Commun. math. Phys. 10, 311—324 (1968)

Analyticity Properties of the Anisotropic Heisenberg Model

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Received June 1, 1968

Abstract. An upper bound $\bar{T}_c$ for the critical temperature of a class of spin systems which includes the Heisenberg ferromagnet is derived. The analyticity of the free energy as a function of the temperature, the external magnetic field, and the interaction potentials, is demonstrated in a domain which includes all temperatures $T > \bar{T}_c$. For the isotropic Heisenberg ferromagnet in ν dimensions we find the poor estimate $2\nu J/k\bar{T}_c = 0.0001$. Some analyticity and cluster properties of the reduced density matrices are also derived.

1. Introduction

It was first shown by RUELLE [1, 2] that, in the low density region, one could derive analyticity and cluster properties for the correlation functions of a classical statistical mechanical system from the Kirkwood-Salzburg integral equations. His method consisted in interpreting the latter equations as integral equations on a suitably chosen Banach space. The method was developed by GINIBRE [3] who obtained similar results for continuous quantum systems and, more recently [4], for certain quantum spin systems or lattice gases. The idea behind GINIBRE's innovation is to reduce the quantum mechanical problem to a problem formally identical to the classical one by the use of Wiener integral techniques. These latter techniques allow one to represent the quantum mechanical reduced density matrices in terms of classical correlation functions over a space of quantum mechanical configurations which physically consist of "clouds" of classical configurations.

In the case of classical lattice systems RUELLE's method was generalized to incorporate many-body forces [5] and improved to give a much larger region of analyticity in terms of the thermodynamical variables [6, 7].

The latter improvement originates from two sources. Firstly one remarks that by taking into account the presence of a hard core condition and the fact that the configurations form a discrete set one may partially invert the Kirkwood-Salzburg type equations and obtain more powerful relations. Secondly, one may use a "hole-particle" or "spin-reversal" symmetry to extend domains of analyticity. It is the purpose of this paper to show that similar improvements can be made to Ginibre's approach to the quantum mechanical case because the classical configurations occur as a discrete subset of the quantum mechanical configurations. Thus we have succeeded in proving for the anisotropic Heisenberg model that, at high temperature, i.e., for $|\mathrm{Re}\beta| < \beta_0$ (β_0 is a fixed number) and $\mathrm{Im}\beta$ sufficiently small, one has analyticity of the free energy in the thermodynamic variables.

2. General Formulation

Let Z^ν be a ν dimensional cubic lattice and suppose that at each point $x \in Z^\nu$ there is a spin $\boldsymbol{\sigma}_x = (\sigma_x^{(1)}, \sigma_x^{(2)}, \sigma_x^{(3)})$. Consider a system confined in a cubic box Λ containing $N(\Lambda)$ points. We assume that the Hamiltonian of this system is given by

$$H_\Lambda = \frac{1}{2} \sum_{\substack{x,y \in \Lambda \\ x \neq y}} [K(x-y)\{\sigma_x^{(1)}\sigma_y^{(1)} + \sigma_x^{(2)}\sigma_y^{(2)}\} + J(x-y)\,\sigma_x^{(3)}\sigma_y^{(3)}] + H \sum_{x \in \Lambda} \sigma_x^{(3)} \tag{1}$$

where the potentials $J(x)$ and $K(x)$ are taken to satisfy

$$\|J\| = \sum_{x \neq 0} |J(x)| < +\infty\,, \quad \|K\| = \sum_{x \neq 0} |K(x)| < +\infty\,.$$

We also introduce J and K through the definitions

$$J = -\sum_{x \neq 0} J(x)\,, \quad K = -\sum_{x \neq 0} K(x)\,.$$

Following Ginibre we introduce annihilation and creation operators (2×2 matrices) a_x, a_x^+ by

$$a_x = \frac{1}{2}(\sigma_x^{(1)} - i\sigma_x^{(2)})\,, \quad a_x^+ = \frac{1}{2}(\sigma_x^{(1)} + i\sigma_x^{(2)})$$

and then note that in terms of these operators H_Λ is equivalent to the Hamiltonian

$$H_\Lambda = -\sum_{\substack{x,y \in \Lambda \\ x \neq y}} K(x-y)\,(a_x^+ - a_y^+)\,(a_x - a_y) + 2 \sum_{\substack{x,y \in \Lambda \\ x \neq y}} J(x-y)\, a_x^+ a_x a_y^+ a_y - \mu \sum_{x \in \Lambda} a_x^+ a_x \tag{2}$$

where

$$\mu = -2H - J + K\,.$$

[Actually (1) and (2) differ by a term $-(H + K/2)\,N(\Lambda)$ and a second surface term Σ_Λ i.e., by a term such that $\|\Sigma_\Lambda\|/N(\Lambda) \to 0$ as $\Lambda \to \infty$, neither of which affect the following discussion of analyticity properties of the free energy.]

