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# A Remark on Asymptotic Completeness of Local Fields

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**Abstract.** Assuming the existence of an asymptotically complete Wightman field with non-trivial $S$-matrix, we construct a local field such that the Haag-Ruelle scattering theory applied to this field leads to $\mathfrak{H}_{\text{in}} \neq \mathfrak{H}$ and $\mathfrak{H}_{\text{in}} \neq \mathfrak{H}_{\text{out}}$.

In the framework of local field theory one can define, using the HAAG-RUELLE [1] scattering theory, incoming and outgoing states and the corresponding Hilbert spaces $\mathfrak{H}_{\text{in}}$ and $\mathfrak{H}_{\text{out}}$. It is well-known that the axiom of asymptotic completeness ($\mathfrak{H}_{\text{in}} = \mathfrak{H}$) is independent of the other axioms of field theory. In order to have an unitary $S$-matrix, it is sufficient to require $\mathfrak{H}_{\text{in}} = \mathfrak{H}_{\text{out}}$. Starting from an asymptotically complete Wightman field with non-trivial $S$-matrix we shall construct a field which does not fulfill this requirement. The construction will show that in our case asymptotic completeness and unitarity of the $S$-matrix are destroyed by the fact that the functional of truncated vacuum expectation values can be decomposed into a sum of two such (truncated) functionals.

In the following we consider real scalar Wightman fields. We denote the field operator by $A(x)$, the vacuum state by $\Omega$, the representation of the inhomogeneous Lorentz group by $U(a, \Lambda)$ and the Hilbert space by $\mathfrak{H}$.

In addition to the usual postulates of field theory we require [2]:

(I) Let $\sigma(P)$ be the spectrum of the energy momentum operator $P$. Then $\sigma(P)$ has the form:

$$\sigma(P) = \{p|p = 0\} \cup \{p|p_0 > 0, p^2 = m^2\} \cup \{p|p_0 > 0, p^2 \geqq 4m^2\}; m > 0 .$$

(II) Let $\mathfrak{H}_1$ be defined by $\mathfrak{H}_1 = \{\Phi|\Phi \in \mathfrak{H}, (P^2 - m^2)\Phi = 0\}$, and let $U_1(a, \Lambda)$ be the representation of the inhomogeneous Lorentz group in $\mathfrak{H}_1$. Then $U_1(a, \Lambda)$ is an irreducible representation and has spin 0.

(III) Let $P_1$ be the projection on $\mathfrak{H}_1$. Then the following is true:

$$(A(x)\Omega, P_1 A(y)\Omega) = i\Delta^{(+)}(m^2, x - y) .$$

With the notation (taken from a paper by HEPP [3])

$$G = \left\{p|p_0 < 0, |p^2 - m^2| < \frac{m^2}{2}\right\}$$

and

$$S(G) = \{g \mid g \in S(R^4), \operatorname{supp} g \subset G\}$$

we define

$$A(f, t) = \int \tilde{A}(p)\, \tilde{f}(p)\, e^{-i(p_0+\omega)t}\, d^4p;\ \omega = \sqrt{m^2 + |\boldsymbol{p}|^2},\ f \in S(G)\,. \quad (1)$$

HAAG and RUELLE [1] have shown that the strong limits

$$\lim_{t \to \mp\infty} \prod_{j=1}^{n} A(f_j, t)\, \Omega = \Phi_{\substack{\text{in}\\ \text{out}}}(f_1, \ldots f_n)$$

exist and define incoming and outgoing states. The Hilbert space spanned by $\Omega$ and the states $\Phi_{\substack{\text{in}\\ \text{out}}}$ is denoted by $\mathfrak{H}_{\substack{\text{in}\\ \text{out}}}$. Asymptotic completeness of the field $A$ means: $\mathfrak{H}_{\text{in}} = \mathfrak{H}$.

If we have two fields $A_1(x)$, $A_2(x)$ with the vacuum states $\Omega_1$, $\Omega_2$ and with the representations $U_1(a, \Lambda)$, $U_2(a, \Lambda)$ of the inhomogeneous Lorentz group, we can construct a new field $B(x)$ by

$$\begin{aligned} B(x) &= A_1(x) \otimes 1 + 1 \otimes A_2(x); \\ \Omega &= \Omega_1 \otimes \Omega_2; \\ U(a, \Lambda) &= [U_1(a, \Lambda) \otimes U_2(a, \Lambda)]_B. \end{aligned} \quad (2)$$

$[U_1(a, \Lambda) \otimes U_2(a, \Lambda)]_B$ is the restriction of $U_1(a, \Lambda) \otimes U_2(a, \Lambda)$ to the space $\mathfrak{H}^{(B)} = \overline{\mathfrak{A}_B \Omega} \subseteq \overline{\mathfrak{H}^{(A_1)} \otimes \mathfrak{H}^{(A_2)}}$. ($\mathfrak{A}_B$ is the polynomial algebra of $B$).

