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On the Quantum Logic Approach to Quantum Mechanics

S. GUDDER
Department of Mathematics
University of Wisconsin; Madison, Wisconsin

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Abstract. A quantum logic structure for quantum mechanics which contains the concepts of a physical space, localizability, and symmetry groups is formulated. It is shown that there is an underlying Hilbert space which mirrors much of this axiomatic structure. Quantum fields are defined and shown to arise naturally from the quantum logic structure. The fields of HAAG and WIGHTMAN are generalized to this theory and an attempt is made to find a local equivalence for these fields.

1. Introduction

The two main methods of attack in axiomatic quantum mechanics have been the c^*-algebra [1] and the quantum logic [2] approaches. The first of these inequivalent approaches uses the algebra of bounded observables as the main axiomatic elements while the basic constituents of the second are the quantum propositions or, as they are also called, experimental questions or events. In this paper we are concerned with the second approach.

One of the main goals of the quantum logic approach is to postulate enough physically verifiable axioms so that the structure of the proposition system reduces to the usual von Neumann Hilbert space model for quantum mechanics. In the author's opinion this goal has not been achieved. In all such attempts axioms, such as completeness and atomicity, have been imposed [3] although these axioms have little physical justification. Even the lattice structure of the proposition system seems questionable [4]. For this reason we shall not impose these questionable axioms and use only postulates which seem physically reasonable and justifiable and which hopefully can be tested in the laboratory. Even so, as we shall see, we are very close to a Hilbert space theory as far as the structure of the axiomatic system is concerned. We will also show that even in this abstract context many of the constructs, such as localizability, symmetry groups, and quantum fields, which are used in general quantum theory may be formulated and in fact arise quite naturally.

2. The Axiomatic Structure

Suppose S is a quantum mechanical system. Let us now analyze S and attempt to extract from it the physically relevant properties it possesses. First of all one can ask many questions concerning S, the answers to which must either be yes or no. These correspond to the most elementary observations on the system and the collection $L = \{a, b, \ldots\}$ of these questions or propositions will be called a *proposition system* or *logic*. We postulate that L forms an *orthocomplemented partially ordered set*. That is, L possesses a partial order relation $\leqq$ under which L has a *first* and *last element* 0 and 1 respectively, and a *complementation* $'$ which satisfies (i) $(a')' = a$, (ii) $a \leqq b$ implies $b' \leqq a'$, (iii) $a \vee a' = 1$, and (iv) $\vee a_i$ exists if $a_i \perp a_j$ (i.e. $a_i \leqq a_j'$) $i \neq j = 1, 2, \ldots$ If $a, b \in L$, then a and b are *compatible* (written $a \leftrightarrow b$) if there are mutually disjoint propositions $a_1, b_1, c \in L$ such that $a = a_1 \vee c$ and $b = b_1 \vee c$. It is easily seen that if $a \leftrightarrow b$ then $a \vee b$ exists. Finally we postulate (v) if a, b, c are mutually compatible then $a \leftrightarrow b \vee c$. The significance of these axioms has been amply demonstrated in the literature. By performing repeated experiments on these propositions one can determine the probabilities that they are true. These probabilities will depend upon the initial conditions or preparation or state of our system. We thus define a *state* as a map $m: L \to [0, 1]$ such that (i) $m(1) = 1$, (ii) $m(\vee a_i) = \Sigma m(a_i)$ if $a_i \perp a_j$, $i \neq j = 1, 2, \ldots$ We assume that there is a *full set of states*. That is, (i) if $a \neq b$ there is a state m such that $m(a) \neq m(b)$, (ii) if $a \neq 0$ there is a state m such that $m(a) = 1$. We denote some full set of states by M and call the pair $\mathscr{L} = (L, M)$ a *quantum logic*.

It seems to the author that one cannot get very far by considering $\mathscr{L}$ alone. One must study the propositions L more closely and investigate the properties with which these propositions are concerned. First of all, our system S is concerned with a phenomena that takes place in some sort of physical arena which we call physical space. Mathematically we shall assume that *physical space* $\mathscr{S}$ is a locally compact Hausdorff space with second countability. In a concrete physical situation $\mathscr{S}$ might be three-dimensional Euclidean space, the surface of a sphere, or perhaps four-dimensional space-time. Now many of the propositions in L are concerned with the location of our system in $\mathscr{S}$. If such propositions can be verified in the laboratory we call S localizable. Let $B(\mathscr{S})$ denote the Borel sets in $\mathscr{S}$ and if $E \in B(\mathscr{S})$ let the proposition that S is located in E be denoted by $X(E)$. Thus X is a map from $B(\mathscr{S})$ into L. We must now decide what properties X should possess. It is clear that our system S will be located somewhere so $X(\mathscr{S}) = 1$ and if $E, F \in B(\mathscr{S})$ with $E \perp F$ (here this is the same as $E \cap F = \emptyset$) then $X(E) \perp X(F)$. That is, the propositions that $\mathscr{S}$ is in E and F respectively are mutually exclusive. If $E \perp F$ then $X(E \cup F)$ is the proposition that S is in E or F and should be the proposition which is true if S is in E or if S is in F; i.e. $X(E \cup F) = X(E) \vee X(F)$. Finally we strengthen this last axiom, for mathematical convenience mainly, to $X(\cup E_i) = \vee X(E_i)$, if $E_i \perp E_j$,

$i \neq j = 1, 2, \ldots$ A map $X : B(\mathscr{S}) \to L$ satisfying these three properties is called a *σ-homomorphism*. Notice that if $m \in M$ then $E \to m(X(E))$ is a probability measure which gives the probability that S is in $E \in B(\mathscr{S})$ in the state m.