The states of the finite system form a vector space $\mathscr{H}_\Lambda$ in which one may introduce a normalized basis labelled by the finite subsets of Λ as follows. The vector $|\emptyset\rangle_\Lambda$ is defined by the condition:

$$a_x|\emptyset\rangle_\Lambda = 0 \quad \text{for all} \quad x \in \Lambda$$

and $|X\rangle_\Lambda$, with $X \subset \Lambda$, is defined by

$$|X\rangle_\Lambda = \prod_{x\in X} a_x^+ \; |\emptyset\rangle_\Lambda .$$

If ψ is a function on the subsets of Λ a general vector $|\psi\rangle_\Lambda \in \mathscr{H}_\Lambda$ is given by

$$|\psi\rangle_\Lambda = \sum_{X\subset\Lambda} \psi(X)\,|X\rangle_\Lambda .$$

Denoting the first term in (2) by T_Λ we see that its action on $\mathscr{H}_\Lambda$ is given by

$$(T_\Lambda\psi)(X) \\ = -\sum_y K(y) \sum_{x\in X} \{\psi((X/x)\cup(x+y)) - 2\,\psi(X) + \psi((X/x)\cup(x-y))\}$$

where the first summation is restricted such that all sets occurring are subsets of Λ. If we denote the second two terms in (2) by U_Λ i.e., $U_\Lambda = H_\Lambda - T_\Lambda$ we find

$$(U_\Lambda\psi)(X) = U(X)\,\psi(X)$$

where

$$U(X) = 2 \sum_{\substack{x,y\in X\\ x\neq y}} J(x-y) - \mu N(X) .$$

Whilst T_Λ is a double difference operator and a natural analogue of the Laplace operator which would occur as the kinetic energy term in the Hamiltonian of a continuous gas, U_Λ is the natural analogue of the potential energy arising from a two-body interaction $4J(x)$ and a chemical potential μ.

The equilibrium statistical mechanics of the system, considered as a lattice gas, is described with the aid of the pressure

$$P_\Lambda(\beta\mu, \beta J, \beta K) = \frac{1}{\beta N(\Lambda)} \log \operatorname{Tr}(e^{-\beta H_\Lambda})$$

and the reduced density matrices

$$\varrho_\Lambda(X, Y) = \frac{1}{\operatorname{Tr}(e^{-\beta H_\Lambda})} \operatorname{Tr}\left(e^{-\beta H_\Lambda} \prod_{x\in X} a_x^+ \prod_{y\in Y} a_y\right) . \tag{3}$$

The thermodynamic predictions of the theory are determined by the limits $P(\beta\mu, \beta J, \beta K)$ and $\varrho(X, Y)$ of these functions as $\Lambda \to \infty$. In the sequel we will derive properties of the pressure P and the reduced density matrices ϱ of the lattice gas. Our results can of course be immediately translated into statements concerning the free energy, etc., of the spin system by a simple change of variables and terminology.

The well-known symmetry of a spin system between spin "up" and spin "down" becomes, in the lattice gas language, a "hole-particle" symmetry and leads to the following result.

Symmetry relation. *The thermodynamic pressure $P(\beta\mu, \beta J, \beta K)$ satisfies the symmetry relation*

$$P(\beta\mu, \beta J, \beta K) = P(-\beta(\mu - 2J + 2K), \beta J, \beta K) + \mu - J + K .$$

3. Reduced Density Matrices

In this Section we establish the Ginibre representation [4] of the reduced density matrices in terms of correlation functions on a space of trajectories.

Consider the space $K^{(j)}$ of functions f from $[0, j]$ to Z^ν. We refer to elements ω of $K^{(j)}$ as trajectories. Each trajectory can be parametrized by giving the starting point $\omega(0) = x$, the number n of jumps of the trajectory, the successive jumps $s_1, s_2, \ldots, s_n$, the "times" $t_0, t_1, \ldots, t_n$ spent in the positions $x, x + s_1, \ldots, x + s_1 + \cdots + s_n$ respectively.

Let us introduce on $K^{(j)}$ a measure $P^j_{xy}(d\omega)$, concentrated on the trajectories ω such that $\omega(0) = x$ and $\omega(j) = y$, through the definition

$$\int P^j_{xy}(d\omega) \cdot = \sum_{n \geqq 0} \sum_{\substack{s_1, \ldots, s_n \\ \sum_i s_i = y - x}} \int_{t_i \geqq 0} dt_0 \ldots dt_n \, \delta\Big(\sum_i t_i - j\Big) e^{-2j\beta K} (-2\beta)^n \prod_{i=1}^n K(s_i) \tag{4}$$

where for $n = 0$ the product occurring on the right-hand side is replaced by unity. We have

$$\begin{aligned} &\sum_Y \int |P^j_{xy}(d\omega)| \leqq e^{2j(\|K\| - K)} \\ &\sum_Y \int P^j_{xy}(d\omega) = 1 \end{aligned} \tag{5}$$

and the equality in (5) is valid if, and only if, $\|K\| = K$ i.e., if $K(x) \leqq 0$.

Let $\mathscr{K}$ and $\tilde{\mathscr{K}}$ be the spaces defined by

$$\mathscr{K} = \bigcup_{\substack{j \geqq 1 \\ j \text{ integer}}} K^{(j)}, \quad \tilde{\mathscr{K}} = \bigcup_{n \geqq 1} \mathscr{K}^{\otimes n} .$$

For $\omega \in K^{(j)}$ we define the length $l(\omega)$ of the trajectory ω by $l(\omega) = j$ and call a trajectory ω simple if $l(\omega) = 1$ and composite if $l(\omega) > 1$.