This construction was introduced by BORCHERS [4]. From (2) we obtain for the truncated vacuum expectation values (TVEV):

$$\begin{aligned} &(\Omega, B(x_1) \ldots B(x_n)\Omega)^T \\ &\quad = (\Omega_1, A_1(x_1) \ldots A_1(x_n)\Omega_1)^T + (\Omega_2, A_2(x_1) \ldots A_2(x_n)\Omega_2)^T. \end{aligned} \quad (3)$$

(2) and (3) are equivalent statements, and we shall use both of them. We are now prepared for the following

**Theorem.** *Let $\{A(x), \Omega_A, U_A(a, \Lambda)\}$ be a local field theory which satisfies the conditions (I), (II), (III). Let $A$ be asymptotically complete, and let $S_A$ be the corresponding $S$-matrix. Then the field theory defined by*

$$\begin{aligned} B(x) = \frac{1}{\sqrt{2}}(A(x) \otimes 1 + 1 \otimes A(x));\quad \Omega_B = \Omega_A \otimes \Omega_A; \\ U_B(a, \Lambda) = [U_A(a, \Lambda) \otimes U_A(a, \Lambda)]_B \end{aligned} \quad (4)$$

*has the following properties:*

1. *The theory* (4) *satisfies the conditions (I), (II), (III).*
2. $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}^{(B)}$ *if and only if $A$ is a free field.*
3. $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}_{\text{out}}^{(B)}$ *if and only if* $S_A = 1$ .

*Proof.* 1. The requirements (I), (III) are fulfilled by construction. Hence we have only to show that (II) is fulfilled.

We define

$$\mathfrak{H}_1^{(B)} = \{\Phi \mid \Phi \in \mathfrak{H}^{(B)}, (P^2 - m^2)\,\Phi = 0\};$$

$$\hat{\mathfrak{H}}_1^{(B)} = \overline{\{B(f)\,\Omega_B; \tilde{f} \in S(G)\}}\,.$$

The representation of the inhomogeneous Lorentz group in $\hat{\mathfrak{H}}_1^{(B)}$ is irreducible and has spin 0. We want to show: $\mathfrak{H}_1^{(B)} = \hat{\mathfrak{H}}_1^{(B)}$.

Let us consider $\mathfrak{H}^{(B)}$ as a subspace of $\mathfrak{H}' = \overline{\mathfrak{H}^{(A)} \otimes \mathfrak{H}^{(A)}}$ with $U_A(a,\Lambda) \otimes U_A(a,\Lambda)$ as the representation of the inhomogeneous Lorentz group. We define:

$$\mathfrak{H}_1' = \{\Phi \mid \Phi \in \mathfrak{H}', (P^2 - m^2)\,\Phi = 0\}\,.$$

Since $A$ satisfies condition (II), we get

$$\mathfrak{H}_1' = \mathfrak{H}_1^{(A)} \otimes \Omega_A \oplus \Omega_A \otimes \mathfrak{H}_1^{(A)}$$

$$= \overline{\{A(f_1)\,\Omega_A \otimes \Omega_A + \Omega_A \otimes A(f_2)\,\Omega_A; \tilde{f}_1, \tilde{f}_2 \in S(G)\}}\,.$$

$\hat{\mathfrak{H}}_1^{(B)}$ and $\mathfrak{H}_1^{(B)}$ are subspaces of $\mathfrak{H}_1'$. Let $\hat{\mathfrak{H}}_1^{(B)\perp}$ be the orthogonal complement of $\hat{\mathfrak{H}}_1^{(B)}$ with respect to $\mathfrak{H}_1'$. Then we have:

$$\hat{\mathfrak{H}}_1^{(B)\perp} = \overline{\{A(f)\,\Omega_A \otimes \Omega_A - \Omega_A \otimes A(f)\,\Omega_A; \tilde{f} \in S(G)\}}\,.$$