Because of their great importance, the propositions $X(E)$, $E \in B(\mathscr{S})$ should occupy a prominent place in L. We say that a Boolean σ-algebra B in L is *state determining* with respect to M if $m_1(a) = m_2(a)$ for every $a \in B$ implies $m_1 = m_2$, $m_1, m_2 \in M$. A σ-homomorphism $X : B(\mathscr{S}) \to L$ is *state determining* if its range is state determining. For example, let L be the logic of all orthogonal projections on the Hilbert space $L_2(R^1)$ of Lebesgue square integrable functions on the real line R^1. Let M be the pure states corresponding to the non-negative functions in $L_2(R^1)$ of norm 1; i.e., if $m \in M$ then $m(a) = \langle \phi, a\phi \rangle$, $a \in L$ for some $\phi \in L_2(R^1)$ with $\phi \geqq 0$, $\|\phi\| = 1$. Then (L, M) is a quantum logic. If we define the projections $X(E) f(\lambda) = \chi_E(\lambda) f(\lambda)$, $f \in L_2(R^1)$, $E \in B(R)$ it is easy to check that X is state determining with respect to M.

We would also like $X(E)$ to satisfy another property. Suppose we know that S is located in $E \in B(\mathscr{S})$ when the system is in the state $m \in M$. Knowing that S is in E we now ask the probability that S is in $F \in B(\mathscr{S})$. From elementary probability considerations, if $m(X(E)) \neq 0$, we would expect this probability to be $m_1(X(F)) = m(X(F \cap E))/m(X(E))$. However m_1 may not be a state on L. More generally let B be a Boolean σ-algebra in L, $a \in B$ and m a state for which $m(a) \neq 0$. Then m_1 defined on B by $m_1(b) = m(b \wedge a)/m(a)$ is a state on B. It corresponds to the conditional probability in the state m given that the proposition a is true and is called a *conditional state* on B. B is a *conditional Boolean σ-algebra* if every conditional state on B is the restriction of some state in M (on L) to B. Conditions of this type have been studied in [5]. A σ-homomorphism $X : B(\mathscr{S}) \to L$ is a *conditional σ-homomorphism* if its range is a conditional Boolean σ-algebra. For example, let L be the orthogonal projections on $L_2(R^1)$ and let B be the Boolean σ-algebra generated by the projections $X(E) = \chi_E$, $E \in B(R^1)$ as in the previous example. Let m_ψ be a pure state corresponding to the unit vector ψ and suppose $m_\psi(X(E)) \neq 0$ for some fixed $E \in B(R^1)$. Then $m(X(E)) = m_\psi(X(E \cap F))/m_\psi(X(E))$ is a conditional state on B. Define the function ϕ by $\phi(\lambda) = \chi_E(\lambda)\, \psi(\lambda)/[m_\psi(X(E))]^{1/2}$ and let m_ϕ be the corresponding state. Then m_ϕ is a state on L and

$$m_\phi(X(F)) = \int_F \phi^2(\lambda)\, d\lambda = \frac{1}{m_\psi(X(E))} \int_{F \cap E} \psi^2(\lambda)\, d\lambda$$
$$= m_\psi(X(E \cap F))/m_\psi(X(E)) = m_\psi(X(F)) .$$

In this way we see that B is a conditional Boolean σ-algebra. We thus make the following

Definition. A quantum mechanical system is *localizable* if there is a state determining conditional σ-homomorphism (called a *position* σ-homomorphism) $X : B(\mathscr{S}) \to L$.

Of course there are systems that are not localizable [6]. However, as indicated by the work of JAUCH and PIRON [7] these systems may have a similar structure involving a weaker kind of σ-homomorphism than the one we have postulated. In this paper we shall only consider localizable systems.

So far, we have extracted three important concepts describing our quantum mechanical system S. These were the quantum logic $\mathscr{L} = (L, M)$ physical space $\mathscr{S}$, and the position σ-homomorphism $X : B(\mathscr{S}) \to L$. There is one more concept which all quantum systems seem to possess and which we feel is important enough to single out, the concept of symmetry. Intuitively, a symmetry on S involves some kind of a transformation on S that rearranges some of the properties of the system. If a is a proposition in L then after the symmetry transformation we get a new proposition $W(a)$ which is the proposition a concerning the transformed system. Thus the symmetry induces a map $W : L \to L$. One can convince himself that W should have the properties of a bijective σ-homomorphism on L which we shall call an *automorphism*. The group of automorphisms on L will be denoted by aut (L). In general a symmetry usually comes from a transformation on our physical space $\mathscr{S}$. We say that a group G is a *transformation group* on $\mathscr{S}$ if G is a locally compact Hausdorff topological group with second countability for which there is a map from $G \times \mathscr{S}$ onto $\mathscr{S}$ denoted by $(g, s) \to gs$, $g \in G$, $s \in \mathscr{S}$ which satisfies:

(i) if $s_1, s_2 \in \mathscr{S}$, then there is a $g \in G$ such that $s_1 = gs_2$ (transitivity);

ii) for all $g \in G$, $s \to gs$ is a homeomorphism of $\mathscr{S}$ with itself;

(iii) $g_1(g_2(s)) = (g_1 g_2)(s)$ for all $g_1, g_2 \in G$;

(iv) $g(s) = s$ for all $s \in \mathscr{S}$ if and only if $g = e$ (effectiveness) where e is the group identity.

Now if a transformation group G is a symmetry for S it must induce automorphisms on L. We are thus led to our next

Definition. A *symmetry group* on $(\mathscr{L}, \mathscr{S}, X)$ is a pair $\mathscr{G} = (G, W)$ where G is a transformation group on $\mathscr{S}$ and W is a group homomorphism $W : G \to$ aut (L) (i.e. $W_{g_1 g_2} = W_{g_1} W_{g_2}$) such that

(1) $g \to m(W_g(a))$ is continuous for every $m \in M$, $a \in L$;

(2) $X(gE) = W_g(X(E))$ for all $g \in G$, $E \in B(\mathscr{S})$ (invariance).

Condition (1) is a natural continuity requirement while (2) is an invariance condition which gives the natural interpretation that $W_g(X(E))$ is the proposition that S is located in the set gE.

This completes the background for our axiomatic structure. We shall call a four-tuple $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ as defined above a *quantum system*. We take the viewpoint that the important physical properties of a laboratory experiment are described by a quantum system $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$.

