Commun. math. Phys. 11, 198—213 (1969)

Green Functions of the Anisotropic Heisenberg Model*

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Received June 25, 1968

Abstract. The Green functions of the anisotropic Heisenberg model are studied by a method which was applied previously to the reduced density matrices. Integral equations are used to prove the existence of the infinite volume limit of the Green functions, and some analyticity properties with respect to the fugacity (or magnetic field), the potentials, and the complex times.

Introduction

In a previous paper (1, hereafter referred to as I), we studied the reduced density matrices (RDM) of a quantum lattice gas which is equivalent to the anisotropic Heisenberg ferromagnet. We used a functional integral representation based on a generalized Poisson process and proved the existence of the infinite volume limit and the analyticity of the RDM with respect to the fugacity and the potentials in the low fugacity (and by symmetry, high fugacity) region. In this paper, we extend these results to the Green functions (GF) of the system. We prove for the GF the existence of the infinite volume limit and analyticity with respect to the same parameters, as well as analyticity with respect to the complex times, in some domain. With respect to the fugacity and potentials, the analyticity domain is the same as that of the RDM when all real parts of the times are equal, and decreases when their differences increase.

In Section 1, for the sake of comparison, we present general arguments that prove the analyticity of the GF with respect to the times for general systems, including the present model with physical values of the parameters. In Section 2, we describe the functional representation of the GF first for purely imaginary, and then complex times. In Section 3, we extend to the GF integral equations that were used in I for the RDM, and deduce from them our main results. We conclude in Section 4 with a brief description of the corresponding results for related models that were considered in I.

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* Research supported by the National Science Foundation.

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1. General Properties of Green Functions

The purpose of this section is to make apparent which additional information is obtained by the investigation of the following sections. To do this, we derive general properties of the Green functions of quantum systems in equilibrium, in the framework of Ref. [2].

We consider a system of identical particles in configuration space E, which is either the ν-dimensional euclidian space ($E = R^\nu$) or a ν-dimensional lattice ($E = Z^\nu$). Let $\mathscr{H}$ be the Fock space, namely the (symmetric for bosons, antisymmetric for fermions) tensor algebra constructed over $L^2(E)$. To any bounded (open if $E = R^\nu$) region $\Lambda \subset E$, we associate the Fock space $\mathscr{H}(\Lambda)$, namely the tensor algebra (with appropriate symmetry) over $L^2(\Lambda)$, and the C^*-algebra $\mathfrak{A}(\Lambda) = \mathscr{L}(\mathscr{H}(\Lambda))$ of all bounded operators in $\mathscr{H}(\Lambda)$. $\mathfrak{A}(\Lambda)$ is in a natural way a subalgebra of $\mathscr{L}(\mathscr{H})$. We now define $\mathfrak{A}_0$ by $\mathfrak{A}_0 = \bigcup\limits_{\Lambda} \mathfrak{A}(\Lambda)$ and $\mathfrak{A}$ as the norm closure of $\mathfrak{A}_0$ in $\mathscr{L}(\mathscr{H})$. $\mathfrak{A}$ is the C^*-algebra associated with the system.

We now define an equilibrium state at inverse temperature β and a time evolution of the system as follows. To each bounded (open) $\Lambda \subset E$, we associate a self adjoint operator $H(\Lambda)$ in $\mathscr{H}(\Lambda)$ such that, for any $\alpha > 0$, $\exp[-\alpha H(\Lambda)]$ has a finite trace.

In particular, $H(\Lambda)$ is bounded from below and has a discrete spectrum with finite multiplicities. We then define a state ϱ_Λ on $\mathfrak{A}(\Lambda)$ and a time evolution in $\mathfrak{A}(\Lambda)$ by:

$$\varrho_\Lambda(A) = \mathrm{Tr}(A\, e^{-\beta H(\Lambda)})\,(\mathrm{Tr}\, e^{-\beta H(\Lambda)})^{-1}\,, \tag{1.1}$$

$$A \to A_\Lambda(t) = \exp[itH(\Lambda)]\, A \exp[-itH(\Lambda)]\,. \tag{1.2}$$

We now make the following assumptions. Let Λ be any bounded (open) region and $A \in \mathfrak{A}(\Lambda)$. Let $\Lambda' \supset \Lambda$, so that $\mathfrak{A}(\Lambda) \subset \mathfrak{A}(\Lambda')$. Then, when Λ' becomes infinite in a reasonable sense, the following limits exist:

$$\varrho(A) = \lim_{\Lambda' \to \infty} \varrho_{\Lambda'}(A)\,, \tag{1.3}$$

$$A(t) = \lim_{\Lambda' \to \infty} A_{\Lambda'}(t) \tag{1.4}$$

the second limit being taken in the sense of the norm in $\mathfrak{A}$. We assume furthermore that the limit is a (norm) continuous function of t. Here, "reasonable sense" has to be defined in each specific case, depending presumably on the method of proof of (1.3) and (1.4). It must imply that for any fixed Λ, Λ' must contain Λ for Λ' sufficiently large.

For lattice systems, (1.4) has been proved under general circumstances [3]. (1.3) has been proved under general circumstances for almost all β [4], and for the specific models considered here, for all β in the region we are interested in [I]. It follows from (1.3) and (1.4) that ϱ can be extended to a state on $\mathfrak{A}$ and $A \to A(t)$ can be extended to an automorphism of $\mathfrak{A}$. Moreover, for any family $A_1, \ldots, A_n$ of elements of $\mathfrak{A}_0$, the following limit exists:

$$\varrho(A_1(t_1) \ldots A_n(t_n)) = \lim_{\Lambda \to \infty} \varrho_\Lambda(A_{1\Lambda}(t_1) \ldots A_{n\Lambda}(t_n)) . \tag{1.5}$$

