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A Remark on C^*-Algebras

HORST BEHNCKE*

Department of Physics, Indiana University, Bloomington, Indiana
Courant Institute of Mathematical Sciences, New York University, New York

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Abstract. It is shown that any complex Banach algebra with hermitean involution and the weak C^*-property $|x|^2 = |x^2|$ for all $x = x^*$ is a C^*-algebra.

Among all complex Banach algebras with involution the C^*-algebras are singled out by a very strong condition on the norm. ONO [1], VIDAV [2] and more recently BERKSON [3] were able to replace the original Gelfand-Naimark axioms by considerably weaker assumptions. In this note we present a new proof for generalizations of the results in [2] and [3]. In particular we shall study complex Banach algebras $\mathfrak{A}$ with involution * and norm $|\ |$, which also satisfy:

a) the involution is hermitean and

b) $|x^2| = |x|^2$ for all $x \in \mathfrak{A}_h$.

Here $\mathfrak{A}_h$ denotes the system of hermitean elements of $\mathfrak{A}$. We shall show that in this case $\mathfrak{A}$ is a C^*-algebra. In contradistinction to [2] and [3] we shall not assume the existence of an identity. We also do not assume the involution to be isometric. This result is of importance in algebraic physics, because we only assume conditions on the hermitean elements, which correspond to observables. Mathematically our result shows that the C^*-property is a local property.

The terminology of [4] will be used throughout this paper. We begin with the simpler case of a commutative Banach algebra.

Theorem 1. *Any complex commutative Banach algebra* $\mathfrak{A}$ *with involution, which satisfies* a) *and* b) *is a* C^**-algebra.*

Proof. i) Because of a) $\mathfrak{A}$ is symmetric. Thus its *-radical $\mathfrak{R}$ consists of all x with $\mathrm{Sp}\, xx^* = \{0\}$. This implies by b) $\mathfrak{R} = \{x \mid xx^* = x^*x = 0\} = \{0\}$. Thus $\mathfrak{A}$ is *-semi simple, and the Gelfand representation defines an isomorphism of $\mathfrak{A}$ with an algebra of functions $\mathfrak{B}$. The supremum norm of $\mathfrak{B}$ can be carried back to $\mathfrak{A}$ and defines an auxiliary norm $|\ |_0$, which makes $\mathfrak{A}$ an A^*-algebra. Because of b) we have $|x|_0 = |x|$ for all $x \in \mathfrak{A}_h$. Thus $|x|_0 \leqq |x| \leqq \frac{1}{2}|x + x^*| + \frac{1}{2}|x - x^*| = \frac{1}{2}|x + x^*|_0 + \frac{1}{2}|x - x^*|_0 \leqq 2\,|x|_0$ and we see that $|\ |_0$ and $|\ |$ are equivalent. In particular $\mathfrak{A}$ equipped with $|\ |_0$ is a C^*-algebra.

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ii) Therefore $\mathfrak{A}$ possesses an approximate identity $\{e_\lambda\}$ of hermitean elements with $|e_\lambda|_0 = |e_\lambda| = 1$. Using the left regular representation it is therefore possible to embed $\mathfrak{A}$ *-isomorphically and isometrically in a complex Banach algebra $\mathfrak{A}_1$ with identity. It is easy to verify that also $\mathfrak{A}_1$ has the properties a) and b). The C^*-norm on $\mathfrak{A}_1$ we will again denote by $|\ |_0$. Then we see that $|x^*| \leqq 2\,|x^*|_0 = 2\,|x|_0 \leqq 2\,|x|$, and the involution on $\mathfrak{A}_1$ is continuous. This allows us to consider a new norm $|\ |_1$ on $\mathfrak{A}_1$ by $|x|_1 = \max |x|, |x^*|$. The advantage of $|\ |_1$ is, that properties a) and b) remain valid for $|\ |_1$ and that in addition $|\ |_1$ satisfies $|x|_1 = |x^*|_1$ for all x in $\mathfrak{A}_1$. We have obviously $|x|_0 \leqq |x| \leqq |x|_1 \leqq 2\,|x|_0$ for all $x \in \mathfrak{A}_1$, and all norms are equivalent.