3. The Underlying Hilbert Space

In this section we show that corresponding to any quantum system $S = (\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ there is an underlying Hilbert space that mirrors most of the structure of S. A crucial reason for the existence of this Hilbert space is that there exists a non-trivial σ-finite quasi-invariant measure μ on $\mathscr{S}$. [A measure μ is *quasi-invariant* if $\mu(E) = 0$ if and only if $\mu(gE) = 0$, $g \in G$, $E \in B(\mathscr{S})$.] This measure is constructed by transferring the Haar measure on G to $\mathscr{S}$ in a fairly straightforward way. As a result of this transference, the invariance of the Haar measure is lost leaving however the weaker property of quasi-invariance. It can be shown that any other quasi-invariant measure on $\mathscr{S}$ must have the same null sets as μ[1]. Let $L_2(\mu)$ be the Hilbert space of μ square integrable functions on $\mathscr{S}$. If μ_1 is any other quasi-invariant measure then it is easily seen that $L_2(\mu)$ and $L_2(\mu_1)$ are unitarily equivalent via multiplication by the square root of the appropriate Radon-Nikodym derivative. One can show that this unitary transformation preserves all the structure that we shall construct. For this reason the structure is determined by any quasi-invariant measure μ which we shall keep fixed throughout the sequel.

If $m \in M$ is a state, then it has been shown in [9] that the measure $m(X(\cdot))$ on $B(\mathscr{S})$ is absolutely continuous with respect to μ. This follows using (2) and does not require that X be state determining or conditional, only that X be a σ-homomorphism. Thus by the Radon-Nikodym theorem there is a function $f \in L_1(\mu)$ such that $f \geqq 0$ and $m(X(E)) = \int_E f \, d\mu$ for all $E \in B(\mathscr{S})$. If we define $\hat{m} = + f^{1/2}$, then $\hat{m} \in L_2(\mu)$ and $\hat{m}$ is the unique (a.e. $[\mu]$) non-negative function in $L_2(\mu)$ such that for all $E \in B(\mathscr{S})$,

$$m(X(E)) = \int_E \hat{m}^2 \, d\mu \, . \tag{3.1}$$

It easily follows from the state determining nature of X that the map $m \to \hat{m}$ from M to $L_2(\mu)$ is one-one; however it is not onto. From (3.1) we see that statistically the location of our system is given by vectors in a Hilbert space. Since this statistical information is all that can be obtained in quantum mechanics the Hilbert space describes the location completely.

[1] For a discussion on quasi-invariant measures and their relation to group representations see [8].

Now let V_0 be the complex vector space generated by $\hat{M}$; that is, V_0 is the linear span of the vectors of the form $\hat{m}$. Now $W_g \in \text{aut}\,(L)$ can also be thought of as a map from M onto M defined by $(W_g m)\,(a) = m(W_g(a))$, $m \in M$, $a \in L$. Then W_g induces a natural transformation $\hat{W}_g$ on $\hat{M}$ defined by $\hat{W}_g \hat{m} = (W_g m)^\wedge$. This map is well-defined since $m \to \hat{m}$ is one-one. We next extend $\hat{W}_g$ to V_0 by linearity.

Theorem 3.1. *The map $g \to \hat{W}_g$ is a continuous unitary representation of G on V_0 and $\hat{W}_g f(\lambda) = f(g^{-1}\lambda)\,(d\mu_g/d\mu)^{1/2}$ for all $f \in V_0$. (The notation will be explained in the proof.)*

Proof. To show that $g \to \hat{W}_g$ is a representation we have

$$\hat{W}_{g_1 g_2}\hat{m} = (W_{g_1 g_2} m)^\wedge = (W_{g_1} W_{g_2} m)^\wedge = \hat{W}_{g_1}(W_{g_2} m)^\wedge = \hat{W}_{g_1}\hat{W}_{g_2}\hat{m}\,.$$

The unitarity follows from

$$\begin{aligned}\langle \hat{W}_g \hat{m}, \hat{W}_g \hat{m}\rangle &= \int (\hat{W}_g \hat{m})^2\, d\mu = \int ((W_g m)^\wedge)^2\, d\mu \\ &= (W_g m)\,(X(\mathscr{S})) = 1 = \langle \hat{m}, \hat{m}\rangle\,.\end{aligned}$$

We now find the explicit form of $\hat{W}_g$. For all $E \in B(\mathscr{S})$

$$\begin{aligned}\int_E (\hat{W}_g \hat{m})^2\, d\mu &= \int_E ((W_g m)^\wedge)^2\, d\mu = (W_g m)\,(X(E)) = m\,[W_g(X(E))] \\ &= m(X(gE)) = \int_{gE} \hat{m}^2\, d\mu = \int_E \hat{m}^2(g^{-1}\lambda)\, d\mu(g^{-1}\lambda) \\ &= \int_E \hat{m}^2(g^{-1}\lambda)\, d\mu_g(\lambda)\,.\end{aligned}$$

Where μ_g is the measure defined by $\mu_g(E) = \mu(g^{-1}(E))$ for all $E \in B(\mathscr{S})$. Now since μ is quasi-invariant we see that μ_g is absolutely continuous with respect to μ, so again by the Radon-Nikodym theorem

$$\mu_g(E) = \int_E \frac{d\mu_g}{d\mu}\, d\mu \quad \text{for all} \quad E \in B(\mathscr{S})\,.$$

We therefore obtain

$$\int_E (\hat{W}_g \hat{m})^2\, d\mu = \int_E \hat{m}^2(g^{-1}\lambda)\frac{d\mu_g}{d\mu}(\lambda)\, d\mu = \int_E [\hat{m}(g^{-1}\lambda)\,(d\mu_g/d\mu)^{1/2}(\lambda)]^2\, d\mu\,.$$

It then follows that $\hat{W}_g \hat{m}(\lambda) = \hat{m}(g^{-1}\lambda)\,(d\mu_g/d\mu)^{1/2}(\lambda)$. Continuity now follows by a standard argument.

Now let $V = \bar{V}_0$ the closure of V_0. Then V is a Hilbert space in $L_2(\mu)$ and since $\hat{W}_g$ is continuous on a dense subset of V it can be uniquely extended to a unitary operator on V. We thus see that the states M of our quantum system are represented by certain vectors $\hat{M}$ in a Hilbert space V and that the symmetry group $\mathscr{G}$ is represented by a unitary representation $\hat{W}_g$ on V.