We now consider complex times $\zeta_j = t_j + i\theta_j (1 \leqq j \leqq n)$. Let $A_j \in \mathfrak{A}_0$ $(1 \leqq j \leqq n)$ and Λ be sufficiently large, so that $A_j \in \mathfrak{A}(\Lambda)$ for $j = 1, \ldots, n$. We define:

$$\varrho_\Lambda\left(\prod_{j=1}^{n} A_{j\Lambda}(\zeta_j)\right) = (\operatorname{Tr} e^{-\beta H(\Lambda)})^{-1} \cdot \operatorname{Tr}\left[e^{-\beta H(\Lambda)} \prod_{j=1}^{n} (e^{i(\zeta_j - \zeta_{j-1}) H(\Lambda)} A_j)\right] \tag{1.6}$$

where $\zeta_0 = \zeta_n$. This is an analytic function of the ζ_j [5, 6] in the domain $\mathcal{D}$ defined by:

$$\operatorname{Im}\zeta_1 < \operatorname{Im}\zeta_2 < \cdots < \operatorname{Im}\zeta_n < \operatorname{Im}\zeta_1 + \beta . \tag{1.7}$$

It is continuous on the boundary and satisfies the Kubo Martin Schwinger (KMS) boundary condition in the form [2, 6]

$$\varrho_\Lambda\left(\prod_{j=1}^{n} A_{j\Lambda}(\zeta_j)\right) = \varrho_\Lambda\left(\prod_{j \geqq k} A_{j\Lambda}(\zeta_j) \prod_{j<k} A_{j\Lambda}(\zeta_j + i\beta)\right) \tag{1.8}$$

for $1 \leqq k \leqq n$.

We shall now prove the following extension of some results in Ref. [2].

Theorem 1. *Suppose that the limits* (1.3, 1.4) *exist. Then, when Λ becomes infinite, $\varrho_\Lambda(\prod A_{j\Lambda}(\zeta_j))$ tends to a well defined limit $F_{A_1 \ldots A_n}(\zeta_1, \ldots, \zeta_n)$ which is an analytic function of the ζ_j in $\mathcal{D}$. The limit is uniform in $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ on the compact sets in $\mathcal{D}$. The limiting function is continuous on the boundary, and satisfies the KMS boundary condition:*

$$F_{A_1 \ldots A_n}(\zeta_1, \ldots, \zeta_n) = F_{A_k \ldots A_n A_1 \ldots A_{k-1}}(\zeta_k, \ldots, \zeta_n, \zeta_1 + i\beta, \ldots, \zeta_{k-1} + i\beta). \tag{1.9}$$

An essential element of the proof is the following lemma:

Lemma 1. *Let $A_1, \ldots, A_n$ be bounded operators in some Hilbert space. Let H be a self adjoint operator such that $\operatorname{Tr} e^{-\alpha H} < \infty$ for any $\alpha > 0$. Then for any $\alpha_1, \ldots, \alpha_n$ such that $\alpha_j \geqq 0$ for all j and $\sum \alpha_j = 2\pi$, the following inequality holds:*

$$\left|\operatorname{Tr}\left(\prod_{j=1}^{n} e^{-\alpha_j H} A_j\right)\right| \leqq \prod_{j=1}^{n} \|A_j\| \cdot \operatorname{Tr} e^{-2\pi H} . \tag{1.10}$$

Proof of the Lemma[1]. The proof goes by induction. The result is true for $n = 1$. We assume it up to n and prove it for $n + 1$. To each A_j we

[1] Another and more concise proof of the lemma is obtained by using a generalization of HÖLDER's inequality to operators. [Cf. N. DUNFORD and J. SCHWARTZ, Linear Operators, Interscience, New York (1963), Lemma XI. 9-9-d, p. 1093 and Lemma XI. 9-20-b, p. 1105.] We are grateful to D. RUELLE for pointing this out to us.

associate a point (which we call again A_j) on a circle, in such a way that the successive angles are $\widehat{A_j A_{j+1}} = \alpha_{j+1}$. We treat separately the cases n even and n odd.

$n = 2p$: There are $2p + 1$ points on the circle. Then there is at least one j such that the diameter through A_j leaves p points on each open semi-circle. In fact, let $N(\alpha)$ be the number of points A_k on the semi-circle $[\alpha, \alpha + \pi)$. When α varies, $N(\alpha)$ is constant except for jumps of ± 1. When α increases from some α_0 to $\alpha_0 + \pi$, $N(\alpha)$ varies from some $q \leqq 2p + 1$ to $2p + 1 - q$ and therefore passes at least once in succession through the values p and $p + 1$. The value of α for which $N(\alpha)$ switches from p to $p + 1$ or vice-versa solves the problem. We suppose, after a circular permutation of the indices if necessary, that $j = n + 1 = 2p + 1$. We now apply Schwarz inequality:

$$|\mathrm{Tr}(U\,V)|^2 \leqq \mathrm{Tr}(U^+ U)\,\mathrm{Tr}(V^+ V) \tag{1.11}$$

where:

$$U = A_{n+1}\, e^{-\alpha_1 H} A_1 \ldots A_p\, e^{-(\pi - \alpha_1 - \cdots - \alpha_p) H} \tag{1.12}$$

$$V = e^{-(\pi - \alpha_{p+2} - \cdots - \alpha_{2p+1}) H} A_{p+1} \ldots A_{2p}\, e^{-\alpha_{2p+1} H}\,. \tag{1.13}$$

The second factor in the RHS of (1.11) is of the same type as the original expression, with only $n = 2p$ factors. The first term is of the form:

$$\mathrm{Tr}(A_{n+1}^+ A_{n+1} W) \leqq \|A_{n+1}\|^2\, \mathrm{Tr}\, W \tag{1.14}$$

because W is a positive operator, Furthermore, $\mathrm{Tr}\, W$ is again an expression of the original type, with only $n = 2p$ factors. Using (1.10) for n factors then proves it for $n + 1$.

