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Cosets and Ferromagnetic Correlation Inequalities

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Abstract. Consideration of subgroups of the group of all subsets of $N = \{1, 2, \ldots, n\}$ with symmetric difference as the operation leads to new inequalities on correlations for a generalized Ising ferromagnet. An upper bound for the rate of change of $\langle\sigma^R\rangle$ with respect to J_S in terms of correlations and a new, brief proof for the monotonicity of $\langle\sigma^R\rangle$ as a function of J_S are given.

§ 1. Introduction

The success of Griffiths [1, 2] in establishing correlation inequalities (the non negativity of correlation and the monotonicity of correlation as a function of interaction) for Ising ferromagnets suggests the problem of getting other, if not all, correlation inequalities [3, Appendix, (1)] for generalized Ising ferromagnets.

In this paper other correlation inequalities are deduced as a consequence of considering a subgroup $\mathscr{G}_0$ of the group $\mathscr{G} = (2^N, \Delta)$ of subsets of $N = \{1, 2, \ldots, n\}$, the set of spin locations, under the operation Δ of symmetric difference. In particular it is shown (in the notation of [3] which is used in the sequel) that

$$\frac{\beta}{2}(1 + \langle\sigma^R \sigma^S\rangle^2 - \langle\sigma^R\rangle^2 - \langle\sigma^S\rangle^2) \geqq \frac{d\langle\sigma^R\rangle}{dJ_S}.$$

A brief proof of the monotonicity of correlation as a function of interaction yields new correlation inequalities as a result of the need for using a sufficiently strong inductive hypothesis.

§ 2. Cosets

Let $\mathscr{G} =_{\mathrm{df}}$ the group $(2^N, \Delta)$. Consider $J : \mathscr{G} \to R$ and $\mathscr{G}_0 \prec \mathscr{G}$ ($\mathscr{G}_0$ a subgroup of $\mathscr{G}$). For all $A \in \mathscr{G}$, let

$$\tilde{J}_{A\mathscr{G}_0} =_{\mathrm{df}} \sum_{B \in A\mathscr{G}_0} J_B$$

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thus getting $\tilde{J}: \mathscr{G}/\mathscr{G}_0 \to R$. The mapping $J \to \tilde{J}$ is a homomorphism of $L^1(\mathscr{G})$ into $L^1(\mathscr{G}/\mathscr{G}_0)$. If $\pi = \exp J$, in the real group algebra $\mathscr{A}(\mathscr{G})$, then

$$\exp \tilde{J} = \tilde{\pi} = \sum_{D \in \mathscr{G}/\mathscr{G}_0} \tilde{\pi}_D D$$

in $\mathscr{A}(\mathscr{G}/\mathscr{G}_0)$.

Special case: $\mathscr{G}_0 = \{\phi, R\}$

$$\tilde{\pi}_{\mathscr{G}_0} = \pi_\phi + \pi_R ,$$
$$\tilde{\pi}_{S\mathscr{G}_0} = \pi_S + \pi_{RS} .$$

J ferromagnetic, i.e., $(\forall A \neq \phi)\ J_A \geqq 0 :\Rightarrow: \tilde{J}$ ferromagnetic. J ferromagnetic $\Rightarrow \pi_\phi + \pi_R \geqq \pi_S + \pi_{RS}$, since $\tilde{\pi}_{\mathscr{G}_0} \geqq \tilde{\pi}_{S\mathscr{G}_0}$.

Theorem 1. $\dfrac{\beta}{2}(1 + \langle \sigma^R \sigma^S \rangle^2 - \langle \sigma^R \rangle^2 - \langle \sigma^S \rangle^2) \geqq \dfrac{d\langle \sigma^R \rangle}{dJ_S}$.

Proof. From $\pi_\phi + \pi_R \geqq \pi_S + \pi_{RS}$ it follows $\pi_\phi - \pi_{RS} \geqq \pi_S - \pi_R$. From $\pi_\phi + \pi_S \geqq \pi_R + \pi_{RS}$ it follows $\pi_\phi - \pi_{RS} \geqq \pi_R - \pi_S$. Thus

$$(\pi_\phi - \pi_{RS})^2 \geqq (\pi_S - \pi_R)^2 \quad \text{and} \quad \pi_\phi^2 + \pi_{RS}^2 - \pi_R^2 - \pi_S^2 \geqq 2\pi_\phi \pi_{RS} - 2\pi_R \pi_S .$$

§ 3. Monotonicity of Correlation

For $\pi = \exp J$,

$$\frac{d\pi}{dJ_D} = \pi D$$

and

$$\frac{d\pi_A}{dJ_D} = \pi_{AD} .$$

Let $\delta(\mathscr{G}_0; R, S) =_{\mathrm{df}} \sum_{B \in \mathscr{G}_0} (\pi_B \pi_{BR} - \pi_{BS} \pi_{BRS})$.

