially for the time-evolution operator $e^{-itH(\mu)}$ and the resolvent $\dfrac{1}{\mathrm{z} - \mathrm{H}(\mu)}$.

Proof.

$$f(H(\mu)) = \text{strong} \sum_{E \in \operatorname{spec} H(\mu)} f(E(\mu))\, |\phi_E(\mu)\rangle\, \langle \phi_E(\mu)|$$

with $E(\mu)$ continuous and $|\phi_E(\mu)\rangle$ strongly continuous $\forall E \in \operatorname{spec} H(\mu)$ and $\forall \mu \in [0, 1]$ according to Propositions 1 and 2.

We can therefore say that $H(\mu)$ converges towards $H(0)$ in the sense that every continuous and bounded function of $H(\mu)$ converges strongly. If we define

$$H(0) = \text{“lim”}\, H(\mu) \equiv \text{strong} \sum_E \lim_{\mu \to 0} E(\mu) \lim_{\mu \to 0} P_E(\mu) \tag{10}$$

then

$$H(0)_{gg'} = (\text{“lim”}\, H(\mu))_{gg'} = \sum_E E(0)\, \phi_E^*(g, 0)\, \phi_E(g', 0) = \lim (H(\mu)_{gg'}) \tag{11}$$

2. The Three-Dimensional Case

Starting with the decomposition of the Hilbert space we now have for the free case, $\lambda = 0$, an N_g-fold degeneracy for each eigenvalue $E = \boldsymbol{n}^2 = g$ where $N_g =$ number of different possibilities of decomposing the non-negative integer g into a sum of three squares of integers (zeroes and permutations included) i.e. N_g is the number of points with integer coordinates on a sphere with integer radius g. Thus $N_g = 0$ for all $g = 4^a(8b + 7)$ [2], $a, b \geqq 0$, integer. We again introduce the g-basis as in Section 1 and concentrate on $\mathfrak{H}^S(\mu) =$ closed linear span of the $|g, \mu\rangle$

for fixed μ. For $\mu \neq 0$ eigenfunctions, eigenvalue equation and matrix elements are formally the same as in the one-dimensional case, see Eqs. (4), (5), (6), and (7). Going now to the limit $\mu \to 0$ we encounter the difficulty that

$$\lim_{\mu \to 0} D(E, \mu)$$

is not defined since

$$\sum_{g \geqq 0} \frac{N_g}{g - E}$$

does not converge. Nevertheless we can show:

Proposition 2.1.

a) $\lim_{\mu \to 0} E_g(\mu) = g$ and in addition if $\lambda < 0$ $\lim_{\mu \to 0} E_b(\mu) = -\infty$

b) strong $\lim_{\mu \to 0} \phi_{E_g}(\mu) = |g\rangle$ (= free solution) and if $\lambda < 0$ weak $\lim_{\mu \to 0} \phi_{E_b}(\mu) = 0$, where $g \neq 4^a(8b + 7)$.

$E_g(\mu)$ denotes the unique solution of $D(E, \mu) = 0$ between g and the next possible integer g'.

Proof. a) From

$$D(E, \mu) \equiv 1 + \lambda \sum_{g \geqq 0} \frac{\sigma_g^2(\mu)}{g - E} = 0$$

we have

$$\frac{1}{g - E_g(\mu)} = \frac{1}{\sigma_g^2(\mu)} \left\{ -\frac{1}{\lambda} - \sum_{0 \leqq g' < g} \frac{\sigma_{g'}^2(\mu)}{g' - E_g(\mu)} - \sum_{g' > g} \frac{\sigma_{g'}^2(\mu)}{g' - E_g(\mu)} \right\}.$$

As $\mu \to 0$ the first sum remains bounded while the second sum diverges to $+\infty$; therefore

$$\frac{1}{g - E_g(\mu)} \to -\infty$$

i.e. $E_g(\mu) \downarrow g$. If $\lambda < 0$ then for any $x < 0$ there $\exists \mu_0 > 0 \Rightarrow \forall \mu \; 0 < \mu \leqq \mu_0$ $D(x, \mu) < 0$ and thus the unique solution of the eigenfunction moves arbitrarily far to the left.

b) Convergence of each component and boundedness of the norm imply weak convergence of a sequence in ℓ_P; weak convergence and convergence of the norm imply strong convergence in ℓ_2 [3]. Since we have

$$\|\phi_{E_g}(\mu)\|^2 = \langle g | g \rangle = 1,$$

it suffices to show that

$$\langle g' | \phi_{E_g}(\mu) \rangle \to \langle g' | g \rangle = \delta_{gg'} \; \forall g'.$$

