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A Note on Extended Locality*

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Abstract. Conditions for the validity of extended locality are derived within the framework of the theory of local observables. It is shown that extended locality is equivalent to the condition that the algebra of all local observables contain no non-trivial translationally invariant operators. It is then shown that the derivation of extended locality can be generalized so as to apply to a wider class of regions.

1. Introduction

Our discussion is within the framework of the theory of von Neumann algebras of local observables [1]. The measurements performed by laboratory apparatus confined to an open bounded region $\mathcal{O}$ of space-time give rise to a weakly-closed algebra of bounded operators $R(\mathcal{O})$ acting on a Hilbert space $\mathcal{H}$. The axiom of locality states that the measurements in space-like separated regions are compatible: the algebra associated with one region commutes with the algebra associated with a space-like separated region. A further plausible notion of the physical independence of space-like separated regions, *extended locality*, was introduced by A. SCHOCH [2] in connection with a study of the simplicity of the algebra of quasi-local observables. Extended locality is the condition that space-like separated regions have no common operators other than multiples of the identity: $R(\mathcal{O}_1) \cap R(\mathcal{O}_2) = \{\alpha I\}$. We will derive conditions for this to hold.

$\mathcal{O}'$ denotes the interior of the set of points space-like to the region $\mathcal{O}$, and $R(\mathcal{O})'$ denotes the commutant of the von Neumann algebra $R(\mathcal{O})$. We denote the union of the algebras associated with all open bounded regions by U, the algebra of all local observables:

$$U = \bigcup_{\mathcal{O}} R(\mathcal{O}) .$$

$R(\infty)$ denotes the von Neumann algebra generated by all the local observables:

$$R(\infty) = U'' = \Big\{\bigcup_{\mathcal{O}} R(\mathcal{O})\Big\}'' .$$

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For an unbounded region $\mathcal{O}$ we denote by $U(\mathcal{O})$ the union of the algebras associated with open bounded regions contained in $\mathcal{O}$:

$$U(\mathcal{O}) = \bigcup_{\mathcal{O}_i \subset \mathcal{O}} R(\mathcal{O}_i).$$

We base our discussion on the following axioms:

i) *Translation Invariance*

The Hilbert space $\mathscr{H}$ carries a strongly continuous unitary representation of the translation group with elements $T(x) = \int \exp(ix \cdot p)\, dE(p)$. The local algebras transform among themselves under translations: $T(x)\, R(\mathcal{O})\, T(x)^{-1} = R(\mathcal{O} + x)$ where $\mathcal{O} + x$ denotes the region obtained by translating each point in $\mathcal{O}$ by the amount x.

ii) *Spectral Condition*

The support of the spectral measure $E(p)$ is contained in the closed forward cone $\overline{V}_+ : p^2 \geqq 0,\ p^4 \geqq 0$.

iii) *Locality*

$$\mathcal{O}_2 \subset \mathcal{O}_1' \Rightarrow R(\mathcal{O}_2) \subset R(\mathcal{O}_1)'$$

iv) *Isotony*

$$\mathcal{O}_2 \subset \mathcal{O}_1 \Rightarrow R(\mathcal{O}_2) \subset R(\mathcal{O}_1)$$

v) *Weak Additivity*

Let $\mathcal{O}$ be an arbitrary open bounded region. Then

$$R(\infty) = \left\{ \bigcup_x R(\mathcal{O} + x) \right\}''.$$

For simplicity, in this note we deal only with regions known as "diamonds": Let $(\boldsymbol{x}_0, t_1)$ and $(\boldsymbol{x}_0, t_2)$ be two points with $t_1 > t_2$. Then the open bounded region formed by the intersection of the interior of the backward cone from $(\boldsymbol{x}_0, t_1)$ and the interior of the forward cone from $(\boldsymbol{x}_0, t_2)$ is a diamond. The diamond is said to be generated by the time-like line connecting $(\boldsymbol{x}_0, t_1)$ to $(\boldsymbol{x}_0, t_2)$, and the base of the diamond is the sphere $|\boldsymbol{x} - \boldsymbol{x}_0| < (t_1 - t_2)/2,\ t = (t_1 + t_2)/2$. In fact, this diamond is of a special kind, with its base perpendicular to the time axis. We will work only with diamonds of this kind and denote them with the symbol D.