$n = 2p + 1$. Similarly, there is at least one j such that the diameter through A_j leaves p points on one open semi-circle and at most $p + 1$ on the other. Take $j = n + 1 = 2p + 2$. Applying (1.11) as above gives 2 factors with $n + 1$ operators in each. One of them is of the type (1.14) and can be majorized similarly in terms of an expression with $2p$ operators A_k only, which therefore satisfies (1.10). The second one has $2p + 2 = n + 1$ operators and is therefore bounded only by:

$$M = \sup \prod_{j=1}^{n+1} \|A_j\|^{-1} \left| \mathrm{Tr} \left(\prod_{j=1}^{n+1} e^{-\alpha_j H} A_j \right) \right| \tag{1.15}$$

where the supremum is taken over all possible bounded A_j and positive α_j such that $\Sigma \alpha_j = 2\pi$. Now M is certainly finite. In fact, suppose for simplicity that $H \geqq 0$. Then:

$$M \leqq \mathrm{Tr} \exp[-(\max \alpha_j)\, H] \leqq \mathrm{Tr} \exp\left(-\frac{2\pi}{n+1} H\right). \tag{1.16}$$

We obtain therefore:

$$\left[\prod_{j=1}^{n+1} \|A_j\|\right]^{-2} \left|\mathrm{Tr}\left(\prod_{j=1}^{n+1} e^{-\alpha_j H} A_j\right)\right|^2 \leqq M \,\mathrm{Tr} \exp(-2\pi H) \,. \quad (1.17)$$

Taking the supremum over all possible A_j and α_j in the LHS gives:

$$M^2 \leqq M \,\mathrm{Tr}\, e^{-2\pi H} \quad (1.18)$$

and therefore:

$$M \leqq \mathrm{Tr}\, e^{-2\pi H} \,, \quad (1.19)$$

since $M > 0$. Taking $A_j = 1$ for all j shows that (1.19) is in fact an equality. This concludes the proof of the lemma.

Proof of the Theorem. The function defined by (1.6) depends only on the differences between the ζ_j. It is therefore sufficient to consider

$$\varrho_\Lambda\left(A_0 \prod_{j=1}^{n} A_{j\Lambda}(\zeta_j)\right),$$

for which the domain $\mathscr{D}$ reduces to:

$$0 < \mathrm{Im}\,\zeta_1 < \cdots < \mathrm{Im}\,\zeta_n < \beta \,. \quad (1.20)$$

This function has the following properties:

(1) It is analytic in $\mathscr{D}$ and continuous on the boundary.

(2) It is bounded on the closure of $\mathscr{D}$ by $\prod_{j=0}^{n} \|A_j\|$. In particular, this holds uniformly with respect to Λ. This is a consequence of the previous lemma and the definitions.

(3) It converges pointwise to a continuous function of the t_j, for $\zeta_j = t_j$ or $\zeta_j = t_j + i\beta$ according to whether $j \leqq k$ or $j > k$, for $0 \leqq k \leqq n$. This follows from (1.5) and the KMS boundary condition (1.8).

Let now $f(\zeta)$ be a function of ζ which is analytic and non-zero, and decreases sufficiently fast as $|\mathrm{Re}\,\zeta|$ tends to infinity, in a neighborhood of the strip $0 < \mathrm{Im}\,\zeta < \beta$. Suitable functions would be $(\zeta + i)^{-n}$ with $n \geqq 1$, or $\exp(-\zeta^2)$. From (1) and (2) it follows that the function:

$$\varrho_\Lambda\left(A_0 \prod_{j=1}^{n} A_{j\Lambda}(\zeta_j)\right) \prod_{j=1}^{n} f(\zeta_j) \quad (1.21)$$

has a multiple Cauchy integral representation for which the domain of integration lies in the region described above in property (3). This is seen easily by induction. As an illustration, we write the representation for $n = 2$. We drop the subscript Λ.

$$\begin{aligned}
\varrho\,(A_0 A_1(\zeta_1)\, A_2(\zeta_2))\, f(\zeta_1)\, f(\zeta_2) &= -\frac{1}{4\pi^2}\int \frac{f(u)\,du}{u-\zeta_1} \\
&\cdot\left\{\int \frac{f(v)\,dv}{v-\zeta_2}\,\varrho\,(A_0 A_1(u)\, A_2(v)) - \int \frac{f(v+i\beta)\,dv}{v+i\beta-\zeta_2}\,\varrho\,(A_2(v)\, A_0 A_1(u))\right\} \\
&+ \frac{1}{4\pi^2}\int \frac{du}{u+\zeta_2-\zeta_1}\left\{\int \frac{f(u-v+i\beta)\, f(i\beta-v)\,dv}{v-i\beta+\zeta_2}\;\varrho(A_1(u)\, A_2 A_0(v))\right. \\
&\left. -\int \frac{dv}{v-\zeta_2}\, f(u-v)\, f(-v)\,\varrho(A_0(v)\, A_1(u)\, A_2)\right\}. \quad (1.22)
\end{aligned}$$

All integrations run from $-\infty$ to $+\infty$.

The proof of the theorem is now immediate. The existence of the limit (pointwise) follows from Lebesgue's theorem, and the analyticity of the limiting function from the integral representation. The other properties follow easily.

Uniform boundedness of the functions was essential in the derivation. The sequence of functions $F_n(\zeta) = \exp[-n \cosh \zeta]$ in the strip $0 < \operatorname{Im} \zeta < 2\pi$ provides an example of the kind of pathology which is thereby excluded. Failure of the lemma to hold for complex β or complex values of the parameters in H prevents the extension of the method to these situations, even if we know that the limit (1.3) exists under such circumstances. This limitation will be circumvented by the method of the following sections.

In the sequel, we shall be interested in Green functions. These are functions of the type:

$$G(\zeta_1, \ldots, \zeta_n) = \varrho(T A_1(\zeta_1) \ldots A_n(\zeta_n)) , \tag{1.23}$$

where T means time ordered product: more precisely the product is to be taken in such an order that $\operatorname{Im} \zeta$ increases from left to right. Such functions are piecewise analytic for $\max(\operatorname{Im} \zeta_j) - \min(\operatorname{Im} \zeta_j) < \beta$ with discontinuities whenever two $\operatorname{Im} \zeta_j$ coincide. Due to the KMS boundary condition they can be continued as periodic functions of all arguments, with period $i\beta$, and discontinuities whenever $\operatorname{Im}(\zeta_j - \zeta_k) = 0 \pmod{\beta}$ for some (j, k) [6].

The Green functions are obtained by piecing together functions of the type (1.6), and Theorem 1 therefore applies in each connected component of the analyticity domain.

2. The Green Functions for Complex Times and the Integral Equations

In this section, we apply the methods of (I) to obtain a representation of the Green functions of the same model as functional integrals over the paths of a generalized Poisson process. A similar representation has already been obtained [7] for the temperature Green functions of continuous quantum gases, using the Wiener process. It constitutes a natural extension of that used for the reduced density matrices.