We now consider the position σ-homomorphism X. It seems that the most natural way to transfer X to V is to define the projection-valued measure $\hat{X}$ from $B(\mathscr{S})$ into the set of projections on V by $\hat{X}(E) =$ projection on the closed span of $\{\hat{m} : m(X(E)) = 1\} = \text{proj } \overline{\text{sp}} \{\hat{m} : m(X(E)) = 1\}$. Notice $\hat{X}(E) \leqq \text{proj}\{f \in L_2(\mu) : f = 0 \text{ a.e. } [\mu] \text{ on } E'\}$. The following properties of $\hat{X}$ are easily verified. (i) $\hat{X}(\mathscr{S}) = V$, (ii) if $E \perp F$, then $\hat{X}(E) \perp \hat{X}(F)$, (iii) if $E \subset F$, then $\hat{X}(E) \leqq \hat{X}(F)$, (iv) $\hat{X}(E') \leqq \hat{X}(E)'$, (v) if $E \perp F$ then $\hat{X}(E \cup F) \geqq \hat{X}(E) \vee \hat{X}(F)$.

Unfortunately, $\hat{X}$ need not be a σ-homomorphism on V. In the following case it is not only a σ-homomorphism but a position σ-homomorphism. A σ-homomorphism $X : B(\mathscr{S}) \to L$ is said to be *smooth* if for any $E \in B(\mathscr{S})$ with $\mu(E) \neq 0$ we have $X(E) \neq 0$. Since quasi-invariant measures have the same null sets, this definition is independent of the quasi-invariant measure μ. In many physical situations this condition can be arranged to hold if it does not hold already. For example, if our system is confined to a box we can take our space to be the interior of the box instead of the entire three dimensional space. A quantum system $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ is *smooth* if X is smooth. In the next theorem we use the fact that X is conditional for the first time.

Theorem 3.2. *If $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ is a quantum system then the following statements are equivalent*: (1) X *is smooth*; (2) $V = L_2(\mu)$; (3) $\hat{X}(E) = \text{proj}\{f \in L_2; f = 0 \text{ a.e. } [\mu] \text{ on } E'\}$.

Proof. Suppose (1) holds and $E \in B(\mathscr{S})$ satisfies $\mu(E) \neq 0$. Let $m \in M$ satisfy $m(X(E)) \neq 0$. Then m_1 defined by

$$m_1(X(F)) = m(X(F \cap E))/m(X(E))$$

is a conditional state on the range $\mathscr{R}(X)$ of X and is therefore the restriction to $\mathscr{R}(X)$ of a state which we again denote by m_1. Then $\hat{m}_1$ satisfies

$$\int_F \hat{m}_1^2 \, d\mu = m_1(X(F)) = m(X(E \cap F))/m(X(E))$$
$$= \frac{1}{m(X(E))} \int_{F \cap E} \hat{m}^2 \, d\mu = \int_E \hat{m}^2/m(X(E)) \cdot \chi_E \, d\mu \, .$$

Therefore $\hat{m}_1 = \hat{m}/[m(X(E)]^{1/2} \cdot \chi_E \in V$, so $\hat{m}\chi_E \in V$. If $m(X(E)) = 0$, then $\hat{m} = 0$, a.e. on E so again $\hat{m}\chi_E = 0 \in V$. We have thus shown that if $E \in B(\mathscr{S})$, $m \in M$, then $\hat{m}\chi_E \in V$. Thus if $f \in V$ we have $\chi_E f \in V$. Since multiplication by characteristic functions form a maximal set of projections it follows that V is either $L_2(\mu)$ or all the functions in $L_2(\mu)$ which vanish on some measurable set Λ. Suppose $\mu(\Lambda) \neq 0$ and finite. Let $m \in M$ satisfy $m(X(\Lambda)) = 1$. Then $\int_\Lambda \hat{m}^2 \, d\mu = 1$ so $\hat{m}$ is not zero a.e. on Λ which is a contradiction. If $\mu(\Lambda) = \infty$ we can use a standard σ-finiteness argument. Thus $V = L_2(\mu)$ and (1) implies (2). To show (1) implies (3) again suppose (1) holds. We have seen before that

$$\hat{X}(E) \subset \text{proj}\{f \in L_2(\mu) : f = 0 \text{ a.e. } [\mu] \text{ on } E'\} \, .$$

Now suppose $h \in L_2(\mu)$ and $h = 0$ a.e. $[\mu]$ on E'. Then since (2) holds h is a point-wise limit of a sequence f_i of linear combinations of $\hat{m}$'s. But now $\chi_E f_i \to h$. Since from before $\hat{m}\chi_E$ is in the range of $\hat{X}(E)$ we have h in the range of $\hat{X}(E)$ and thus (3) holds. We leave it to the reader to show that (2) and (3) imply (1) so that all the statements are equivalent.

Denote the logic of all orthogonal projections on $L_2(\mu)$ by $\hat{L}$. We see by our examples in Section 2 that if X is smooth then $\hat{X}$ is a position σ-homomorphism on $\hat{L}$.

Now $\hat{W}_g$ induces an automorphism on $\hat{L}$ defined by $\hat{W}^g P = \hat{W}_g P \hat{W}_g^{-1}$ for all $P \in \hat{L}$.

Theorem 3.3. *If* $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ *is a smooth quantum system then* $\hat{X}(gE) = \hat{W}^g \hat{X}(E)$ *for all* $g \in G$, $E \in B(\mathscr{S})$ *and* $(G, \hat{W}^g)$ *forms a symmetry group on* $\hat{L}$.

The proof of this theorem depends upon the following useful.

Lemma 3.4. *If* $g_1, g_2 \in G$, *then* $\frac{d\mu_{g_1 g_2}}{d\mu}(\lambda) = \frac{d\mu_{g_2}}{d\mu}(g_1^{-1}\lambda) \cdot \frac{d\mu_{g_1}}{d\mu}(\lambda)$ a.e. $[\mu]$.