2. A Useful Theorem

It is the purpose of this section to prove the following theorem:

Theorem 1. *Let D_1 and D_2 be space-like separated diamonds. Then $U(D_1')' \cap U(D_2')' = R(\infty)'$.*

This means that if an operator A commutes with all the algebras associated with regions space-like to D_1 and regions space-like to D_2 then A commutes with the entire algebra $R(\infty)$.

It is clear that $R(\infty)' = U' \subset U(D_1')' \cap U(D_2')'$ and so we must only show that $U(D_1')' \cap U(D_2')' \subset R(\infty)'$.

Without loss of generality we take D_1 and D_2 with base in $t = 0$: We can clearly assume the base of D_1 is in $t = 0$, and if the base of D_2 is not also in $t = 0$, we can find a larger diamond $\bar{D}_2$ containing D_2 which will have its base in $t = 0$ (see the symbolic Fig. 1). Then $\bar{D}_2 \supset D_2$ implies

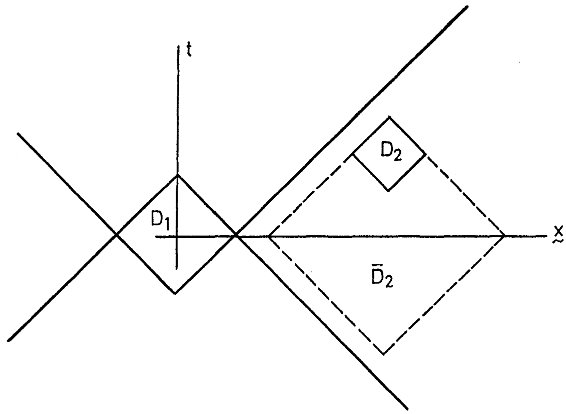

Fig. 1. D_1 and $\bar{D}_2$ are centered about $t = 0$

$\bar{D}_2' \subset D_2'$ which implies $U(D_2')' \subset U(\bar{D}_2')'$, and if the theorem holds for D_1 and $\bar{D}_2$, it will hold for D_1 and D_2: $U(D_1')' \cap U(D_2')' \subset U(D_1')' \cap U(\bar{D}_2')' \subset R(\infty)'$.

Take a "small" diamond D_ϵ centered about the origin $x = 0$. Denote by $\tilde{D}_1$ the set of points $D_\epsilon + x$ with x in D_1. $\tilde{D}_2$ is defined similarly. D_ϵ is chosen small enough so that $\tilde{D}_1$ and $\tilde{D}_2$ are space-like separated (see Fig. 2). Let b be any operator from $R(D_\epsilon)$ and define $b(x) = T(x)\, b\, T(x)^{-1}$.

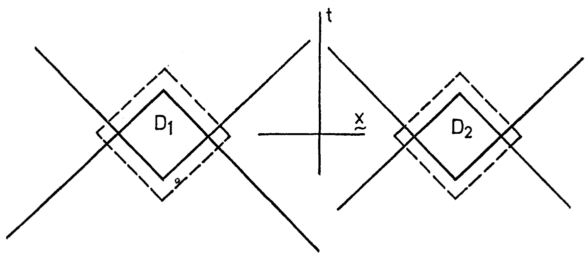

Fig. 2. $\tilde{D}_{1,2}$ (the dashed diamonds) are obtained by placing a small diamond over each point in $D_{1,2}$

Then $b(x) \in R(D_\epsilon + x)$. In particular if $x \in \tilde{D}_1'$, $b(x) \in R(D_1')$. So if A is any bounded operator that commutes with the local algebras associated with regions contained in D_1' and D_2', then A will commute with $b(x)$ for x in $\tilde{D}_1'$ or $\tilde{D}_2'$.