Each composite trajectory of length l can be regarded as the union of l successive simple trajectories. If ω_1, ω_2, are two simple trajectories we define the mean potential energy between ω_1 and ω_2 by

$$J(\omega_1 - \omega_2) = \int_0^1 dt \, J(\omega_1(t) - \omega_2(t))$$

where we take the convention $J(0) = +\infty$ in the integrand. We define the mean potential between two composite trajectories by

$$J(\omega_1 - \omega_2) = \sum_{\omega_1^s,\, \omega_2^s} J(\omega_1^s - \omega_2^s)$$

where the summation runs over simple trajectories ω_1^s, ω_2^s which occur in the decomposition of ω_1 and ω_2 respectively. Let $\boldsymbol{\omega} = \{\omega_1, \ldots, \omega_n\} \in \tilde{\mathscr{K}}$ with $\omega_i \in \mathscr{K}$ then we define the mean potential energy $U(\boldsymbol{\omega})$ of $\boldsymbol{\omega}$ by

$$U(\boldsymbol{\omega}) = 4 \sum_{i<j} J(\omega_i - \omega_j) - \mu \sum_{i=1}^{n} l(\omega_i)\,.$$

Next we introduce a measure on $\mathscr{K}$ by the definition:

$$d\omega = \sum_{j \geqq 1} \sum_{x \in Z^\nu} \frac{1}{j} P^j_{xx}\,(d\omega) \tag{6}$$

and the corresponding measure $d\boldsymbol{\omega}$ on $\tilde{\mathscr{K}}$ by

$$d\boldsymbol{\omega} = \sum_{n \geqq 0} \frac{1}{n!}\, d\omega_1 \ldots d\omega_n\,. \tag{7}$$

A second useful measure on $\tilde{\mathscr{K}}$ is defined by

$$\tilde{P}_{XY}(d\boldsymbol{\omega}) = \delta_{N(X),N(Y)} \sum_{j_1,\ldots,j_n \geqq 1} \sum_{\mathscr{P}} P^{j_1}_{x_1 \mathscr{P}_1(y)}(d\omega_1) \ldots P^{j_n}_{x_n \mathscr{P}_n(y)}(d\omega_n) \tag{8}$$

where $n = N(X)$ and the second sum runs over all permutations $\mathscr{P}$ of the points of the set Y.

The usefulness of the above definitions is demonstrated by the following representations due to GINIBRE [4]. Define $Z_\Lambda(\beta\mu, \beta J, \beta K)$ by

$$Z_\Lambda(\beta\mu, \beta J, \beta K) = \sum_{X \subset \Lambda} \int' \tilde{P}_{XX}(d\boldsymbol{\omega})\, e^{-\beta U(\boldsymbol{\omega})}$$

where the prime on the integration symbol denotes that the integration is restricted to $\boldsymbol{\omega} = \{\omega_1, \ldots, \omega_n\}$ such that each ω_i is simple and contained in Λ. Ginibre has shown that this function can be written as

$$Z_\Lambda(\beta\mu, \beta J, \beta K) = \mathrm{Tr}\,(e^{-\beta(H_\Lambda + \Sigma_\Lambda)})$$

where Σ_Λ is a surface term [i.e., $\|\Sigma_\Lambda\|/N(\Lambda) \to 0$ as $\Lambda \to \infty$].

If one further defines reduced density matrices using the Hamiltonian $\beta(H_\Lambda + \Sigma_\Lambda)$ one finds [4]

$$\varrho_\Lambda(X, Y) = \frac{1}{Z_\Lambda} \sum_{S \subset \Lambda} \int' \tilde{P}_{X \cup S, Y \cup S}\,(d\boldsymbol{\omega})\, e^{-\beta U(\boldsymbol{\omega})}\,. \tag{9}$$

Introducing functions $\hat{\varrho}_\Lambda$ by

$$\hat{\varrho}_\Lambda(\boldsymbol{\omega}) = \frac{1}{Z_\Lambda} \int_{\boldsymbol{\omega}' \subset \Lambda} d\boldsymbol{\omega}'\, e^{-\beta U(\boldsymbol{\omega} \cup \boldsymbol{\omega}^1)} \tag{10}$$

one has $\hat{\varrho}_\Lambda(\emptyset) = 1$ and after some rearrangement one finds that

$$\varrho_\Lambda(X, Y) = \int' \tilde{P}_{XY}\,(d\boldsymbol{\omega})\, \hat{\varrho}_\Lambda(\boldsymbol{\omega})\,.$$

Note that the $\hat{\varrho}_A$ defined by (10) are identical to classical correlation functions defined for a system whose configuration space is $\tilde{\mathscr{K}}$. Further note that this configuration space has the physical interpretation as a space of "clouds" of classical configurations and the set of classical configurations is contained in $\tilde{\mathscr{K}}$ as a discrete subset namely the subset of families of simple trajectories with no jumps.