Let us now consider the scalar products:

$$\left(A(f)\,\Omega_A \otimes \Omega_A - \Omega_A \otimes A(f)\,\Omega_A, \left[\prod_{j=1}^{n} B(g_j)\right] \Omega_A \otimes \Omega_A\right); \qquad (5)$$

$$n = 1,2,\ldots; \qquad f, g_j \in S(R^4)\,.$$

Since $B$ is symmetric in $A \otimes 1$ and $1 \otimes A$, we get:

$$\left([A(f) \otimes 1]\,\Omega_A \otimes \Omega_A, \left[\prod_{j=1}^{n} B(g_j)\right] \Omega_A \otimes \Omega_A\right)$$

$$= \left([1 \otimes A(f)]\,\Omega_A \otimes \Omega_A, \left[\prod_{j=1}^{n} B(g_j)\right] \Omega_A \otimes \Omega_A\right).$$

From this we conclude that the scalar products (5) vanish for arbitrary $f, g_j \in S(R^4)$. Therefore $\hat{\mathfrak{H}}_1^{(B)\perp}$ is orthogonal to $\mathfrak{H}^{(B)}$. This implies $\hat{\mathfrak{H}}_1^{(B)} = \mathfrak{H}_1^{(B)}$.

Hence $B$ satisfies condition (II).

2. If $A$ is a free field, $B$ is also a free field and, of course, asymptotically complete. It remains to show that asymptotic completeness of $B$ implies that $A$ is a free field.

We now suppose that $B$ is asymptotically complete. With the operators $A(f_j, t)$, $B(f_j, t)$ given by (1) we construct the states

$$\Phi_{\text{in}}^{(B)}(f_1, \ldots f_n) = \lim_{t \to -\infty} \prod_{j=1}^{n} B(f_j, t)\Omega_B;$$

$$\Phi_{\text{in}}^{(A)}(f_1, \ldots f_n) = \lim_{t \to -\infty} \prod_{j=1}^{n} A(f_j, t)\Omega_A .$$

The linear hull of all such states is called $D_{n|\text{in}}^{(B)}$ resp. $D_{n|\text{in}}^{(A)}$. With $D_{0|\text{in}}^{(B)} = \{\lambda\Omega_B\}$, $D_{0|\text{in}}^{(A)} = \{\lambda\Omega_A\}$ we define

$$D_{\text{in}}^{(B)} = \bigoplus_{n=0}^{\infty} D_{n|\text{in}}^{(B)}; \; D_{\text{in}}^{(A)} = \bigoplus_{n=0}^{\infty} D_{n|\text{in}}^{(A)} .$$

$D_{\text{in}}^{(B)}$ (resp. $D_{\text{in}}^{(A)}$) is dense in $\mathfrak{H}_{\text{in}}^{(B)}$ (resp. $\mathfrak{H}_{\text{in}}^{(A)}$). Finally we remark that the mapping $\Phi_{\text{in}}^{(B)}(f_1, \ldots f_n) \to \Phi_{\text{in}}^{(A)}(f_1, \ldots f_n)$ can be extended to an isometric mapping of $\mathfrak{H}_{\text{in}}^{(B)}$ onto $\mathfrak{H}_{\text{in}}^{(A)}$. With

$$j_B(x) = (\Box + m^2)B(x) \quad \text{and} \quad j_A(x) = (\Box + m^2)A(x)$$

we get:

$$\lim_{t \to -\infty} \left(j_B(g)\Omega_B, \prod_{j=1}^{n} B(f_j, t)\Omega_B\right) = \lim_{t \to -\infty} \left(j_B(g)\Omega_B, \prod_{j=1}^{n} B(f_j, t)\Omega_B\right)^T,$$

$$\lim_{t \to -\infty} \left(j_A(g)\Omega_A, \prod_{j=1}^{n} A(f_j, t)\Omega_A\right) = \lim_{t \to -\infty} \left(j_A(g)\Omega_A, \prod_{j=1}^{n} A(f_j, t)\Omega_A\right)^T,$$

$$g \in S(R^4) .$$

Due to (3), we obtain

$$(\Omega_B, B(x_1) \ldots B(x_n)\Omega_B)^T = \frac{2}{\sqrt{2^n}} (\Omega_A, A(x_1) \ldots A(x_n)\Omega_A)^T$$

and

$$(j_B(g)\Omega_B, \Phi_{\text{in}}^{(B)}(f_1, \ldots f_n)) = \frac{2}{\sqrt{2^{n+1}}} (j_A(g)\Omega_A, \Phi_{\text{in}}^{(A)}(f_1, \ldots f_n)) .$$

Let $\Phi_{\text{in}}^{(B)} = \sum_n \Phi_{n|\text{in}}^{(B)}$, $\Phi_{n|\text{in}}^{(B)} \in D_{n|\text{in}}^{(B)}$, and let $\Phi_{\text{in}}^{(A)}$ be the corresponding state in $D_{\text{in}}^{(A)}$. Then we get:

$$\sum_n (j_B(g)\Omega_B, \Phi_{n|\text{in}}^{(B)}) = \sum_n \frac{2}{\sqrt{2^{n+1}}} (j_A(g)\Omega_A, \Phi_{n|\text{in}}^{(A)}) . \tag{6}$$