The model is the same as in (I), as defined by (I, 1.1 to 1.6), namely a quantum lattice system with point hard cores. The Green functions are obtained by analytic continuation of the temperature Green functions (TGF), which we consider first. They are defined for a finite system as follows. Let $x_j = (r_j, i\alpha_j)$ and $y_j = (s_j, i\alpha_j')$ (where $1 \leqq j \leqq m$) be two families of space-time points, where $0 \leqq \alpha_j, \alpha_j' \leqq \beta$. The variables α_j, α_j' are real and have the meaning of inverse temperatures for the TGF. We introduce factors i in the notation and refer to the $i\alpha$ as times, in anticipation of the general case. Meanwhile, the α_j, α'_j will be referred to as "times". We define:

$$\begin{cases} a(x_j) = \exp(-\alpha_j H)\, a_{r_j} \exp(\alpha_j H) \\ a^+(y_j) = \exp(-\alpha'_j H)\, a^+_{s_j} \exp(\alpha'_j H) \end{cases} \tag{2.1}$$

$$\begin{cases} a(x^m) = \prod_{j=1}^{m} a(x_j) \\ a^+(y^m) = \prod_{j=1}^{m} a^+(y_j)\,. \end{cases} \tag{2.2}$$

The TGF are now defined by:

$$\overline{G}_\Lambda(x^m, y^m) = \varrho_\Lambda(T a^+(y^m)\, a(x^m)) \tag{2.3}$$

where ϱ_Λ is defined by (1.1), and the time ordering T in Section 1. As a function of the variables α_j and α'_j, $\overline{G}_\Lambda$ is continuous whenever all are different. If for some (j, k), $\alpha_j = \alpha_k$ or $\alpha'_j = \alpha'_k$, $\overline{G}_\Lambda$ is still continuous, due to the commutation relations (including the hard core condition). $\overline{G}_\Lambda$ has discontinuities whenever for some (j, k), $\alpha_j = \alpha'_k$. At such points, it is convenient to define $\overline{G}_\Lambda$ as the limit for $\alpha_j \to \alpha'_k$ with $\alpha_j > \alpha'_k$. This means that we choose the normal order in any dubious case, and has the advantage that when all α_j and α'_j are equal, $\overline{G}_\Lambda$ become the reduced density matrices of the system.

We associate with the model a generalized Poisson process as described by (I, 1.11 to 1.21). The TGF then have the following representation as functional integrals [7]:

$$\begin{aligned} \overline{G}_\Lambda(x^m, y^m) = \mathscr{S} \prod_{k=1}^{m} \Big\{ \sum_{j_k} \theta(\alpha'_k - \alpha_k + j_k\beta) \\ \cdot \int P_{r_k s_k}^{\alpha'_k - \alpha_k + j_k\beta}(d\omega_k) \exp[\mu(\alpha'_k - \alpha_k + j_k\beta)] \Big\}\, G_\Lambda(X) \end{aligned} \tag{2.4}$$

where $\mathscr{S}$ means sum over all permutations of the y_j; the step function $\theta(x)$ is normalized to be left continuous ($\theta(0) = 0$), and X is a family of m paths $\omega_k (1 \leqq k \leqq m)$ of the Poisson process, starting from the points r_k at "times" α_k and ending at the points s_k at "times" $\alpha'_k + j_k\beta$.

The family of functions $G_\Lambda(X)$ is given by:

$$G_\Lambda(X) = Z^{-1} \int d\,Y\, z^q \exp[-U(X + Y)]\, \alpha_\Lambda(X + Y) \tag{2.5}$$

where the notation is the same as in (I, 2.4 to 2.6). In the present case, the interaction term $U(X + Y)$ is the integral from 0 to β of the classical potential energy of whatever points are available at "time" t (mod β) on the paths that constitute X and Y. The number of such points increases (resp. decreases) by one whenever t passes through one of the α_j (resp. α'_j). Notice also that the definition of G_Λ differs from the corresponding ϱ_Λ (I, 2.4) in that the contribution of the open paths of X to the factor that contains μ has been taken outside of G_Λ, which therefore depends on μ only through $z = e^{\beta\mu}$.

We now consider the GF of the system for complex times. They are obtained from the TGF by analytic continuation from $i\alpha_j$ to $\zeta_j = t_j + i\alpha_j$ and from $i\alpha_j'$ to $\zeta_j' = t_j' + i\alpha_j'$, in each region where no two of the α_j and α_j' are equal (mod β). We consider one such region. The TGF $\overline{G}_\Lambda(x^m, y^m)$ is then a sum of path integrals, where the paths have the α_j and α_j' (mod β) as starting and ending "times". Introducing intermediate summations over the positions on the various paths at *all* the α_j and α_j' (mod β), we can write $\overline{G}_\Lambda$ as a sum of path integrals in such a way that in each term of the sum, *all* the paths have the same "time" interval, starting at some α_j or α_j' (mod β) and ending at some other one. This interval is at most β. In the present case, due to the presence of hard cores and the discreteness of the configuration space, this sum is finite for a finite system, so that no convergence problem arises. In related models [I] where there is no hard core, the convergence follows from the stability condition on $\varphi_{\|}$. This will still hold after analytic continuation. In any case, the analytic continuation of $\overline{G}_\Lambda$ reduces to that of integrals of the type:

$$I(\gamma, \gamma') = \int P_{rs}^{\gamma'-\gamma}(d\omega) \int P_{uv}^{\gamma'-\gamma}(d\overline{\omega}) \cdot \exp\left[2\mu(\gamma' - \gamma) - \int_{\gamma}^{\gamma'} dt\, \varphi_{\|}(\omega(t) - \overline{\omega}(t))\right] \tag{2.6}$$

where $\gamma < \gamma' < \gamma + \beta$. Here we have considered for illustration the case where there is only one piece of internal path $\overline{\omega}$ (to be integrated over in order to get G_Λ) and one piece of external path ω (to be integrated over in order to obtain $\overline{G}_\Lambda$ from G_Λ). The argument extends straightforwardly to the case where there are several paths of each type.