4. Integral Equations

We next proceed to derive integral equations for the correlation functions $\hat{\varrho}_A$. Let us order the trajectories of $\boldsymbol{\omega}$ in the lexicographic order of their starting points. If ω_1 is the first simple trajectory of $\boldsymbol{\omega}$ we may then write

$$\begin{aligned} U(\boldsymbol{\omega} \cup \boldsymbol{\omega}') = U(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\omega}') + 4 \sum_{\omega \notin \boldsymbol{\omega}^{(1)}} J(\omega_1 - \omega) \\ + 4 \sum_{\omega' \in \boldsymbol{\omega}'} J(\omega_1 - \omega') - \mu \end{aligned} \tag{11}$$

where $\boldsymbol{\omega}^{(1)}$ is the set of composite trajectories obtained by subtracting ω_1 from $\boldsymbol{\omega}$.

If $J(\omega_1 - \omega_2) = +\infty$ we will say that the trajectories ω_1 and ω_2 overlap and we will write $\omega_1 \cap \omega_2 \neq \emptyset$; if $J(\omega_1 - \omega_2) < +\infty$ we write $\omega_1 \cap \omega_2 = \emptyset$. If $J(\omega) = +\infty$, namely if ω overlaps the empty trajectory, we say that ω is self-overlapping.

Now, using (10) and (11), we find

$$\hat{\varrho}_A(\boldsymbol{\omega}) = \frac{z\, e^{-\beta U^1(\boldsymbol{\omega})}}{Z_A} \int_{\boldsymbol{\eta} \subset A} d\boldsymbol{\eta}\, K^1(\boldsymbol{\eta}) \int_{\substack{\omega_1 \cap \boldsymbol{\omega}' = \emptyset \\ \boldsymbol{\omega}' \subset A}} d\boldsymbol{\omega}'\, e^{-\beta U(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}')} \quad \text{for} \quad \boldsymbol{\omega} \subset A \tag{12}$$

where

$$U^1(\boldsymbol{\omega}) = 4 \sum_{\omega \in \boldsymbol{\omega}^{(1)}} J(\omega_1 - \omega)\,, \qquad z = e^{\beta\mu}$$

and $K^1(\boldsymbol{\eta})$ is defined as follows

$$\begin{aligned} &K^1(\emptyset) = 1 \\ &K^1(\boldsymbol{\eta}) = 0 && \text{if} \quad \omega_1 \cap \boldsymbol{\eta} \neq \emptyset \\ &K^1(\boldsymbol{\eta}) = \prod_{\eta \in \boldsymbol{\eta}} (e^{-4J(\omega_1 - \eta)} - 1) && \text{if} \quad \omega_1 \cap \boldsymbol{\eta} = \emptyset \quad \text{and} \quad \boldsymbol{\eta} \neq \emptyset\,. \end{aligned}$$

Next, to obtain integral equations we use the following identity

$$\begin{aligned} &\int_{\substack{\omega_1 \cap \boldsymbol{\omega}' = \emptyset \\ \boldsymbol{\omega}' \subset A}} d\boldsymbol{\omega}'\, e^{-\beta U(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}')} \\ &= \int_{\boldsymbol{\omega}' \subset A} d\boldsymbol{\omega}'\, e^{-\beta U(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}')} \\ &\quad + \int_{\substack{\text{all} \\ \omega_1 \cap \omega'' \neq \emptyset \\ \boldsymbol{\omega}'' \subset A}} d\boldsymbol{\omega}''(-1)^{N(\boldsymbol{\omega}'')} \int_{\boldsymbol{\omega}' \subset A} d\boldsymbol{\omega}'\, e^{-\beta U(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}' \cup \boldsymbol{\omega}'')} \end{aligned} \tag{13}$$

where $N(\boldsymbol{\omega}'')$ is the number of composite trajectories of $\boldsymbol{\omega}''$ and $\omega_1 \cap \boldsymbol{\omega}'' \overset{\text{all}}{\neq} \emptyset$ means that each of the composite trajectories of $\boldsymbol{\omega}''$ must overlap ω_1.

Combining (12) and (13) we find

$$\begin{aligned}\hat{\varrho}_\Lambda(\boldsymbol{\omega}) = z e^{-\beta U^1(\boldsymbol{\omega})} \Big\{ \hat{\varrho}_\Lambda(\boldsymbol{\omega}^{(1)}) + \int\limits_{\substack{\omega_1 \cap \boldsymbol{\omega}' \overset{\text{all}}{\neq} \emptyset \\ \boldsymbol{\omega}' \subset \Lambda}} d\boldsymbol{\omega}' (-1)^{N(\boldsymbol{\omega}')} \hat{\varrho}_\Lambda(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\omega}') \\ + \int\limits_{\substack{\boldsymbol{\eta} \neq \emptyset \\ \boldsymbol{\eta} \subset \Lambda}} d\boldsymbol{\eta}\, K^1(\boldsymbol{\eta}) \Big[\hat{\varrho}_\Lambda(\boldsymbol{\eta} \cup \boldsymbol{\omega}^{(1)}) \\ + \int\limits_{\substack{\omega_1 \cap \boldsymbol{\omega} \overset{\text{all}}{\neq} \emptyset \\ \boldsymbol{\omega}' \subset \Lambda}} d\boldsymbol{\omega}' (-1)^{N(\boldsymbol{\omega}^1)} \hat{\varrho}_\Lambda(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}') \Big] \Big\} .\end{aligned} \tag{14}$$

These integral equations are basic to the rest of our analysis and differ from those used by GINIBRE insofar we have explicitly taken into account the fact that $\hat{\varrho}_\Lambda(\boldsymbol{\omega}) = 0$ if the trajectories contained in $\boldsymbol{\omega}$ overlap. This distinction gives rise to the second and fourth term in (14); in a classical lattice gas similar terms arise when one takes the hard core conditions explicitly into account [6, 7].