Proof. Using standard properties of Radon-Nikodym derivatives and integral change of variables we have for $E \in B(\mathscr{S})$

$$\int_E \frac{d\mu_{g_2}}{d\mu}(g_1^{-1}\lambda)\frac{d\mu_{g_1}}{d\mu}(\lambda)\,d\mu = \int_E \frac{d\mu_{g_2}}{d\mu}(g_1^{-1}\lambda)\,d\mu_{g_1}(\lambda)$$

$$= \int_{g_1^{-1}E} \frac{d\mu_{g_2}}{d\mu}(\lambda)\,d\mu_{g_1}(g_1\lambda) = \int_{g_1^{-1}E} \frac{d\mu_2}{d\mu}(\lambda)\,d\mu(\lambda)$$

$$= \int_{g_1^{-1}E} d\mu_{g_2}(\lambda) = \mu_{g_2}(g_1^{-1}E) = \mu(g_2^{-1}g_1^{-1}(E))$$

$$= \int_E \frac{d\mu_{g_1 g_2}}{d\mu}\,d\mu\,.$$

The lemma then follows.

Proof of Theorem. For $f \in L_2(\mu)$ we have

$$(\hat{W}^g\hat{X}(E)f)(\lambda) = (\hat{W}_g\hat{X}(E)\hat{W}_g^{-1}f)(\lambda)$$
$$= (\hat{X}(E)\hat{W}_{g^{-1}}f)(g^{-1}\lambda)(d\mu_g/d\mu)^{1/2}(\lambda)$$
$$= (d\mu_g/d\mu)^{1/2}(\lambda)(\hat{W}_{g^{-1}}f)(g^{-1}\lambda)\chi_E(g^{-1}\lambda)$$
$$= (d\mu_g/d\mu)^{1/2}(\lambda)\chi_E(g^{-1}\lambda)f(\lambda)(d\mu_{g^{-1}}/d\mu)(g^{-1}\lambda)\,.$$

By the above lemma $\frac{d\mu_g}{d\mu}(\lambda)\frac{d\mu_{g^{-1}}}{d\mu}\cdot(g^{-1}\lambda) = \frac{d\mu_e}{d\mu} = 1$ a.e. $[\mu]$. Hence $(\hat{W}^g\hat{X}(E)f)(\lambda) = \chi_E(g^{-1}\lambda)f(\lambda) = \chi_{gE}(\lambda)f(\lambda) = \hat{X}(gE)f(\lambda)$. It is now easy to see that the other axioms are satisfied so that $(G, \hat{W}^g)$ forms a symmetry group on $\hat{L}$.

Letting $\hat{\mathscr{L}} = (\hat{L}, \hat{M})$ and $\hat{\mathscr{G}} = (G, \hat{W}^g)$ we see that the structure of a smooth quantum system $(\mathscr{L}, \mathscr{S}, X, \mathscr{G})$ is fairly accurately mirrored by the Hilbert space quantum system $(\hat{\mathscr{L}}, \mathscr{S}, \hat{X}, \hat{\mathscr{G}})$ induced by it. This mirror is not perfectly accurate however, for the following two reasons. Except for the propositions in the range of X there is no isomorphism between L and $\hat{L}$ so the propositions in general are not represented by $\hat{L}$. Secondly, there is no provision for distinguishing between pure and mixed states since all of the states in M are mapped into pure states in $\hat{M}$.

Let us now consider the simple example of a three dimensional non-relativistic quantum mechanical particle. In this case the physical space $\mathscr{S}$ is taken to be four-dimensional Euclidean space $\mathscr{S} = \{(x, y, z, t) : x, y, z, t \in R^1\}$. The transformation group of symmetries G is taken to be the group of rotations and reflections in space together with the space-time translations. The quasi-invariant measure on $\mathscr{S}$ becomes the actually invariant Lebesgue measure μ and the underlying Hilbert space becomes $L_2(R^4)$. The *coordinate functions* f_x, f_y, f_z, f_t are defined by $f_x(x, y, z, t) = x, \ldots, f_t(x, y, z, t) = t$. If X is the position σ-homomorphism, the *coordinate observables* $f_x(X), \ldots, f_t(X)$ are σ-homomorphisms based on $B(R^1)$ defined by $f_x(X)(E) = X(f_x^{-1}(E)), \ldots, f_t(X)(E) = X(f_t^{-1}(E))$, $E \in B(R^1)$. The induced coordinate observables on $L_2(R^4)$ become $f_x(\hat{X})(E) = \hat{X}(f_x^{-1}(E)) = \chi_E, \ldots, f_z(\hat{X}) = \chi_E$. (The time coordinate observable is usually not considered.) Now $E \to \chi_E$ is a projection-valued measure on $L_2(R^4)$ whose corresponding self-adjoint operator is multiplication by the independent variable. We thus have the coordinate operators A_x, A_y, A_z given formally by $A_x f(x, y, z, t) = x f(x, y, z, t,)$ $A_y f = y f$, $A_z f = z f$. The representation $\hat{W}_g$ on $L_2(R^4)$ is given by $\hat{W}_g f(s) = f(g^{-1}s)$, $f \in L_2(R^4)$, $g \in G$ since in this case $\frac{d\mu_g}{d\mu} = 1$. Thus the representation of the group of translations is $\hat{W}_g f(x_0, y_0, z_0, t_0) = f(x_0 - x, y_0, z_0, t_0)$, $\ldots, \hat{W}_t f(x_0, y_0, z_0, t_0) = f(x_0, y_0, z_0, t_0 - t)$. These are one parameter group representations of unitary operators and by Stone's theorem they have the form $\hat{W}_x = e^{-ixP_x}, \ldots, \hat{W}_t = e^{-itP_t}$ where the infinitesimal generators $P_x = -i\frac{\partial}{\partial x}, \ldots, P_t = -i\frac{\partial}{\partial t}$ are the momentum-energy operators.