For $g \neq g'$ $D_{,E}(E_g, \mu) \to \infty$ and $\dfrac{\sigma_{g'}(\mu)}{g' - E_g(\mu)}$ is bounded; for $g = g'$ $D_{,E}(E_g, \mu)\,(g - E_g(\mu))^2 \to \lambda \sigma_g^2(0) = \lambda N_g$. A similar argument for $\lambda < 0$ shows that $\langle g' | \phi_{E_b}(\mu) \rangle \to 0 \, \forall g'$. By application of Theorem A we conclude that for every bounded and continuous function f $f(H(\mu)) \to f(T)$. If we repeat the construction of "lim" $H(\mu)$ according to Eq. (10) we end up with the free Hamiltonian T but the analogue of Eq. (11) is wrong. In order to obtain a physically non-trivial limit we have to make λ dependent on μ in such a way that a certain function of λ and μ remain finite. We rewrite $D(E, \mu)$ as

$$D(E, \mu) = \left\{ 1 + \lambda_R \left(-\frac{1}{E} + E \sum_{g>0} \frac{\sigma_g^2(\mu)}{g(g-E)} \right) \right\} \tilde{D}(0, \mu) \equiv D^R(E, \mu)\, \tilde{D}(0, \mu) \quad (12)$$

with

$$\tilde{D}(0, \mu) \equiv 1 + \lambda \sum_{g>0} \frac{\sigma_g^2(\mu)}{g} \neq 0$$

for μ sufficiently close to zero and

$$\lambda_R = \frac{\lambda}{\tilde{D}(0, \mu)} . \quad (13)$$

Thus for $\mu \neq 0$ $D(E, \mu) = 0$ is equivalent to $D^R(E, \mu) = 0$, which we call the renormalized eigenvalue equation. It has the advantage that for fixed λ_R the limit $\mu \to 0$ exists. By a similar argument on the analytic structure as in the one-dimensional case we find exactly one solution E_g^R between two succeeding allowed values of g; in addition we obtain independently of the sign of λ a unique solution $E_b^R < 0$. We can repeat the argument of Proposition 1.1 and find that $E^R(\mu)$ is continuous in $[0, 1]$. Straightforward calculation shows that

$$\lambda / D_{,E}(E, \mu) = \lambda_R / D^R_{,E}(E, \mu) . \quad (14)$$

Replacing in Eq. (4) λ, D and E by the renormalized quantities λ^R, D^R and E^R resp., we obtain the renormalized eigenvectors $\phi^R_{E^R}(\mu)$ which are strongly continuous in $\mu \in [0, 1]$; their completeness and orthogonality (for fixed μ) can be shown in analogy to the one-dimensional case.

We emphasize that in distinction to the one-dimensional case $|\phi_E^R(\mu)\rangle \notin \ell_1$ and the corresponding eigenfunctions in x-space will not converge uniformly. Expressing λ in terms of λ_R yields

$$\lambda = \lambda_R \Big/ \left[1 - \lambda_R \sum_{g>0} \frac{\sigma_g^2(\mu)}{g} \right] . \quad (15)$$

We now define a renormalized Hamiltonian $H^R(\mu)$, $\mu \neq 0$, by $H^R(\mu) = H(\mu; \lambda = \lambda(\lambda_R, \mu))$ and apply again Theorem A to obtain

$$e^{-itH^R(\mu)} \to e^{-itH^R(0)} \quad (16)$$

with

$$H^R(0) = \text{“lim”}\, H^R(\mu) \equiv \text{strong} \sum_{E^R} \lim E^R(\mu) \lim P_{E^R}(\mu)\,. \tag{17}$$