To use the above integral equations we follow the method developed by RUELLE and GINIBRE. Introduce a Banach space $\mathscr{E}_{\sigma\tau}$ of functions φ over the non-empty sets of (composite) trajectories $\boldsymbol{\omega}$ vanishing on overlapping trajectories with the norm

$$\|\varphi\|_{\sigma\tau} = \sup_{\boldsymbol{\omega}} \frac{|\varphi(\boldsymbol{\omega})|}{\sigma^{l(\boldsymbol{\omega})}\, \tau^{n(\boldsymbol{\omega})}}, \qquad 0 < \sigma < 1, \qquad 1 < \tau .$$

where $l(\boldsymbol{\omega})$ and $n(\boldsymbol{\omega})$ denote the number of simple trajectories and the number of jumps in the set $\boldsymbol{\omega}$ respectively. Clearly, if $\boldsymbol{\omega} = \boldsymbol{\omega}' \cup \boldsymbol{\omega}''$ then $l(\boldsymbol{\omega}) = l(\boldsymbol{\omega}') + l(\boldsymbol{\omega}'')$ and $n(\boldsymbol{\omega}) = n(\boldsymbol{\omega}') + n(\boldsymbol{\omega}'')$.

Define the following operators on $\mathscr{E}_{\sigma\tau}$:

$$(e^{-\beta U^1}\varphi)(\boldsymbol{\omega}) = e^{-\beta U^1(\boldsymbol{\omega})}\varphi(\boldsymbol{\omega})$$

$$\begin{aligned}(\chi_\Lambda \varphi)(\boldsymbol{\omega}) &= \varphi(\boldsymbol{\omega}) \qquad \text{if} \quad \boldsymbol{\omega} \subset \Lambda \\ &= 0 \qquad\quad\ \text{if} \quad \boldsymbol{\omega} \not\subset \Lambda\end{aligned}$$

$$(E\varphi)(\boldsymbol{\omega}) = \int\limits_{\omega_1 \cap \boldsymbol{\omega}' \overset{\text{all}}{\neq} \emptyset} d\boldsymbol{\omega}' (-1)^{N(\boldsymbol{\omega}')} \varphi(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\omega}')$$

$$\begin{aligned}(F\varphi)(\boldsymbol{\omega}) &= \varphi(\boldsymbol{\omega}^{(1)}) \quad \text{if} \quad \boldsymbol{\omega}^{(1)} \neq \emptyset \\ &= 0 \qquad\quad\ \ \text{if} \quad \boldsymbol{\omega}^{(1)} = \emptyset\end{aligned}$$

and

$$(G\varphi)(\boldsymbol{\omega}) = \int_{\eta \neq \emptyset} d\boldsymbol{\eta}\, K^1(\boldsymbol{\eta})\, [\varphi(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta})$$

$$+ \int_{\substack{\text{all} \\ \omega_1 \cap \omega' \neq \emptyset}} d\boldsymbol{\omega}' (-1)^{N(\omega')}\, \varphi(\boldsymbol{\omega}^{(1)} \cup \boldsymbol{\eta} \cup \boldsymbol{\omega}')] .$$

Further we introduce the vector $\alpha \in \mathscr{E}_{\sigma\tau}$ by

$$\alpha(\boldsymbol{\omega}) = 1 \quad \text{if} \quad \boldsymbol{\omega}^{(1)} \neq \emptyset$$

$$= 0 \quad \text{if} \quad \boldsymbol{\omega}^{(1)} = \emptyset .$$

These definitions allow us to write (14) as an integral equation on the space $\mathscr{E}_{\sigma\tau}$ in the form

$$\hat{\varrho}_\Lambda = z\chi_\Lambda \alpha + z\chi_\Lambda e^{-\beta U^1} [-E + F + G]\, \hat{\varrho}_\Lambda . \tag{15}$$

Next, we wish to rewrite these equations by introducing a decomposition of E into two parts E_1 and E_2; the decomposition is chosen such that $(1 + ze^{-U^1} E_1)$ is invertible and E_2 is small for small values of β. Explicitly we have $E = E_1 + E_2$ where

$$(E_1 \varphi)(\boldsymbol{\omega}) = e^{-2\beta K}\, \delta_{n(\omega_1),0}\, \varphi(\boldsymbol{\omega}) .$$

A rearrangement of (15) then yields

$$\left.\begin{aligned} \hat{\varrho}_\Lambda &= z\chi_\Lambda \frac{1}{1 + ze^{-\beta U^1} E_1}\, \alpha + \chi_\Lambda \mathscr{H} \hat{\varrho}_\Lambda \\ \text{where} \qquad \mathscr{H} &= \frac{ze^{-\beta U^1}}{1 + ze^{-\beta U^1} E_1} (-E_2 + F + G) \end{aligned}\right\} \tag{16}$$

In order to apply to these integral equations the methods of Ruelle it is necessary to find the region of z and β for which the norm of $\mathscr{H}$ in $\mathscr{E}_{\sigma\tau}$ is less than unity. In this region one can then demonstrate that as $\Lambda \to \infty$ lim $\hat{\varrho}_\Lambda(\boldsymbol{\omega})$ exists uniformly for $\boldsymbol{\omega}$ contained in any bounded region and also that $\|\hat{\varrho}_\Lambda\|_{\sigma\tau}$ is uniformly bounded.