4. Quantum Fields

In this section we show that the concept of a quantum field arises naturally from that of a σ-homomorphism and in fact under certain circumstance these concepts are equivalent. We first need some preliminary definitions. A σ-homomorphism based on $B(R^1)$ is called an *observable*. A collection of observables is *compatible* if their ranges are compatible. It can be shown [10] that a collection of observables is compatible if and only if their ranges are contained in a common Boolean

σ-algebra. In this section we shall assume that L is *separable*; i.e., every Boolean σ-algebra is generated by a countable number of elements. It can then be shown [10] that any Boolean σ-algebra is the range of some observable. A Boolean σ-algebra in L is *maximal* if it is not properly contained in a larger Boolean σ-algebra. A set of compatible observables is *maximal* if the union of their ranges generates a maximal Boolean σ-algebra. A set of compatible observables $\mathcal{O}$ is a *complete set of compatible observables* if an observable x satisfying $x \leftrightarrow \mathcal{O}$ implies $x \in \mathcal{O}$. Note that a complete set of compatible observables is maximal but the converse need not hold. It is easily seen that any set of compatible observables is contained in a complete set (and hence maximal) of compatible observables. The above definitions can also be applied to σ-homomorphisms based on any Boolean σ-algebra.

Let $\mathcal{S}$ be a physical space and let $X: B(\mathcal{S}) \to L$ be a σ-homomorphism (not necessarily a position σ-homomorphism). If $f: \mathcal{S} \to R^1$ is a measurable function we define the observable $f(X)$ by $f(X)(E) = X(f^{-1}(E))$, $E \in B(R^1)$. It is shown in [10] that all the observables in a compatible set are functions of a single observable and if the range of an observable x is contained in the range of an observable y then x is a function of y. Now suppose that x and y are compatible observables. Then there is an observable z and Borel functions u and v such that $x = u(z)$, $y = v(z)$. We define the sum and product of x and y by $x + y = (u + v)(z)$ and $x \cdot y = (u \cdot v)(z)$ respectively. It is shown in [11] that these are well defined, i.e., independent of u, v, and z. The *expectation* or *average value* of x in the state m is $m(x) = \int \lambda m[x(d\lambda)]$ if the integral exists. An observable x is *bounded* if $m(x)$ exists and is finite for every state m. It is shown in [11] that $m(x + y) = m(x) + m(y)$ whenever these exist.

Now if $X: B(\mathcal{S}) \to L$ is a σ-homomorphism then X induces a map $f \to X_f = f(X)$ from the set of measurable functions on $\mathcal{S}$ into the set of observables on L. The following three properties are easily verified.

(i) $X_f \leftrightarrow X_g$ for any measurable f and g;

(ii) X is maximal if and only if $\{X_f : f \text{ measurable}\}$ is a maximal set of compatible observables;

(iii) $X_{\alpha f + \beta g} = \alpha X_f + \beta X_g$, $X_{f \cdot g} = X_f X_g$, for any $\alpha, \beta \in R^1$, f, g measurable. Now let f_i be a pointwise Cauchy and dominated sequence of measurable functions. (f_i is dominated if there is a $g \in L_1(\mu)$ such that $|f_i| \leqq g$.) Using the dominated convergence theorem we obtain;

(iv) $m[|X_{f_i} - X_{f_j}|] = \int |f_i(\lambda) - f_j(\lambda)| \, m[X(d\lambda)] \to 0$ as $i, j \to \infty$ for all $m \in M$.

If K is an algebra of continuous functions on $\mathcal{S}$ which is dense relative to the supremum norm in the space of all continuous functions on $\mathscr{S}$ we define a *K-field* to be a map $f \to Y_f$ from K into the set of observables on L which satisfies

(i′) $Y_f \leftrightarrow Y_g$ for all $f, g \in K$;

(ii′) $\{Y_f : f \in K\}$ is a maximal set of compatible observables;

(iii′) $Y_{\alpha f + \beta g} = \alpha Y_f + \beta Y_g$, $Y_{f \cdot g} = Y_f Y_g$, $\alpha, \beta \in R^1$, $f, g \in K$;

(iv′) if f_i is a pointwise Cauchy and dominated sequence in K, then Y_{f_i} is weakly Cauchy [i.e. (iv) holds].

We have seen in the previous paragraph that if X is a maximal σ-homomorphism, then $f \to X_f = f(X)$ is a K-field for $f \in K$. We shall now show that under a particular circumstance any K-field comes from a σ-homomorphism in this way. A quantum logic (L, M) is *state separable* if there is a sequence of states $m_i \in M$ with the following property: for every sequence x_j of bounded observables on L which satisfy $\lim\limits_{j\to\infty} m_i(x_j) = 0$ for $i = 1, 2, \ldots$, we have $\lim\limits_{j\to\infty} m(x_j) = 0$ for all $m \in M$.

For example if L is the logic of all orthogonal projections on a separable Hilbert space H then L is state separable. To see this let ϕ_i be a complete orthonormal sequence in H. Then if A_i is a sequence of bounded self-adjoint operators and $\lim\limits_{j\to\infty} \langle \phi_i, A_j \phi_i \rangle = 0$, $i = 1, 2, \ldots$, it follows that $\lim\limits_{j\to\infty} \langle \phi, A_j \phi \rangle = 0$ for every $\phi \in H$.

In [11, 12] it is shown that a complete set of bounded compatible observables $\mathcal{O}$ is metrically complete with respect to the spectral norm. We now show $\mathcal{O}$ is weakly complete in the state separable case.

Lemma 4.1. *If (L, M) is state separable, then a complete set of bounded compatible observables $\mathcal{O}$ is weakly complete. That is,*

$$\lim_{i,j\to\infty} m(|x_i - x_j|) = 0\,, \qquad x_i, x_j \in \mathcal{O}\,,$$

for every $m \in M$ implies there exists $x \in \mathcal{O}$ such that $\lim\limits_{i\to\infty} m(|x_i - x|) = 0$ for every m.