The analytic continuation of (2.6) with respect to γ and γ' is obtained most easily by using a parametrization of the paths which does not depend on γ and γ'. Let $t = \gamma' t' + \gamma(1 - t')$ and $\omega'(t') = \omega(t)$, $\overline{\omega}'(t') = \overline{\omega}(t)$. Let $n(\omega)$ ($\equiv n(\omega')$) and $n(\overline{\omega})$ ($\equiv n(\overline{\omega}')$) be the number of jumps of ω and $\overline{\omega}$ respectively. Then from the definition of the process and relations such as (I, 1.21), it follows that:

$$\begin{aligned} I(\gamma, \gamma') = \int P_{r,s}^1(d\omega') \int P_{u,v}^1(d\overline{\omega}') & \\ \cdot \exp[-2M_0(1 + \gamma - \gamma')]\,(\gamma' - \gamma)^{n(\omega) + n(\overline{\omega})} & \\ \cdot \exp\left[2\mu(\gamma' - \gamma) - (\gamma' - \gamma)\int_0^1 dt'\, \varphi_{\|}(\omega'(t') - \overline{\omega}'(t'))\right] & \end{aligned} \tag{2.7}$$

where M_0 is given by (I, 1.3).

This is in an obvious way an analytic function of γ and γ', since now the path integration no longer depends on γ, γ'. The analytic continuation to complex ζ and ζ' with $\operatorname{Im}\zeta = \gamma$ and $\operatorname{Im}\zeta' = \gamma'$ then gives:

$$\begin{aligned}\tilde{I}(\zeta, \zeta') = \int P^1_{r,s}(d\omega') \int P^1_{u,v}(d\overline{\omega}') \exp[-2M_0(1 + i(\zeta' - \zeta))] \\ \cdot [i(\zeta - \zeta')]^{n(\omega)+n(\overline{\omega})} \exp[-2i\mu(\zeta' - \zeta) \\ + i(\zeta' - \zeta)\int_0^1 dt'\, \varphi_{\|}(\omega'(t') - \overline{\omega}'(t'))]\,.\end{aligned} \tag{2.8}$$

It is convenient to come back to a parametrization of the paths where the (now complex) time runs from ζ to ζ', thereby obtaining:

$$\begin{aligned}\tilde{I}(\zeta, \zeta') = \int \tilde{P}^{\zeta'-\zeta}_{r,s}(d\omega) \int \tilde{P}^{\zeta'-\zeta}_{u,v}(d\overline{\omega}) \\ \cdot \exp\left[-2i\mu(\zeta' - \zeta) + i\int_\zeta^{\zeta'} dt\, \varphi_{\|}(\omega(t) - \overline{\omega}(t))\right]\end{aligned} \tag{2.9}$$

where the new time t is defined by:

$$t = \zeta' t' + \zeta(1 - t') \tag{2.10}$$

and where:

$$\omega(t) = \omega'(t')\,, \tag{2.11}$$

$$\begin{aligned}\int \tilde{P}^{\zeta'-\zeta}_{r,s}(d\omega) = \int P^1_{r,s}(d\omega')\, [i(\zeta - \zeta')]^{n(\omega)} \\ \cdot \exp[-M_0(1 + i(\zeta' - \zeta))]\,.\end{aligned} \tag{2.12}$$

Notice that for fixed ω', this provides independent analytic continuation of both the integrand and the measure.

Piecing together integrals of the type (2.8, 2.9) gives the representation of the GF we are looking for. Let $\zeta_j = t_j + i\alpha_j$ and $\zeta'_j = t'_j + i\alpha'_j$ be the complex times associated with the points r_j and s_j to build the space-time points $x_j = (r_j, \zeta_j)$ and $y_j = (s_j, \zeta'_j)$. We assume that all the α_j, α'_j are different and lie in $[0, \beta]$. Let Γ be the polygonal contour on the complex cylinder ($\equiv$ complex plane mod $i\beta$) of the time variable ζ, with vertices at the points ζ_j and ζ'_j taken in the order of increasing imaginary parts. Γ is oriented in the sense of increasing $\operatorname{Im}\zeta$. The representation of $\overline{G}_\Lambda$ is then obtained by modifying (2.4, 2.5) as follows:

$$\begin{aligned}\overline{G}_\Lambda(x^m, y^m) = \mathscr{S} \prod_{k=1}^m \Big\{\sum_{j_k} \theta(\alpha'_k - \alpha_k + j_k\beta) \int \tilde{P}^{\zeta'_k - \zeta_k + ij_k\beta}_{r_k, s_k}(d\omega_k) \\ \cdot \exp[-i\mu(\zeta'_k - \zeta_k + ij_k\beta)]\Big\}\, G_\Lambda(X)\end{aligned} \tag{2.13}$$

where:

$$G_\Lambda(X) = Z^{-1} \int \tilde{d}Y\, z^q \exp[i\tilde{U}(X + Y)]\, \alpha_\Lambda(X + Y)\,. \tag{2.14}$$

Now the time parameter of *all* the paths runs along the contour Γ, *all* the path integrations are correspondingly defined by (2.12) on each segment of Γ, and $\tilde{U}$ is obtained from U by replacing the integral from 0 to β by the contour integral along Γ. The integration associated with the contour Γ as described above will be called hereafter integration along Γ. It satisfies the following invariance property, which is obvious when expressions as (2.6) are written in their original operator form. Let Γ and Γ' be two polygonal contours on the complex cylinder of the variable ζ, with $\operatorname{Im}\zeta$ always increasing along Γ and Γ'. We say that Γ' is a refinement of $\Gamma(\Gamma' \supset \Gamma)$ if all the vertices of Γ belong to Γ'. To each X is then associated a minimal contour $\Gamma(X)$ which is the one described above. Now the representations (2.13), 2.14) still hold if the integration is taken along any $\Gamma' \supset \Gamma(X)$. (In particular the integration in Z is contour independent.) Intuitively, the only important property of the contour is that it passes through all end times of X.