In fact as $\|\alpha\|_{\sigma\tau} = 1/\sigma$ it follows from (16) that if $\|\mathscr{H}\|_{\sigma\tau} < 1$ then

$$\|\hat{\varrho}_\Lambda\|_{\sigma\tau} \leqq \frac{|z|}{\sigma(1 - \|\mathscr{H}\|_{\sigma\tau})} . \tag{17}$$

Estimation of the norms of the various integral operators introduced above can be made in a straightforward, but nevertheless complicated, manner using the explicit form of the measure $d\boldsymbol{\omega}$ in terms of the parametrization of the trajectories, i.e., Eqs. (4), (6) and (7) are used (see also GINIBRE [4]). For example one finds

$$\begin{aligned}\|E\|_{\sigma\tau} &\leqq \sup_{\omega_1} \frac{1}{\sigma\,\tau^{n(\omega_1)}} \exp\left\{\int_{\omega\cap\omega_1\neq\emptyset} |d\omega|\,\sigma^{l(\omega)}\,\tau^{n(\omega)}\right\}\\ &\leqq \sup_{\omega_1} \frac{1}{\sigma\,\tau^{n(\omega_1)}} \exp\left\{(n(\omega_1)+1)\sum_{j=1}^{\infty}\frac{1}{j}\,|P_{00}^{\,j}(d\omega)|\,\sigma^j z^{n(\omega)}(n(\omega)+1)\right\}\\ &\leqq \frac{e^\lambda}{\sigma}\sup_{\omega_1}\left(\frac{e^\lambda}{\tau}\right)^{n(\omega_1)} \qquad (18)\\ &\leqq \frac{e^\lambda}{\sigma}\end{aligned}$$

where we have introduced

$$\lambda = -\log\left(1-\sigma e^{2\beta(\tau\|K\|-K)}\right) + \frac{2\,\beta\tau\,\|K\|\,\sigma\,e^{2\beta(\tau\|K\|-K)}}{1-\sigma\,e^{2\beta(\tau\|K\|-K)}}$$

and the third step in (18) is valid if $\sigma e^{2\beta(\tau\|K\|-K)} < 1$, the fourth if $e^\lambda < \tau$.

One finds the following results:

$$\|\chi_\Lambda\|_{\sigma\tau} = 1$$

$$\left\|\frac{z\,e^{-\beta U^1}}{1+z\,e^{-\beta U^1}E_1}\right\|_{\sigma\tau} \leqq |z|\,e^{4\beta\|J\|}$$

and

$$\|E_2\|_{\sigma\tau} \leqq f_E(\sigma,\tau)\,,\quad \left\|\frac{z\,e^{-\beta U^1}}{1+z\,e^{-\beta U^1}E_1}F\right\|_{\sigma\tau} \leqq f_F(\sigma,\tau)\,,\quad \|G\|_{\sigma\tau} \leqq f_G(\sigma,\tau)$$

where

$$f_E(\sigma,\tau) = \frac{(1+\sigma\,e^{-2\beta K})^2}{\sigma\tau} + \begin{cases}\dfrac{\tau}{\sigma}\,\dfrac{\lambda-\lambda_0}{(\log\tau-\lambda)\,e} & \text{if } 0<\log\tau-\lambda\leqq 1\\[2mm] \dfrac{e^\lambda}{\sigma}(\lambda-\lambda_0) & \text{if } 1\leqq\log\tau-\lambda\,,\end{cases}$$

$$f_F(\sigma,\tau) = \mathrm{Max}\left(\frac{1}{\sigma}\sup_{\omega}\left|\frac{z\,e^{-\beta U^1(\omega)}}{1+z\,e^{-2\beta K}\,e^{-\beta U^1(\omega)}}\right|,\ \frac{|z|}{\sigma\tau}\,e^{4\beta\|J\|}\right)$$

$$f_G(\sigma,\tau) = \frac{1}{\sigma}\left(1+\frac{e^\lambda}{\sigma}\right)\left[\exp\left\{(e^{4\beta\|J\|}-1)\frac{\sigma\,e^{2\beta(\tau\|K\|-K)}}{1-\sigma\,e^{2\beta(\tau\|K\|-K)}}\right\}-1\right]$$

where

$$\lambda_0 = \lambda|_{\beta\tau=0} = -\log(1-\sigma\,e^{-2\beta K})$$

and these estimates are valid under the following conditions

$$e^\lambda < \tau\,,\quad \sigma\,e^{2\beta(\tau\|K\|-K)} < 1\,,\quad \text{and}\quad 1+\sigma\,e^{-2\beta K} < \tau\,. \qquad (19)$$