Proof. It follows from our previous discussion that there is an observable x such that $y \in \mathcal{O}$ if and only if there is a Borel function u such that $y = u(x)$. Let $x_i \in \mathcal{O}$ be weakly Cauchy. Then there exist Borel functions u_i such that $x_i = u_i(x)$ and

$$\int |u_i - u_j|\, m[x(d\lambda)] = m[|u_i(x) - u_j(x)|] \to 0$$

as $i, j \to \infty$ for every $m \in M$. In particular this holds for the m_i, $i = 1, 2, \ldots$, in the definition of state separable. Denoting the measure $E \to m_i(x(E))$ by m_{ix}, since $L_1(m_{1x})$ is complete, there is a function v_1 such that $u_i \to v_1$ in mean $[m_{1x}]$. It follows that there is a sub-sequence $u_i^1 \to v_1$ a.e. $[m_{1x}]$. Now again since

$$\int |u_i^1 - u_j^1|\, m_2[x(d\lambda)] = m_2[|u_i^1(x) - u_j^1(x)|] \to 0$$

as $i, j \to \infty$ there is a subsequence u_i^2 of u_i^1, and a function v_2^0 such that $u_i^2 \to v_2^0$ a.e. $[m_{2x}]$. Since $u_i^2 \to v_1$ a.e. $[m_{1x}]$ we must have two measurable sets $R_1, R_2 \subset R^1$ such that $m_{1x}(R_1) = m_{2x}(R_2) = 0$ and $u_i^2 \to v_1$ on R_1' and $u_i^2 \to v_2^0$ on R_2'. Define v_2 to be $v_1 = v_2^0$ on $R_1' \cap R_2'$, v_1 on $R_2 \cap R_1'$, v_2^0 on $R_1 \cap R_2'$ and arbitrary on $R_1 \cap R_2$. Then $u_i^2 \to v_2$ except for a set of m_{1x} and m_{2x} measure zero. Continuing, we get a sequence $u_i^n \to v_n$ except for a set of $m_{1x}, \ldots, m_{nx}$ measure zero. Let $v = \lim_{n\to\infty} v_n$ and notice that this limit exists except for a set N of $m_{1x}, m_{2x}, \ldots$ measure zero. We then have for the diagonal sequence u_n^n, $\lim_{n\to\infty} u_n^n = v$ except on N. If we put $y = v(x)$ we have

$$m_i|u_j^j(x) - y| = \int |u_j^j - v| \, m_i[x(d\lambda)] \to 0$$

as $j \to \infty$ for $i = 1, 2, \ldots$ Thus if $\varepsilon > 0$ is given we have

$$m_i[|x_j - y|] \leqq m_i[|x_j - u_k^k(x)|] + m_i[|u_k^k(x) - y|] < \varepsilon$$

for j sufficiently large and this holds for $i = 1, 2, \ldots$ (j may depend on i, of course). Since (L, M) is state separable we have $m[|x_j - y|] \to 0$ and $x_j \to y$ weakly.

Theorem 4.2. *Let (L, M) be state separable and let $f \to Y_f$ be a K-field. Then there is a unique σ-homomorphism $X : B(\mathscr{S}) \to L$ such that $Y_f = f(X)$.*

Proof. The Boolean σ-algebra B generated by the ranges of Y_f for all $f \in K$ is maximal. There exists an observable x whose range is B such that the Y_f's are Borel functions of x. That is, for $f \in K$ there is a u_f such that $Y_f = u_f(x)$. Let χ_E be the characteristic function of $E \in B(\mathscr{S})$. Then there are functions $f_i \in K$ bounded by χ_E which converge to χ_E pointwise. Since the f_i's satisfy the hypotheses of (iv′) Y_{f_i} is a weakly Cauchy sequence. By Lemma 4.1 there is a Borel function v such that $Y_{f_i} \to v(x)$. We now define $Y_{\chi_E} = v(x)$. It can easily be shown that this extended Y which may now be applied to characteristic functions is well-defined and satisfies (i), (ii), (iii) and (iv). We now define $X(E) = Y_{\chi_E}(\{1\})$. We now indicate how to show that X is a σ-homomorphism. Since one can show

$$X(E \cap F) = Y_{\chi_{E \cap F}}(\{1\}) = Y_{\chi_E \cdot \chi_F}(\{1\}) = Y_{\chi_E} Y_{\chi_F}(\{1\}) = X(E) \wedge X(F)$$

it follows that if $E \perp F$, then $X(E) \perp X(F)$. Also, if $E \perp F$ then

$$X(E \cup F) = Y_{\chi_{E \cup F}}(\{1\}) = Y_{\chi_E + \chi_E}(\{1\}) = (Y_{\chi_E} + Y_{\chi_F})(\{1\})$$
$$= X(E) \vee X(F) .$$

The rest of the proof is left to the reader.

Corollary. *If $X : B(\mathscr{S}) \to L$ is a maximal σ-homomorphism and $Y_f = f(X)$, $f \in K$. Then Y is a K-field and X is the unique σ-homomorphism satisfying $Y_f = f(X)$, $f \in K$.*

Let $\mathscr{G} = (G, W_g)$ be a symmetry group on $(\mathscr{L}, \mathscr{S}, X)$. Recall the invariance of X is given for $g \in G$, $E \in B(\mathscr{S})$ by

$$X(gE) = W_g X(E)\,. \tag{4.1}$$

If f is measurable and $g \in G$ we define the function $g(f)$ by $g(f)(\lambda) = f(g^{-1}\lambda)$. It is now easily seen that X satisfies (4.1) if and only if the corresponding K-field Y satisfies $Y_{g(f)} = W_g Y_f$ for all $f \in K$.

Let us now consider the concrete example of relativistic quantum mechanics. The physical space becomes Minkowski space-time and the transformation group G is taken as the inhomogeneous Lorentz group or Poincare group. In this case K is usually taken as the test function space consisting either of the infinitely differentiable functions with compact support or the infinitely differentiable functions of rapid decrease. We will now generalize the concept of fields as they are usually used in quantum field theory. By a Wightman field [13] on K we mean a collection of maps $A_1, \ldots, A_n$ from K into the set of observables on L with the following properties:

(1) $A_i(f) \leftrightarrow A_i(g)$, $i = 1, 2, \ldots, n$; $f, g \in K$;

(2) $A_i(\alpha f + \beta g) = \alpha A_i(f) + \beta A_i(g)$, $i = 1, 2, \ldots, n$; $f, g \in K$, $\alpha, \beta \in R^1$;

(3) $W_g A_i(f) = A_i(g(f))$ (invariance);

(4) if an observable x satisfies $x \leftrightarrow A_i(f)$, $i = 1, 2, \ldots$, for all $f \in K$, then $x = \alpha I$ for some $\alpha \in R^1$ (irreducibility) (I is the identity observable defined by $I(\{1\}) = 1$);

(5) if the supports of f and g are space-like separated then $A_i(f) \leftrightarrow A_j(g)$, $i = j = 1, \ldots, n$ (local compatibility).