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be vectors with compact support in momentum space and define $f(x) = \langle\psi_1|\, [b(x), A]\, |\psi_2\rangle$. $f(x)$ vanishes for x in

$\tilde{D}_1' \cup \tilde{D}_2'$. We will show that $f(x)$ must vanish for all x. This follows from the nature of the fourier transform $\tilde{f}(p)$, the support of which is contained in $(\overline{V}_+ - \text{supp}(\psi_2)) \cup (\text{supp}(\psi_1) - \overline{V}_+)$, i. e., the union of the closed forward cones from $-\text{supp}(\psi_2)$ and the closed backward cones from $\text{supp}(\psi_1)$. This is the same support property encountered by ARAKI [3] and his results can be taken over directly.

ARAKI showed that $f(x)$ could be associated with the boundary value on $s = 0$ of an infinitely differentiable solution of the five-dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{\partial^2}{\partial s^2}\right) F(x, s) = 0 .$$

His results can be summarized as follows. Suppose $f(x)$ vanishes in some region $\mathcal{O}$. One then considers an infinitely differentiable solution $F(x, s)$ to the wave equation which vanishes together with its normal derivative on $s = 0$ in the region $\mathcal{O}$. Using uniqueness theorems for the wave equation one concludes that $F(x, s)$ must vanish in a certain five-dimensional region $\mathcal{O}_5$ depending on $\mathcal{O}$. In particular, $F(x, 0)$ will vanish in $\hat{\mathcal{O}}$, the intersection of $\mathcal{O}_5$ with $s = 0$. From ARAKI's work it then follows that $f(x)$ *also* vanishes in $\hat{\mathcal{O}}$. $\hat{\mathcal{O}}$ will in general be larger than $\mathcal{O}$, and we can thus conclude that if $f(x)$ vanishes in a region $\mathcal{O}$, it must in fact vanish in the larger region $\hat{\mathcal{O}}$.

We need the following lemma:

Lemma 1. *Let D be a diamond centered at the origin, with base $|\boldsymbol{x}| < r$, $t = 0$. If $F(x, s)$ and its normal derivative vanish in D' on $s = 0$, then the boundary values of $F(x, s)$ on $t = 0$ are non-zero only in the "strip" $S = \{(\boldsymbol{x}, s) \mid |\boldsymbol{x}| \leqq r, \text{all } s\}$.*

To prove the lemma we use the fact that if an infinitely differentiable solution of the five-dimensional wave equation vanishes together with all its derivatives on a segment of the time axis then it will vanish in the five-dimensional diamond generated by that line segment. (See Lemma 6 of [3] for discussion and references on this classical result). Lemma 1 is proved by noting that every point in $t = 0$ can be covered by a five-dimensional diamond generated by a time-like line segment in D' except those points in the strip S of the lemma.

We are now in a position to prove Theorem 1. As discussed earlier, $f(x)$ vanishes for x in $\tilde{D}_1' \cup \tilde{D}_2'$, so we look at a solution to the wave equation which, together with its normal derivative, vanishes in $\tilde{D}_1' \cup \tilde{D}_2'$ on $s = 0$. From the vanishing of the boundary values on $s = 0$ in $\tilde{D}_1'$, we conclude that the boundary values on $t = 0$ vanish except in the strip S_1 containing the base of $\tilde{D}_1$. But from the vanishing of the boundary data on $s = 0$ in $\tilde{D}_2'$ we conclude that the boundary values on $t = 0$ must vanish except in the strip S_2 containing the base of $\tilde{D}_2$. Since these two strips have no common points, the boundary values of $F(x, s)$ vanish identically on $t = 0$. But this implies that $F(x, s)$ is identically zero. Therefore $f(x)$ vanishes for all x.