Thus under these conditions

$$\|\mathscr{H}\|_{\sigma\tau} \leqq f_{\mathscr{H}}(\sigma,\tau)$$

where

$$f_{\mathscr{H}}(\sigma,\tau) = |z|\,e^{4\beta\|J\|}\,\{f_E(\sigma,\tau)+f_G(\sigma,\tau)\} + f_F(\sigma,\tau)\,. \qquad (20)$$

5. Analyticity and Cluster Properties

Let us define $\mathscr{D}$ to be the set of values of the activity z, the inverse temperature β, and the potentials $J(x)$, $K(x)$ for which there exist σ, τ, satisfying the inequalities (19) and such that

$$f_{\mathscr{H}}(\sigma, \tau) < 1 .$$

Theorem 1. *If for $\beta \geqq 0$, $z \geqq 0$ either*

$$(z, \beta, J(\cdot), K(\cdot)) \in \mathscr{D}$$

or

$$\left(\frac{1}{z} e^{-2\beta(J+K)}, \beta, J(\cdot), K(\cdot)\right) \in \mathscr{D}$$

then the thermodynamic pressure is an analytic function of z, β, $J(\cdot)$, $K(\cdot)$. In particular there exists a $B_0 > 0$ such that the pressure is an analytic function of z, β, $J(\cdot)$, $K(\cdot)$ for

$$z \geqq 0 , \quad \beta \|J\| < B_0 , \quad \text{and} \quad \beta \|K\| < B_0 .$$

Proof. The first statement of the theorem follows from the same arguments used by Ginibre; the second statement follows from an

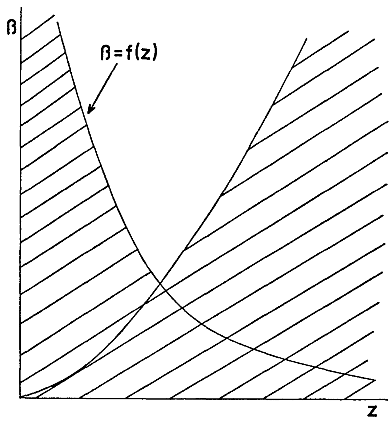

Fig. 1. The analyticity domain of the pressure obtained in Theorem 1 for fixed $J(.)$ and $K(.)$ is indicated by the shaded region

examination of the set $\mathscr{D}$. Let us study the form of $\mathscr{D}$ for $J(\cdot)$, $K(\cdot)$ fixed. Let σ be a function of $|z|$ such that $\sigma < 1$ and

$$\frac{1}{\sigma} \sup_{\omega} \left| \frac{z\, e^{-\beta U^1(\omega)}}{1 + z\, e^{-2\beta K}\, e^{-\beta U^1(\omega)}} \right| < 1$$

and let τ be a function of β such that as $\beta \to 0$ we have $\tau(\beta) \to \infty$ but $\beta\tau(\beta) \to 0$; in this limit $\lambda \to \lambda_0$, $f_E \to 0$ and $f_G \to 0$. It is immediately clear from (20) that given z one can find a positive function $f(z)$ such that if $\beta < f(|z|)$ one has $f_{\mathscr{H}}(\sigma(z), \tau(\beta)) < 1$. The curve $f(|z|)$ can be chosen decreasing. Using the symmetry relation of Section 2 we find the second condition of the theorem and this takes the form

$$|z|^{-1} \exp(-2\beta(J + K)) < f^{-1}(\beta) .$$

The nature of the curves are shown in Fig. 1. The shaded region in this figure depicts the values of β and z for which analyticity has been derived. (In the above, analyticity is understood in the same sense as in GINIBRE's paper, namely there exists a complex neighbourhood of $\mathscr{D}$ where one has analyticity in the usual sense.)

Theorem 2. *For $(z, \beta, J(\cdot), K(\cdot)) \in \mathscr{D}$ the infinite volume reduced density matrices exist, are analytic functions of $z, \beta, J(\cdot), K(\cdot)$, and satisfy the cluster property*

$$\lim_{\substack{d(X_1 \cup Y_1, X_2 \cup Y_2) \to \infty \\ N(X_1) + N(X_2)\ \text{fixed}}} \{\varrho(X_1 \cup X_2, Y_1 \cup Y_2) - \varrho(X_1, Y_1)\, \varrho(X_2, Y_2)\} = 0 ,$$

where $d(X, Y)$ denotes the distance between the sets X and Y.