The invariance condition in (3) is for a so-called scalar field. For a tensor field one would have an $n \times n$ matrix representation of G, $S_{ij}(g)$ such that $W_g A_j(f) = \sum_{k=1}^{n} S_{jk}(g^{-1}) A_k(g(f))$. In this case however one would have to define what is meant by the sum of non-compatible observables [14].

If D is a subset of a logic L we define D^c by $D^c = \{a \in L : a \leftrightarrow D\}$. It is easy to see that D^c is a sublogic of L. The following properties are also easily verified.

(C 1) $D \subset D^{cc}$;

(C 2) $D^c = D^{ccc}$;

(C 3) if $D_1 \subset D_2$, then $D_2^c \subset D_1^c$;

(C 4) if $W \in \operatorname{aut}(L)$, then $(WD)^c = WD^c$.

By a Haag field [1] we mean a map H from the open subsets $\mathscr{U}$ of space-time $\mathscr{S}$ with compact closures into sublogics $H(\Delta)$ of L $(\Delta \in \mathscr{U})$ with the following properties:

(a) if $\Delta_1 \subset \Delta_2$ then $H(\Delta_1) \subset H(\Delta_2)$ (isotony);

(b) if Δ_1 and Δ_2 are space-like separated then $H(\Delta_1) \leftrightarrow H(\Delta_2)$; (local compatibility);

(c) $\left[\bigcup_\Delta H(\Delta)\right]^{cc} = L$ (irreducibility);

(d) $H(g\Delta) = W_g H(\Delta)$ (invariance);

where $\Delta, \Delta_1, \Delta_2 \in \mathscr{U}$.

Now suppose $A_1, \ldots, A_n$ is a Wightman field, and $T(\Delta) = \{\mathscr{R}[A_i(f)] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\} \subset L$, $\Delta \in \mathscr{U}$, where $\mathscr{R}[A_i(f)]$ is the range of $A_i(f)$. We now define a map H from $\mathscr{U}$ into the set of sublogics of L by $H(\Delta) = T(\Delta)^{cc}$ and show that H is a Haag field.

(a) If $\Delta_1 \subset \Delta_2$ then $\{f : \operatorname{supp} f \subset \Delta_1\} \subset \{f : \operatorname{supp} f \subset \Delta_2\}$ so $T(\Delta_1) \subset T(\Delta_2)$. It follows from (C 3) that $H(\Delta_1) \subset H(\Delta_2)$.

(b) If Δ_1 and Δ_2 are space-like separated it follows from (5) that $T(\Delta_1) \subset T(\Delta_2)^c$ and $T(\Delta_2) \subset T(\Delta_1)^c$. Using (C 2) and (C 3) we have $T(\Delta_2)^{cc} \subset T(\Delta_1)^c = T(\Delta_1)^{ccc}$ and $T(\Delta_1)^{cc} \subset T(\Delta_2)^c = T(\Delta_2)^{ccc}$. Hence $T(\Delta_2)^{cc} \leftrightarrow T(\Delta_1)^{cc}$.

(c) It follows from (4) that $\left[\bigcup_\Delta H(\Delta)\right]^c = \{0, 1\}$. We then have $[\cup H(\Delta)]^{cc} = \{0, 1\}^c = L$.

(d) It suffices to show that $T(g\Delta) = W_g T(\Delta)$ since then by (C 4), $H(g\Delta) = T(g\Delta)^{cc} = [W_g T(\Delta)]^{cc} = W_g T(\Delta)^{cc} = W_g H(\Delta)$. Now

$$\begin{aligned} W_g T(\Delta) &= \{W_g \mathscr{R}[A_i(f)] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\} \\ &= \{\mathscr{R}[W_g A_i(f)] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\} \\ &= \{\mathscr{R}[A_i(g(f))] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\} \\ &= \{\mathscr{R}[A_i(h)] : i = 1, \ldots, n, \operatorname{supp} h \subset g\Delta\} = T(g\Delta). \end{aligned}$$

This correspondence between the Wightman field $A_1, \ldots, A_n$ and the above Haag field H is called a *local correspondence*. One might ask if H is a given Haag field, is there a Wightman field which is a local correspondent to H [15]? This appears to be a difficult problem and we shall obtain only a simple partial result. We will need the following.

Assumption. For a given Wightman field $A_1, \ldots, A_n$ there are σ-homomorphisms $X_1, \ldots, X_n$ in $\mathscr{S}$ such that $A_i(f) = f(X_i)$, for any $f \in K$, $i = 1, \ldots, n$.

This assumption is motivated by our correspondence between K-fields and σ-homomorphisms given at the beginning of this section.

If H is a given Haag field let us find a necessary condition for there to be a Wightman field which is a local correspondent to H. Suppose $A_1, \ldots, A_n$ is such a local correspondent. Then

$$\begin{aligned} H(\Delta) &= \{\mathscr{R}[A_i(f)] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\}^{cc} \\ &= \{\mathscr{R}[f(X_i)] : i = 1, \ldots, n, \operatorname{supp} f \subset \Delta\}^{cc} \\ &= \{X_i(E) : i = 1, \ldots, n, E \subset \Delta, E \in B(\mathscr{S})\}^{cc}. \end{aligned}$$

Thus a necessary condition is that there exist σ-homomorphisms $X_1, \ldots, X_n$ such that $H(\Delta) = \{X_i(E) : i = 1, \ldots, n, E \subset \Delta, E \in B(\mathscr{S})\}^{cc}$. W leave if to the reader to show that this condition is also sufficient.

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STANLEY GUDDER
Department of Mathematics
The University of Wisconsin
213 Van Vleck Hall
Madison, Wisconsin 53706, USA