iii) Our aim is to prove that $|\ |_1 = |\ |_0$, which would show Theorem 1. Let f be a hermitean functional of $\mathfrak{A}_1$. Since $\mathfrak{A}_1$ equipped with $|\ |_0$ is a C^*-algebra f has a decomposition into its positive and negative parts $f = f_+ - f_-$ and $|f|_0 = |f_+|_0 + |f_-|_0$ [5]. Here $|f|_0$ is the norm induced on the functionals by $|\ |_0$ on $\mathfrak{A}_1$. Since hermitean functionals on algebras with isometric involution attain their norms on $\mathfrak{A}_h$, where $|\ |_0$ and $|\ |_1$ agree, we also have $|f|_1 = |f_+|_1 + |f_-|_1$. This shows by [5] that $|\ |_1$ is already a C^*-norm. Consequently $|\ |_1$ and $|\ |_0$ agree, which proves that $\mathfrak{A}$ equipped with $|\ | = |\ |_0$ is already a C^*-algebra.

The proof for the non commutative case is fashioned after the one for the commutative case.

Theorem 2. *Any complex Banach algebra $\mathfrak{A}$ with involution, which satisfies* a) *and* b) *is a C^*-algebra.*

Proof. i) By Theorem 1 we know that any maximal commutative subalgebra of $\mathfrak{A}$ is a C^*-algebra. Thus using Theorem 1 by Ono [1] we can assert that $\mathfrak{A}$ is *-semi simple and that $\mathfrak{A}$ has a faithful representation $\mathfrak{B}$ as a C^*-algebra of operators on a Hilbert space. The C^*-norm on $\mathfrak{B}$ can be carried back to $\mathfrak{A}$ and defines there an auxiliary norm $|\ |_0$. As in Theorem 1 (i) one shows that $|x|_0 \leqq |x| \leqq 2\,|x|_0$ for all $x \in \mathfrak{A}$. Thus $\mathfrak{A}$ equipped with $|\ |_0$ is a C^*-algebra.

ii) Therefore $\mathfrak{A}$ possesses an approximate identity of hermitean elements $\{e_\lambda\}$, $|e_\lambda| = |e_\lambda|_0 = 1$, and as before we can embed $\mathfrak{A}$ *-isomorphically and isometrically in a complex Banach algebra $\mathfrak{A}_1$ with identity. Also $\mathfrak{A}_1$ satisfies a) and b).

iii) Again we introduce a new norm $|\ |_1$ by $|x|_1 = \max |x|, |x^*|$ on $\mathfrak{A}_1$, and Grothendieck's result [5] shows as before that $|\ |_1 = |\ |_0$.

Corollary. *A complex Banach algebra $\mathfrak{A}$ with involution is a C^*-algebra if:*

a) $|x^2| = |x|^2$ *for all* $x \in \mathfrak{A}_h$;

b′) $|x|\,|x^*| \leqq K\,|xx^*|$ *for all normal x, K a positive constant.*

Proof. By [4, Theorem 4.2.2] b′) implies b).

We should remark here that condition b) alone is not sufficient to insure that $\mathfrak{A}$ is isomorphic and homeomorphic to a C^*-algebra. In fact let $\mathfrak{A} = \{(x_1, x_2) \mid x_1, x_2$ complex numbers$\}$. Multiplication and addition are defined componentwise. However the involution is defined by $(x_1, x_2)^* = (\bar{x}_2, \bar{x}_1)$. Then the norm $|\ |$, defined by $|(x_1, x_2)| = \max |x_1|, |x|_2$, makes $\mathfrak{A}$ a Banach algebra with isometric involution, in which b) is satisfied. However $(x, 0)\,(x, 0)^* = = (0, 0)$.

In algebraic physics one interpretes the observables as the hermitean elements of a C^*-algebra. In this case our result shows that one gains no generality in assuming the observables to correspond to the hermitean elements of a Banach algebra, because the conditions a) and b) arise quite naturally in a physical interpretation. In fact this was the initial motivation for this paper.

We have shown that any complex Banach algebra with involution in which every maximal abelian subalgebra is a C^*-algebra is itself a C^*-algebra. The analog of this result for W^*-algebras is the following conjecture: Any C^*-algebra in which every maximal abelian subalgebra is a W^*-algebra is itself a W^*-algebra. So far it is only known that these algebras are all $A\,W^*$-algebras, and at least for $A\,W^*$-algebras of type I the conjecture is true.

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Dr. H. Behncke
Courant Institute
of Mathematical Sciences
251 Mercer Street
New York, N.Y. 10012, USA