Since $j_B(g)\Omega_B \perp \Omega_B \oplus \mathfrak{H}_1^{(B)}$ and $j_A(g)\Omega_A \perp \Omega_A \oplus \mathfrak{H}_1^{(A)}$, only terms with $n \geqq 2$ contribute to the sums. This leads to the following estimate for the right hand side of (6):

$$\left|\left(j_A(g)\Omega_A, \sum_{n \geqq 2} \frac{1}{\sqrt{2^{n-1}}} \Phi_{n|\text{in}}^{(A)}\right)\right| \leqq \|j_A(g)\Omega_A\| \frac{1}{\sqrt{2}} \left\|\sum_n \Phi_{n|\text{in}}^{(A)}\right\| .$$

Since $\|\Phi_{\text{in}}^{(B)}\| = \|\Phi_{\text{in}}^{(A)}\|$, we have

$$\frac{|(j_B(g)\Omega_B, \Phi_{\text{in}}^{(B)})|}{\|\Phi_{\text{in}}^{(B)}\|} \leqq \frac{1}{\sqrt{2}} \|j_A(g)\Omega_A\| .$$

$D_{\text{in}}^{(B)}$ is dense in $\mathfrak{H}_{\text{in}}^{(B)}$, and $B$ is supposed to be asymptotically complete. We conclude:

$$\|j_B(g)\Omega_B\| \leqq \frac{1}{\sqrt{2}} \|j_A(g)\Omega_A\| .$$

Due to (3), the 2-point functions of $A$ and $B$ are the same. Therefore we get

$$\|j_A(g)\Omega_A\| \leqq \frac{1}{\sqrt{2}} \|j_A(g)\Omega_A\| .$$

This implies $j_A(g)\Omega_A = 0$. The conclusion holds for arbitrary $g \in S(R^4)$. Since $A$ is local, $A$ is a free field.

3. From $S_A = 1$ it follows:

$$\lim_{t\to\infty} \left( \prod_{j=1}^{n_1} B(f_j, t)\Omega_B, \prod_{k=1}^{n_2} B(g_k, -t)\Omega_B \right)^T$$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{2}^{n_1+n_2}} \left( \prod_{j=1}^{n_1} A(f_j, t)\Omega_A, \prod_{k=1}^{n_2} A(g_k, -t)\Omega_A \right)^T = 0,$$

$$n_1 + n_2 > 2 .$$

This yields

$$\lim_{t\to+\infty} \prod_{j=1}^{n} B(f_j, t)\Omega_B = \lim_{t\to-\infty} \prod_{j=1}^{n} B(f_j, t)\Omega_B .$$

Hence we have $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}_{\text{out}}^{(B)}$. It remains to show that the assumption $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}_{\text{out}}^{(B)}$ implies $S_A = 1$.

We now assume $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}_{\text{out}}^{(B)}$. We want to give a proof by induction. We define:

$$\Psi^{(A)}(f_1, \ldots f_n) = \Phi_{\text{out}}^{(A)}(f_1, \ldots f_n) - \Phi_{\text{in}}^{(A)}(f_1, \ldots f_n) ,$$

$$\Psi^{(B)}(f_1, \ldots f_n) = \Phi_{\text{out}}^{(B)}(f_1, \ldots f_n) - \Phi_{\text{in}}^{(B)}(f_1, \ldots f_n) .$$

For $n = 1$ we have $\Psi^{(A)}(f) = 0$. Suppose now, it has been proved that $\Psi^{(A)}(f_1, \ldots f_n)$ vanishes for all $n < N$ and arbitrary $f_j, \tilde{f}_j \in S(G)$, $j = 1, 2 \ldots n$. Since the TVEV of $B$ are multiples of the TVEV of $A$, $\Psi^{(B)}(f_1, \ldots f_n)$ also vanishes for $n < N$. This has the consequence:

$$(\Psi^{(B)}(f_1, \ldots f_N), \Phi_{\text{in}}^{(B)}(g_1, \ldots g_l))$$

$$= \lim_{t\to\infty} \left( \left\{ \prod_{j=1}^{N} B(f_j, t) - \prod_{j=1}^{N} B(f_j, -t) \right\} \Omega_B, \prod_{k=1}^{l} B(g_k, -t)\Omega_B \right)^T$$

$$= \lim_{t\to\infty} \frac{2}{\sqrt{2}^{N+l}} \left( \left\{ \prod_{j=1}^{N} A(f_j, t) - \prod_{j=1}^{N} A(f_j, -t) \right\} \Omega_A, \prod_{k=1}^{l} A(g_k, -t)\Omega_A \right)^T$$

$$= \frac{2}{\sqrt{2}^{N+l}} (\Psi^{(A)}(f_1, \ldots f_N), \Phi_{\text{in}}^{(A)}(g_1, \ldots g_l)) .$$