Theorem 2. *J ferromagnetic $\Rightarrow$ a) $\delta \geqq 0$, and b) $\dfrac{d\delta}{dJ_D} \geqq 0$.*

Proof. Fix N and let $P =_{\mathrm{df}} \{D : J_D > 0\}$. Proceed by induction on $\# P$. If $\# P = 0$, then

$$B \neq \phi \Rightarrow \pi_B = 0$$

and

$$B = \phi \Rightarrow \pi_B > 0 .$$

Since $\mathscr{G}_0$ is a group, $\phi \in \mathscr{G}_0$. For $R = \phi$.

$$\sum_{B \in \mathscr{G}_0} \pi_B \pi_{BR} = \pi_\phi^2$$

and

$$0 \leqq \sum_{B \in \mathscr{G}_0} \pi_{BS} \pi_{BRS} \leqq \pi_\phi^2 ,$$

so $\delta(\mathscr{G}_0; R, S) \geqq 0$. For $R \neq \phi$, $\delta(\mathscr{G}_0; R, S) = 0$. Thus (a) holds if $\# P = 0$. Suppose (a) holds $(\forall \mathscr{G}_0 \prec \mathscr{G})$ $(\forall R, S \in \mathscr{G})$ if $\# P \leqq k$. Consider $\# P = k+1$. Note that

$$\frac{d\delta(\mathscr{G}_0; R, S)}{dJ_D} = \sum_{B \in \mathscr{G}_0} \pi_{BD}\pi_{BR} + \pi_B \pi_{BRD} - \pi_{BSD}\pi_{BRS} - \pi_{BS}\pi_{BRSD}$$

$$= \begin{cases} \delta([\mathscr{G}_0 \cup \{D\}]; RD, S), & D \notin \mathscr{G}_0 , \\ 2\delta(\mathscr{G}_0; RD, S), & D \in \mathscr{G}_0 \end{cases}$$

where $[\mathscr{D}]$ is group generated by the elements of $\mathscr{D}$. Suppose $D \in P$.

First consider $D \in \mathscr{G}_0$, the possible behavior of δ, where the last argument is J_D, is given by the following table:

Table

	$\delta(\mathscr{G}_0; R, S; 0)$	$\delta(\mathscr{G}_0; RD, S; 0)$
(1)	0	0
(2)	+	0
(3)	0	+
(4)	+	+

In case (1), $(\forall J_D \geqq 0)\, \delta(\mathscr{G}_0; R, S; J_D) \equiv \delta(\mathscr{G}_0; RD, S; J_D) \equiv 0$. In cases (2–4), $(\forall J_D > 0)\, \delta(\mathscr{G}_0; R, S; J_D) > 0$ and $\delta(\mathscr{G}_0; RD, S; J_D) > 0$.

Next consider $D \notin \mathscr{G}_0$. From $D \in [\mathscr{G}_0 \cup \{D\}]$ and the inductive hypothesis, $(\forall J_D \geqq 0)\, \delta(\mathscr{G}_0; \omega, S; J_D) \geqq 0$. Thus (a) holds $(\forall P)$ and from the expression for the derivative of δ with respect to J_D, (b) holds $(\forall P)$. The proof is complete.

Corollary. *J ferromagnetic,*

$$\mathscr{G}_0 \prec \mathscr{G} \Rightarrow \sum_{B \in \mathscr{G}_0} (\langle \sigma^B \rangle \langle \sigma^{BR} \rangle - \langle \sigma^{BS} \rangle \langle \sigma^{BRS} \rangle) \geqq 0.$$

Special case: $\mathscr{G}_0 = \{\phi\}$. Here the inequality becomes

$$\langle \sigma^R \rangle - \langle \sigma^S \rangle \langle \sigma^{RS} \rangle \geqq 0$$

which implies that the correlations are monotone functions of interactions in a generalized ferromagnetic Ising model.

Added in proof. Professor J. Ginibre has an elegant proof of Griffith's second theorem as well as an elegant generalization of Theorem 2 of this paper. These will appear in the lecture notes of the 1969 Cargese NATO Summer School in Theoretical Physics.

References

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