Up to now, we have restricted our attention to Green functions by considering contours Γ along which $\operatorname{Im}\zeta$ increases. Such contours we call monotonous. We now show that this restriction is unnecessary in the present model. In fact, the operator that simulates the kinetic energy and is used to define the stochastic process is bounded (for a finite number of particles). The path integral can be defined for any complex value of the time parameter (in sharp contrast with continuous systems for which the kinetic energy is represented by the Laplace operator). Therefore, in the present model, all path integrations (2.13, 2.14) remain well defined for an arbitrary, i.e., non-monotonous contour.

This would not lead very far in general. In fact, the decrease of $\operatorname{Im}\zeta$ along some part of Γ means that one uses operators of the type $\exp(\gamma H)$ with $\gamma > 0$. Now, if the potential energy becomes unstable under a change of sign, which is the case in related models [I] without hard core, this will behave catastrophically, causing for instance the divergence of the series that define G_Λ and $\overline{G}_\Lambda$.

In the present model however the hard core ensures stability by restricting the configuration space, and the interactions remain stable upon a change of sign, so that operators of the form $\exp(\gamma H)$ $(\gamma > 0)$ are allowed, and one can use non-monotonous contours Γ. For finite systems, one then obtains entire functions of the ζ_j, ζ_j', which are analytic continuations of the GF, as follows. Let ζ_j, ζ_j' be 2 m complex numbers in a prescribed order, and Γ a (polygonal) contour in the complex plane which is periodic with period $i\beta$, oriented in such a way that t increases by $i\beta$ along one period, and which passes *in one period* through all the points ζ_j, ζ_j' in the prescribed order. One then defines $\overline{G}_\Lambda(x^m, y^m)$ by (2.13, 2.14) where now all integrations run along Γ, and with the modification that in the θ-function, $\operatorname{Im}\zeta$ has to be replaced by a parameter that increases along Γ. In other words, the sum in (2.13) runs over all the points ζ_k' (mod $i\beta$) which are later than ζ_k on the periodic contour Γ. If for some k, $\zeta_k = \zeta_k'$, the term where ω_k reduces to a point is to be taken or left aside, according to whether ζ_k' comes after or before ζ_k in the prescribed order of the ζ_j, ζ_j'.

The entire function thereby constructed is the analytic continuation of that part of the GF that is obtained when all the ζ_j, ζ'_j lie in some strip $\gamma < \operatorname{Im}\zeta, \zeta' < \gamma + \beta$, and the prescribed order coincides with that of increasing imaginary parts. It is therefore identical with the thermal average of the product of the $a(x_j)$, $a^+(y_j)$ taken in the prescribed order of the ζ_j and ζ'_j.

The method and results of the next section will apply to these more general functions, for the model with hard cores, and only to the GF for related models without hard core (I, see also Section 4).

3. Integral Equations and Results

In this section we write down integral equations for the G_Λ that generalize those of [I], obtain bounds and analyticity properties of their kernels, and derive from them the results mentioned in the introduction. The model is the same as in [I] and the previous section, and although we shall speak in terms of GF, the results apply to their analytic continuation as described at the end of Section 2.

The Kirkwood Salzburg (KS) Eqs. (I, 2.7) extend straightforwardly from the RDM to the GF and become:

$$G_\Lambda(X+\omega) = \alpha_\Lambda(X+\omega)\exp[i\tilde{F}(\omega, X)] \textstyle\int \tilde{d}\, Y z^q \tilde{K}(\omega, Y)\, G_\Lambda(X+Y)\,. \tag{3.1}$$

Here ω is a path or piece of path starting at time ζ and ending at time ζ'. The integration runs along any contour $\Gamma \supset \Gamma(X+\omega)$, and ζ and ζ' belong to one period of Γ. $\tilde{F}$ and $\tilde{K}$ are defined respectively by:

$$\tilde{F}(\omega, X) = \int_\zeta^{\zeta'} dt \sum_i \varphi_{\|}(\omega(t) - \omega_i(t)) \tag{3.2}$$

where the integral runs along Γ, and the sum over whatever paths are available in X in the time interval (ζ, ζ'), and:

$$\tilde{K}(\omega, \bar{\omega}) = \exp\left[i\int_\Gamma dt \sum_i \varphi_{\|}(\omega_i(t) - \bar{\omega}_i(t))\right] - 1 \tag{3.3}$$

where the sum runs over all elementary paths that build the composite path $\bar{\omega}$. Other notations are the same as in [I]. For a given $X+\omega$, there is a great arbitrariness in the choice of ω for which one can write Eq. (3.1). We shall reduce it by the following rule. Let ω_0 be any path in $X+\omega$, starting at some ζ_0 and ending at some ζ'_0. Then ω shall be the part of ω_0 obtained when the time varies (in the sense of orientation of Γ) from ζ_0 to the first of the points $\zeta'_0 + ij\beta$ which it reaches [and to $\zeta_0 + i\beta$ if $\zeta'_0 = \zeta_0 \pmod{i\beta}$].

We now consider a *fixed* (polygonal) contour Γ and the set off all equations of the type (3.1) such that $\Gamma(X+\omega) \subset \Gamma$, and with the previous restriction on the choice of ω. The time parameter of a typical path ω

varies from some $\zeta \in \Gamma$ to some later time $\zeta' \in \Gamma$. Let $j^+(\omega)$ be the smallest integer j such that ζ' does not occur later than $\zeta + ij\beta$ on Γ (if $\zeta' = \zeta + ij\beta$, then $j^+(\omega) = j$). Let ξ and τ be real strictly positive numbers, to be chosen later, and define for any ω:

$$\Delta(\omega) = \xi^{j^+(\omega)} \exp[\tau\, n(\omega)] \,. \tag{3.4}$$

Let:

$$\Delta(X) = \prod_{\omega \in X} \Delta(\omega) \,. \tag{3.5}$$

Let $\mathscr{E}$ be the complex vector space of functions $h(X)$ of families X of paths with parameter running along Γ. The subspace $\mathscr{E}_\Delta$ of those h for which

$$\|h\| \equiv \sup_X \Delta(X)^{-1} |h(X)| < \infty \tag{3.6}$$

is a Banach space with (3.6) as the definition of the norm. The family of Eqs. (3.1) is then a linear equation in $\mathscr{E}$ of the type:

$$G_\Lambda = A_\Lambda(a + \mathscr{L} G_\Lambda) \tag{3.7}$$

where A_Λ is defined in [I] (cf. I, 2.11), a is the vector in $\mathscr{E}_\Delta$ defined by $a(\omega) = 1$ if $j^+(\omega) = 1$ and $a(X) = 0$ otherwise. It satisfies $\|a\| = \xi^{-1}$. $\mathscr{L}$ is a linear operator, easily extracted from (3.1). Notice that Δ, $\mathscr{E}_\Delta$ and $\mathscr{L}$ depend on Γ, which we keep fixed. We now show that under appropriate circumstances, $\mathscr{L}$ is a bounded operator in $\mathscr{E}_\Delta$. Let $\varphi_\| \in \mathscr{B}$, where $\mathscr{B}$ is defined by (I, 1.4) and let:

$$L = \int_\Gamma d|\zeta| \tag{3.8}$$

where the integral runs over one period of Γ. Let $h \in \mathscr{E}_\Delta$. Then:

$$|\mathscr{L} h(X + \omega)| \leqq \exp[L\phi + \int |d\tilde{\varpi}|\, |z|^j\, \Delta(\varpi)\, |\tilde{K}(\omega, \varpi)|]\, \|h\|\, \Delta(X) \tag{3.9}$$

where j is the number of elementary paths that constitute ϖ ($j \equiv j^+(\varpi)$). Now the choice of ω in $X + \omega$ and the definition of $\Delta(X)$ are such that $\Delta(X + \omega) = \Delta(X)\,\Delta(\omega)$. Moreover $j^+(\omega) = 1$ and $\Delta(\omega) = \xi \exp[\tau n(\omega)]$. We then have to compare the exponential in the RHS of (3.9) with $\Delta(\omega)$. By the same method as in [I], one obtains for the last term in the exponent the bound:

$$\begin{aligned} \int |d\tilde{\varpi}|\, |z|^j\, \Delta(\varpi)|\, \tilde{K}(\omega, \varpi)| &\leqq \sum_{j=1}^{\infty} (|z|\; \xi)^j \int |\tilde{P}^{ij\beta}_{0,r}(d\varpi)| \\ &\cdot \exp[\tau n(\varpi)]\, (n(\omega) + j^{-1}\, n(\varpi) + \exp(L\phi)) \,. \end{aligned} \tag{3.10}$$

$\mathscr{L}$ will be a bounded operator if ξ and τ satisfy:

$$\tau > \sum_j (|z|\; \xi)^j \sum_r \int |\tilde{P}^{ij\beta}_{or}(d\varpi)| \exp[\tau\, n(\varpi)] \tag{3.11}$$

$$= \sum_j (|z|\; \xi)^j \exp[j(\beta \operatorname{Re} M_0 + L M\, e^\tau)] \tag{3.12}$$

where M_0 and M are defined by (I, 1.2, 1.3). This can be written as:

$$\xi < \frac{\tau}{\tau+1} \exp[-(\beta \operatorname{Re}(M_0 + \mu) + LM e^{\tau})]\,. \tag{3.13}$$

Comparing (3.9, 3.10, 3.13), we obtain:

$$\|\mathscr{L}\| < \xi^{-1} \exp[L\phi + \tau(\exp L\phi + LM e^{\tau})]\,. \tag{3.14}$$

In particular $\mathscr{L}$ will have norm less than one provided:

$$\xi > \exp[L\phi + \tau(\exp(L\phi) + LM e^{\tau})]\,. \tag{3.15}$$

We therefore have proved the following result:

Lemma 2. *Let $\mathscr{E}_\Lambda$ be defined by* (3.4, 3.5, 3.6), *for fixed Γ, ξ and τ ($\xi > 0, \tau > 0$). Let $\mathscr{F}_0(L, \xi, \tau)$ be the (open connected) set of those $(\mu, \varphi_\perp)$ that satisfy* (3.13). *Then, for all $(\mu, \varphi_\perp, \varphi_\parallel) \in \mathscr{F}_0(L, \xi, \tau) \times \mathscr{B}$, $\mathscr{L}$ is a bounded operator in $\mathscr{E}_\Lambda$ and its norm satisfies* (3.14).

The following result is proved by the same method as in [I].

Lemma 3. *Under the assumptions of Lemma 1, $\mathscr{L}$ is norm analytic in $(\mu, \varphi_\perp, \varphi_\parallel)$ for $(\mu, \varphi_\perp, \varphi_\parallel) \in \mathscr{F}_0(L, \xi, \tau) \times \mathscr{B}$.*

We now state the main results.

Theorem 2. *Let $\mathscr{E}_\Lambda$ be defined by* (3.4, 3.5, 3.6) *for fixed $\Gamma, \xi > 0$, $\tau > 0$. Let $\mathscr{F} \equiv \mathscr{F}(L, \xi, \tau)$ be the (open connected) subset of $\mathbb{C} \times \mathscr{B} \times \mathscr{B}$ consisting of those $(\mu, \varphi_\perp, \varphi_\parallel)$ that satisfy* (3.13, 3.15). *Then, for $(\mu, \varphi_\perp, \varphi_\parallel) \in \mathscr{F}$:*

(I) *The Eq.* (3.1) *has a unique solution G_Λ in $\mathscr{E}_\Lambda$, obtained by iteration. The solution is a norm analytic function of $(\mu, \varphi_\perp, \varphi_\parallel)$ in $\mathscr{F}$. It coincides within $\mathscr{F}$ with* (2.14), *and satisfies:*

$$\|G_\Lambda\| \leqq \{\xi - \exp[L\phi + \tau(e^{L\phi} + LM e^{\tau})]\}^{-1}\,. \tag{3.16}$$

(II) *The infinite volume equation:*

$$G = a + \mathscr{L} G \tag{3.17}$$

has a unique solution G in $\mathscr{E}_\Lambda$, which is also analytic in $\mathscr{F}$ and satisfies (3.16). *G is invariant under the group $\mathscr{T}$ of translations that leave the lattice invariant.*

(III) *When Λ' becomes infinite in the sense of Theorem* (II, 1), *$\|A_\Lambda(G_{\Lambda'} - G)\|$ tends to zero. The convergence is uniform with respect to Λ for fixed $(\mu, \varphi_\perp, \varphi_\parallel)$ and uniform in μ on the compact sets for fixed $(\varphi_\perp, \varphi_\parallel)$.*

The proof is the same as in [I] and the comments that follow Theorem (I, 1) apply also to the present case. One can furthermore include at this stage analyticity properties of $G_\Lambda(X)$ and $G(X)$ with respect to the end times of X, by using a parametrization of the type

(2.10, 2.11) with fixed ω', and allowing for deformations of Γ, with L as an upper bound on its length.