Let $\Phi_{\text{in}}^{(B)}$, $\Phi_{\text{in}}^{(A)}$ be the states which we used in the proof of statement 2). Then we get:

$$\left(\Psi^{(B)}(f_1, \dots f_N), \sum_n \Phi^{(B)}_{n|\text{in}}\right) = \left(\Psi^{(A)}(f_1, \dots f_N), \sum_n \frac{2}{\sqrt{2}^{N+n}} \Phi^{(A)}_{n|\text{in}}\right).$$

Due to the induction assumption, only terms with $n \geqq N$ contribute to the sums.

This leads to the following estimates:

$$\left|\left(\Psi^{(A)}(f_1, \dots f_N), \sum_{n \geqq N} \Phi^{(A)}_{n|\text{in}} \frac{2}{\sqrt{2}^{N+n}}\right)\right| \leqq \|\Psi^{(A)}(f_1, \dots f_N)\| \frac{1}{2^{N-1}} \left\|\sum_n \Phi^{(A)}_{n|\text{in}}\right\|;$$

$$\frac{|(\Psi^{(B)}(f_1, \dots f_N), \Phi_{\text{in}}^{(B)})|}{\|\Phi_{\text{in}}^{(B)}\|} \leqq \frac{1}{2^{N-1}} \|\Psi^{(A)}(f_1, \dots f_N)\|.$$

Since $\mathfrak{H}_{\text{in}}^{(B)} = \mathfrak{H}_{\text{out}}^{(B)}$, $\Psi^{(B)}(f_1, \dots f_N)$ is a vector in $\mathfrak{H}_{\text{in}}^{(B)}$. We conclude:

$$\|\Psi^{(B)}(f_1, \dots f_N)\| \leqq \frac{1}{2^{N-1}} \|\Psi^{(A)}(f_1, \dots f_N)\|.$$

On the other hand, we have

$$\|\Psi^{(B)}(f_1, \dots f_N)\|^2 = \lim_{t\to\infty} \left(\left\{\prod_{j=1}^N B(f_j, t) - \prod_{j=1}^N B(f_j, -t)\right\} \Omega_B, \right.$$
$$\left. \cdot \left\{\prod_{j=1}^N B(f_j, t) - \prod_{j=1}^N B(f_j, -t)\right\} \Omega_B\right)^T$$
$$= \frac{1}{2^{N-1}} \lim_{\to\infty} \left(\left\{\prod_{j=1}^N A(f_j, t) - \prod_{j=1}^N A(f_j, -t)\right\} \Omega_A, \right.$$
$$\left. \cdot \left\{\prod_{j=1}^N A(f_j, t) - \prod_{j=1}^N A(f_j, -t)\right\} \Omega_A\right)^T$$
$$= \frac{1}{2^{N-1}} \|\Psi^{(A)}(f_1, \dots f_N)\|^2.$$

This yields

$$\frac{1}{\sqrt{2}^{N-1}} \|\Psi^{(A)}(f_1, \dots f_N)\| \leqq \frac{1}{2^{N-1}} \|\Psi^{(A)}(f_1, \dots f_N)\|.$$

That implies $\Psi^{(A)}(f_1, \dots f_N) = 0$. Since the induction assumption is true for $n = 1$, we get for all $n$ $\Psi^{(A)}(f_1, \dots f_n) = 0$. The conclusion holds for arbitrary $f_j, \bar{f}_j \in S(G)$. Hence we obtain $S_A = 1$. This proves the theorem.

Assume now, there is an asymptotically complete Wightmann field $A(x)$ which satisfies the conditions (I), (II), (III). Let the $S$-matrix be non-trivial. Then we construct the field $B(x) = \frac{1}{\sqrt{2}}(A(x) \otimes 1 + 1 \otimes A(x))$. Due to our theorem we get $\mathfrak{H}_{\text{in}}^{(B)} \neq \mathfrak{H}^{(B)}$ and $\mathfrak{H}_{\text{in}}^{(B)} \neq \mathfrak{H}_{\text{out}}^{(B)}$.

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