So the matrix elements of $[b(x), A]$ between states of compact support vanish for all x. Since $b(x)$ and A are bounded, and states of compact support are dense in $\mathscr{H}$, $[b(x), A]$ is zero for all x. But then, since b is an arbitrary operator from $R(D_\epsilon)$, we have by weak additivity $A \in \left\{\bigcup_x R(D_\epsilon + x)\right\}' = R(\infty)'$. So we have shown that $U(D_1')' \cap U(D_2')' \subset R(\infty)'$, which proves Theorem 1.

3. Derivation of Extended Locality

Using the results of Section 2 we can derive conditions for the validity of extended locality which we formulate for diamonds: $R(D_1) \cap R(D_2) = \{\alpha I\}$ for space-like separated diamonds D_1 and D_2.

By locality $R(D_1) \subset U(D_1')'$ and $R(D_2) \subset U(D_2')'$, so that by Theorem 1 $R(D_1) \cap R(D_2) \subset R(\infty)'$. On the other hand $R(D_1)$ and $R(D_2)$ are contained in U. Therefore we have:

Corollary 1. $R(D_1) \cap R(D_2) \subset R(\infty)' \cap U$.

ARAKI has shown (Proposition 1 of [4]) that $R(\infty)' \cap R(\infty)$ contains only translationally invariant operators, and therefore $R(\infty)' \cap U$ contains only translationally invariant operators. From this and Corollary 1 we have.

Lemma 2. *U does not contain non-trivial translationally invariant operators implies extended locality.*

On the other hand, if $A \in R(\mathcal{O})$ is a non-trivial translationally invariant operator, then $A = T(x) A T(x)^{-1}$ is contained in $R(\mathcal{O} + x)$. Since $R(\mathcal{O})$ is contained in some diamond D, and since for large enough space-like x, $D + x$ will be space-like to D, we can conclude that $R(D) \cap R(D + x) \neq \{\alpha I\}$. We thus have:

Lemma 3. *Extended locality implies that U does not contain non-trivial translationally invariant operators.*

Then Lemmas 2 and 3 imply

Theorem 2. *Extended locality is equivalent to the condition that U does not contain non-trivial translationally invariant operators.*

From Corollary 1 and Theorem 2 we can obtain several other results. Since $R(\infty)' \cap U \subset R(\infty)' \cap R(\infty)$, and since if $R(\infty)$ is a factor $R(\infty)' \cap R(\infty) = \{\alpha I\}$ we can conclude:

Corollary 2. *$R(\infty)$ is a factor implies that extended locality holds and U does not contain non-trivial translationally invariant operators.*

We could also derive extended locality by assuming a physically motivated asymptotic condition (see Proposition 4 of [4]):

Asymptotic Condition. *There exists a unique vacuum and*

$$\langle\psi|\,T(x)\,b\,T(x)^{-1}|\psi\rangle \to \langle\mathrm{vac}|b|\mathrm{vac}\rangle\,\langle\psi|\psi\rangle$$

for all $b \in U$ *and all* $|\psi\rangle \in \mathscr{H}$ *as* x *goes to infinity (say in a space-like direction).*

In other words, if we translate the state $|\psi\rangle$ far enough away, any local measurement will find only the vacuum.

This asymptotic condition clearly implies that U does not contain non-trivial translationally invariant operators. So we have:

Corollary 3. *The asymptotic condition implies that extended locality holds.*

4. A Generalization

We have shown that the physically plausible notion of extended locality for space-like separated diamonds can be derived under certain assumptions. In this section we show that the derivation of extended locality can be generalized so as to apply to any pair of *disjoint* diamonds. We prove the following theorem:

Theorem 3. *Let* D_1, D_2 *be disjoint diamonds. Then*

$$R(D_1) \cap R(D_2) = \{\alpha I\}$$

if any one of the following conditions holds:

i) *U does not contain non-trivial translationally invariant operators.*

ii) *$R(\infty)$ is a factor.*

iii) *The asymptotic condition holds.*

The only change from the discussion in the preceding sections is that we must now show $U(D_1')' \cap U(D_2')' = R(\infty)'$ for disjoint diamonds rather than only for space-like separated diamonds. Paralleling the preceding discussion, we choose D_ϵ small enough so that $\tilde{D}_1, \tilde{D}_2$ are disjoint.