We now consider the GF themselves.

Theorem 3. *For* $(\mu, \varphi_\perp, \varphi_\parallel) \in \overline{\mathscr{F}(L)} = \bigcup_{\xi,\tau} \mathscr{F}(L, \xi, \tau)$ *and for any set of complex times that can be picked up by a contour Γ, the length of which does not exceed L, the GF* (2.13) *tend to well defined limits $\overline{G}(x^m, y^m)$ in the sense that:*

$$\sup_{r,s\in\Lambda} \sup_{\zeta_j, \zeta_j'} |\overline{G}_{\Lambda'}(x^m, y^m) - \overline{G}(x^m, y^m)| \to 0 \tag{3.18}$$

when Λ' becomes infinite in the previous sense. Both $\overline{G}_\Lambda$ and $\overline{G}$ are analytic functions of $(\mu, \varphi_\perp, \varphi_\parallel)$ in $\overline{\mathscr{F}}(L)$ and satisfy the inequality:

$$\begin{aligned} |\overline{G}_{(\Lambda)}(x^m, y^m)| \leqq (\xi - \exp[L\phi + \tau(e^{L\phi} + LM\, e^\tau)])^{-1}\, m! \\ \cdot (\tau \exp[-\beta \operatorname{Re}(M_0 + \mu) + L|M_0 + \mu|])^m . \end{aligned} \tag{3.19}$$

Furthermore, $\overline{G}_\Lambda$ and $\overline{G}$ are analytic functions of the complex times under the same restrictions. $\overline{G}$ is invariant under the translation group $\mathscr{T}$.

The last exponential in (3.19) follows from crude estimates of integrals along fractions of the contour Γ. Analyticity (in Sup. norm or for fixed r^m, s^m) with respect to the times follows from the analyticity of G mentioned after Theorem 1. The order in which the times appear on Γ depends on the function considered, as described in Section 2. (It has to be that of increasing imaginary parts (mod β) in models without hard cores). The times appear in the definition of the analyticity domain only through the length L of Γ. For a given family of times, the best Γ is the polygonal contour with vertices at these times in the prescribed order.

Summarizing, the analyticity domain is the set of those $(\mu, \varphi_\perp, \varphi_\parallel)$ and times for which, for some $\tau > 0$:

$$\exp[\beta \operatorname{Re}(\mu + M_0) + LM(\tau + 1)\, e^\tau + L\phi + \tau e^{L\phi}] < \frac{\tau}{\tau + 1} . \tag{3.20}$$

From the symmetry between occupied and empty sites, we obtain:

Theorem 4. *All previous results hold with μ replaced by:*

$$-\mu + \sum_r (\varphi_\parallel(r) - \varphi_\perp(r)) . \tag{3.21}$$

We have not tried to obtain the best possible bounds on $\mathscr{L}$. Minor improvements are obtained easily by using the bound (I, 1.29) instead of (I, 1.30) and/or by separating real and imaginary parts of the potentials and the times.

4. Conclusion

We first compare the results of Sections 1 and 3. In Section 1, we have obtained analyticity in the times in domains of the type (1.7), but only for physical (real) values of μ, $\varphi_\perp$ and $\varphi_\parallel$. In Section 3, we have obtained in addition analyticity with respect to μ, $\varphi_\perp$ and $\varphi_\parallel$, and for the present model, we could dispense with the time ordering condition. On the other hand, the analyticity domain in the times is now much smaller. This is not surprising, since for complex parameters, the operator $\exp(itH)$ is no longer unitary for real t, and therefore not uniformly bounded when $|t|$ becomes large. In any case, even for real parameters, the method of Section 3 does not make use of the unitarity of this operator, and the analyticity domain in the times still does not extend to arbitrarily large values of $|\mathrm{Re}\,\zeta|$. On the other hand, for the TGF ($L = \beta$), the analyticity domain in (μ, $\varphi_\perp$, $\varphi_\parallel$) is the same as for the RDM, as can be seen by comparing (3.20) with (I, 2.22, 2.23).

We now describe the results obtained by the method of Section 3 for the related models described in I.

(1) *Fermi Statistics and Point Hard Core.* The hard core plays no role, since multiple occupancy of a single site is already forbidden by the Pauli principle. The results can be slightly improved by modifying the definition of $\mathscr{E}_\Lambda$ and the choice of equations. In the definition of the analyticity domain, (3.20) can be replaced by:

$$\exp[\beta\,\mathrm{Re}(\mu + M_0) + LM + (\tau + 1)\,L\phi] < \frac{\tau}{\tau + 1}\,. \tag{4.1}$$

(2) *Boltzman Statistics and Point Hard Core.* The method is the same as for Bose statistics. (3.20) should be replaced by:

$$\exp[\beta\,\mathrm{Re}(\mu + M_0) + LM(\tau + 1)\,e^{\tau} + L\phi + \tau\,(1 + \frac{\phi}{c}\,(e^{Lc} - 1))] < \tau$$

where:

$$c = \max_r |\varphi_\parallel(r)|\,. \tag{4.2}$$

In both cases, one can still dispense with the time ordering condition, and the symmetry between occupied and empty sites (expressed by Theorem 4) holds.

(3) *Models Without Hard Core* ($\varphi_\parallel(0) < \infty$). The last two properties are lost in these models, and in particular, the results apply only to Green functions, corresponding to monotonous contours Γ. Moreover, for complex $\varphi_\parallel$, the conditions on the variables ζ which define the analyticity domain are slightly more complicated, and cannot be described in terms of the length L of Γ alone.

Acknowledgements. One of us (J. G.) would like to thank Professor K. SYMANZIK and Professor W. ZIMMERMANN for the hospitality extended to him at the Courant Institute of Mathematical Sciences.

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