Trying to find the analogue of Eq. (11) we run into even worse troubles than in the unrenormalized case. From Eq. (15) we get $\lambda \to -0$ for $\mu \to 0$, λ_R fixed and thus

$$\lim (H(\mu)_{gg'}) = T_{gg'} = g\delta_{gg'}\,;$$

on the other hand $H^R(0)$ is certainly different from T, because $E_g^R(0) \neq g$. Assuming the existence of the matrix elements of $H^R(0)$ in the g-basis an attempt to calculate them will lead to a contradiction: formally we have

$$\begin{aligned} H^R(0)_{gg'} &= \sum_i E_i^R \frac{\lambda_R}{D^R_{,E}(E_i^R)} \frac{\sqrt{N_g N_{g'}}}{(g - E_i^R)(g' - E_i^R)} \\ &= g\delta_{gg'} + \lambda_R \sqrt{N_g N_{g'}} \frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_R} dz \frac{z}{D^R(z)(z-g)(z-g')} \end{aligned}$$

with C_R denoting a circle of radius R around the origin. The limit is independent of g and g' and calling it C_∞ we have the following possibilities:

a) $C_\infty = 0$; then $H^R(0) = T$ which is explicitly wrong.

b) $C_\infty \neq 0$; then $H^R(0)$ would not be defined on its own eigenfunctions $\phi^R_{E^R}$; in fact $\sum_{g'} H^R(0)_{gg'}\, \phi^R_{E^R}(g')$ diverges for all g.

We conclude therefore that C_∞ does not exist and the integral diverges. Hence $\dfrac{1}{D^R(z)}$ is not bounded for $|z| \to \infty$. However, if we calculate the matrix elements of

$$\text{Res}\,(z) = \frac{1}{z - H^R(0)}$$

we have two more powers of z in the denominator and we obtain a finite result which coincides with $\lim (\text{Res}\,(z; \mu)_{gg'})$.

3. Extensions of H_{free}

We are now interested in the domain of $H^R(0)$. The analysis is more complicated than one would expect since we have for $H^R(0)$ no tractable matrix representation at our disposal. We derive that $H^R(0)$ is well defined on the class of vectors $|g, h\rangle \equiv \sqrt{N_h}|g\rangle - \sqrt{N_g}|h\rangle$, ($g \neq h$, both sum of three squares) and coincides with T on them:

$$\begin{aligned} \langle g' | H^R(0) | g, h\rangle &= g\sqrt{N_h}\,\delta_{gg'} - h\sqrt{N_g}\,\delta_{hg'} + \lambda_R \sqrt{N_g N_{g'} N_h} \\ &\quad \cdot \lim_{R\to\infty} \frac{1}{2\pi i} \int_{C_R} \mathrm{d}z \frac{z}{D^R(z)(g'-z)} \left\{ \frac{1}{g-z} - \frac{1}{h-z} \right\} \end{aligned}$$

and the limit is zero due to sufficiently many powers of z in the denominator; in fact we have the same asymptotic behaviour as in the completeness relation. The same is true also for the finite linear span of the $|g, h\rangle$. As self-adjoint operators both $H^R(0)$ and T are closed and we can extend their equality to a domain $\mathfrak{D}$ similar to the one-dimensional case: $\psi \in \mathfrak{D}$ iff $\sum_g \sqrt{N_g}\psi(g) = 0$ and $\sum_g g^2 |\psi(g)|^2 < \infty$. $T/\mathfrak{D} = H^R(0)/\mathfrak{D}$ is a closed symmetric operator and its defect indices are $(1, 1)$: $\phi \perp$ to all $(H^R(0) - iI)\psi$, $\psi \in \mathfrak{D}$, iff $\phi(g) = ic \dfrac{\sqrt{N_g}}{ig+1}$. Because $H^R(0)$ is self-adjoint it must be contained in the set of all possible self-adjoint extensions of $T/\mathfrak{D}$. Applying the standard technique (Smirnov loc. cit.) we first determine the adjoint operator