A space-like plane can be found such that $\tilde{D}_1, \tilde{D}_2$ lie on opposite sides of the plane. Let $\tilde{D}_1$ be the diamonds which lies above the plane. Then the support of a function $f(x)$ which vanishes in $\tilde{D}_1' \cap \tilde{D}_2'$ is contained in the union of the forward cone from the lower apex of $\tilde{D}_1$, the backward cone from the upper apex of $\tilde{D}_2$, plus a bounded region B_1 (see Fig. 3).

We do a Lorentz transformation on the coordinates so that the space-like plane becomes the plane $t = 0$, the lower apex of $\tilde{D}_1$ becomes the point $(a, 0, 0, a^4)$, and the upper apex of $\tilde{D}_2$ becomes the point $(-a, 0, 0, -a^4)$ with $a^4 > 0$ (see Fig. 4). The bounded region B_1 is transformed into another bounded region B_2.

We must now show that if the boundary values on $s = 0$ of a solution $F(x, s)$ of the five-dimensional wave equation have support contained in the forward cone from $(a, 0, 0, a^4)$, the backward cone from $(-a, 0,$

0, $-a^4$), and the bounded region B_2, then $F(x, s)$ must be identically zero. This is accomplished by showing that every point in the $t = 0$ plane can be covered by a five-dimensional diamond generated by a time-like line in the region of the plane $s = 0$ where the boundary values of

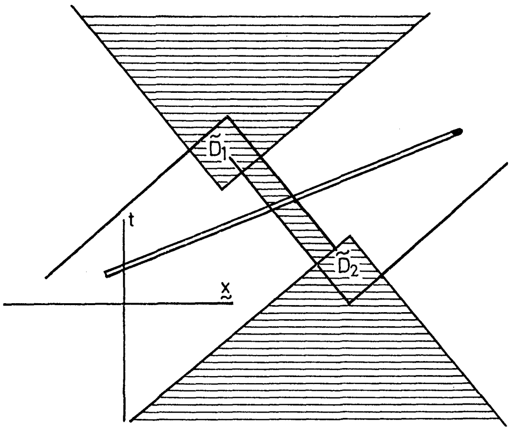

Fig. 3. $\tilde{D}_1$ and $\tilde{D}_2$ are separated by a space-like plane. The forward cone from $\tilde{D}_1$, the backward cone from $\tilde{D}_2$, and the bounded region B_1 are shown shaded

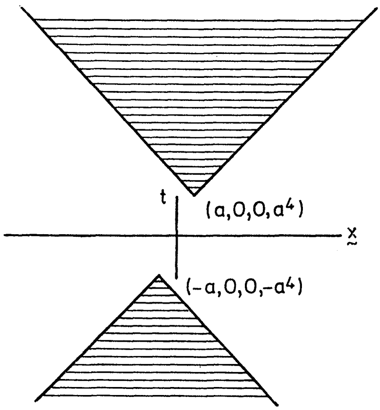

Fig. 4. The space-like plane becomes the plane $t = 0$ after the Lorentz transformation

$F(x, s)$ are zero. The bounded region B_2 does not need to be considered since the time-like lines will actually be chosen far from the origin $x = 0$.

The base of the five-dimensional diamond generated by a time-like line through the point $P_0 = (x_0, y_0, z_0, t = 0, s = 0)$ has a radius r given by the smaller of

$$\sqrt{(x_0 - a)^2 + y_0^2 + z_0^2 + a^4} \quad \text{or}$$

$$\sqrt{(x_0 + a)^2 + y_0^2 + z_0^2 + a^4}\,.$$

The distance of any point in the $t = 0$ plane from P_0 is given by

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + s^2}$$

and this can be made smaller than r by a suitable choice of P_0 sufficiently far from the origin. The rest of the proof of Theorem 3 is the same as in the case of space-like separated diamonds.

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