$$\langle g|(T/\mathfrak{D})^*|\phi\rangle = \sqrt{N_g}C_0 + g\phi(g)$$

with C_0 chosen (uniquely) such that $\|(T/\mathfrak{D})^*|\phi\rangle\| < \infty$; the domain of definition of $(T/\mathfrak{D})^*$ consists exactly of all ϕ for which a C_0 exists. The kernel of $(T/\mathfrak{D})^*$ is the one-dimensional subspace generated by $|0\rangle$. Looking now for solutions of $(T/\mathfrak{D})^{*2}|\phi\rangle = 0$ we find a one-parameter family of subspaces generated by the vectors $|\phi_\theta\rangle$: $\phi(0) = \cos\theta$, $\phi(g) = \dfrac{\sqrt{N_g}}{g}\sin\theta$, $g \neq 0$, $0 \leqq \theta < \pi$. The action of $(T/\mathfrak{D})^*$ on $|\phi\rangle$ is given by $(T/\mathfrak{D})^*|\phi\rangle = -\sin\theta|0\rangle \in \ker(T/\mathfrak{D})^*$. Adjoining to the previous domain $\mathfrak{D}$ the one-dimensional subspace generated by $|\phi_\theta\rangle$ for a fixed θ we obtain a self-adjoint extension of $T/\mathfrak{D} = H^R(0)/\mathfrak{D}$ and every self-adjoint extension (especially $H^R(0)$ itself) can be represented in this form. A relation between λ_R and θ is given by the condition that $H^R(0)$ is defined on its explicitly known eigenvectors: we have $\psi = \phi^R(\lambda^R) - c\phi_\theta \in \mathfrak{D}$ for some c; condition $\sum_g g^2|\psi|^2 < \infty$ yields $c = 1/\sin\theta$ whereas condition $\sum_g \sqrt{N_g}\psi(g) = 0$ implies

$$-1/E^R - \cot\theta + \sum_{g>0} N_g(1/g - E^R - 1/g) = 0\,.$$

By comparison with Eq. (12) we thus obtain $\lambda^R = -\tan\theta$. The value $\theta = \pi/2$ has a precise mathematical meaning: going to the limit $\lambda_R \to \infty$ in the renormalized eigenvalue equation we arrive again at the well-defined expression

$$-\frac{1}{E} + E\sum_{g>0}\frac{N_g}{g(g-E)} = 0\,,$$

from which we can obtain eigenvalues and eigenvectors as for finite λ_R. For $\lambda_R \downarrow 0$ the eigenstate with a corresponding energy $E_b^R < 0$ vanishes weakly and $E_b^R \to -\infty$. The extension for $\theta = 0$ is just the free Hamiltonian T.

4. Conclusions

Collecting all results of the preceding sections we can say:

The limit of a family of separable potentials can only be understood in the sense of Theorem A, i.e. we have convergence of the eigenvalues and strong convergence of the corresponding eigenvectors. The usual separation of H into the sum of T and the potential ceases to be meaningful in the limit: though $H(0)$ is a self-adjoint operator the δ-potential operator $V(0)$ is defined only on those elements of the Hilbert space which actually do not "feel" this potential, i.e. which vanish at its location; we can express this by saying that $V(0)$ is a restriction of the zero operator and has therefore only the zero operator as its unique self-adjoint extensions – but this extension does certainly not lead to $H(0)$. While in the one-dimensional case $H(0)$ generates a non-trivial situation, the situation in three dimensions is quite different. Keeping λ fixed the "A-limit" of $H(\mu)$ is the free Hamiltonian, but the limit of the matrix elements does not define a meaningful self-adjoint operator. In order to obtain a non-trivial limit we renormalize the coupling constant, i.e. we establish a functional dependence between λ and μ as both go to zero, each functional dependence being defined by a particular value of λ_R. For fixed λ_R we arrive at $H^R(0)$ by taking again the "A-limit" of $H^R(\mu)$. We obtain different limits because the eigenvalues as functions of λ and μ are not jointly continuous at the point $\lambda=\mu=0$. As we discussed, this operator cannot be represented in the g-basis as its matrix elements in the g-basis do not exist; on the other hand the matrix elements of $H^R(\mu)$ tend to the free matrix elements and we see again that the "A-limit" and taking the matrix elements do not commute. The process of renormalization can be understood within the framework of operator extensions: Restricting $H^R(0)$ to the common domain for all values of λ_R (including $\lambda_R=0$) we arrive at a symmetric operator with defect indices $(1, 1)$ which is the same for all λ_R. We label the self-adjoint extensions of this operator by the parameter θ and obtain finally a bijection between θ and λ_R. In comparison with the well-known Lee-model [5] our model shows two individual features: due to the linearity of the relation between λ and λ^R the coupling constant remains real in every case and there is no need for the introduction of an indefinite metric. On the other hand, as we have shown explicitely, the matrix elements of $H^R(0)$ in the free basis do not exist and there is no way of writing an explicit Schroedinger-equation in momentum-space as in the paper by Haag-Luzzatto. However their two conditions for the domain of the Hamiltonian correspond to our equations which establish the relation between the renormalized coupling constant and the extension parameter.

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Remark. After finishing this paper we became acquainted with a paper by F. A. Berezin and L. D. Faddejev where the authors study operator extensions in $L^2(-\infty, +\infty)$ [4].

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