Proof. The analyticity of the reduced density matrices in the cited region was implicitly used in the proof of Theorem 1 and is a consequence of the analysis of the integral equations. To prove the cluster property we proceed as follows. Define χ by

$$\chi(X_1, Y_1 | X_2, Y_2) = \varrho(X_1 \cup X_2, Y_1 \cup Y_2) - \varrho(X_1, Y_1)\, \varrho(X_2, Y_2) .$$

From the bound (17) and the definition (8) we find

$$|\varrho(X, Y)| \leqq N(X)! \left(\frac{\sigma\, e^{2\beta(\tau \|K\| - K)}}{1 - \sigma\, e^{2\beta(\tau\|K\| - K)}}\right)^{N(X)} \frac{|z|}{\sigma(1 - \|\mathscr{H}\|_{\sigma\tau})} .$$

[With a suitable introduction of moduli signs this bound and those of the previous Sections hold for $(z, \beta, J(\cdot), K(\cdot))$ in a small complex neighbourhood of $\mathscr{D}$.] Now given $(z, \beta, J(\cdot), K(\cdot)) \in \mathscr{D}$ one sees that there exists a small complex simply connected neighbourhood of the interval $(0, z)$ in which χ is uniformly bounded and z analytic. GINIBRE has however proved that χ has a power series expansion of the form

$$\chi(X_1, Y_1 | X_2, Y_2) = z^{N(X_1) + N(X_2)} \sum_{l \geqq 0} z^l C_l(X_1, Y_1 | X_2, Y_2)$$

whose radius of convergence is independent of $X_1 Y_1, X_2 Y_2$. Further he demonstrated that

$$\lim_{d(X_1 \cup Y_1; X_1 \cup Y_2) \to \infty} \chi(X_1, Y_1 | X_2, Y_2) = 0$$

within the circle of convergence of this power series. This implies that

$$\lim_{d(X_1 \cup X_2; Y_1 \cup Y_2) \to \infty} C_l(X_1, Y_1 | X_2, Y_2) = 0 .$$

We are now in a situation analogous to that of [8] and the proof proceeds as in that reference. (See also [9].)

Note that the spin reversal symmetry relation that we have used to extend the analyticity region of the pressure is quite general but our inability to handle surface effects in the reduced density matrices makes it impossible to extend the analyticity properties of these matrices by symmetry arguments.

Although the cluster property we have derived is much weaker than that of GINIBRE it is valid in a larger domain. For fixed $J(\cdot)$, $K(\cdot)$ GINIBRE's domain of analyticity in β, z and the present domain are indicated in Fig. 2, in the hatched and shaded regions respectively.

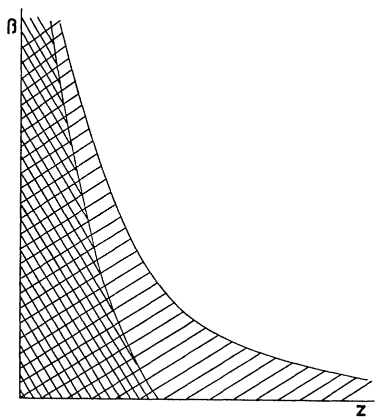

Fig. 2. The domains of analyticity of the reduced density matrices obtained by GINIBRE and by the present authors are indicated by the hatched and shaded regions respectively

In the algebraic approach [10, 11] the reduced density matrices determine a Z^ν invariant state ϱ over the C^* algebra $\mathfrak{A}$ generated by the creation and annihilation operators.

Corollary. *Under the conditions of Theorem 2 ϱ is extremal Z^ν invariant and is a E_I state in the sense of* [12].

6. Comments

In Theorem 1 we have demonstrated that the free energy is an analytic function at high temperatures for the model under consideration. One model of particular interest which is contained in our class is the isotropic Heisenberg ferromagnet. This latter model is defined by setting $J(x) = K(x) \leqq 0$ and in this case a numerical calculation[1] yields analyticity if

$$\beta \|J\| \leqq 0.0001 .$$

[1] We are indebted to W. KLEIN for carrying out this calculation.

Thus in ν dimensions with nearest neighbour interactions of strength J we have an upper bound $\overline{T}_c$ for the critical temperature T_c given by

$$\frac{2\nu J}{k\overline{T}_c} = 0.0001\ .$$

Previously [7] we considered the Ising model, defined by setting $K(\cdot) = 0$, and obtained in the nearest neighbour case [2]

$$\frac{2\nu J}{k\overline{T}_c} = 0.10$$

which may be compared with the value given by numerical calculations [13] in three dimensions $6J/kT_c = 1.3$. One immediately realizes that the complications of the quantum formalism greatly affect the value of the bound $\overline{T}_c$; the above estimates are however independent of the nature of the lattice provided it is a Bravais lattice and 2ν is replaced by q, the number of nearest neighbours.

We conclude with several remarks. Firstly it should be noted that the reduced density matrices for which we have derived results are those defined by (9) as $\Lambda \to \infty$ and not the original ones introduced through (3) as $\Lambda \to \infty$. The difference between these two sets of matrices is due to the presence of the surface term Σ_Λ in (9). The matrices (3) are defined using hard wall boundary conditions and the introduction of Σ_Λ corresponds to a change of boundary conditions. Whilst the free energy can be proved to be independent of boundary conditions no proof exists that the same property holds for the reduced density matrices. In the classical case this independence can be proved but the proof requires the introduction of a space of many-body potentials and the derivation of analyticity of the free energy considered as a functional of these interactions.

To carry through a similar proof in the present setting one would have to consider more general Hamiltonians than those we have discussed, namely one would have to handle Hamiltonians of the form

$$H_\Lambda = \sum_{\substack{X, Y \subset \Lambda \\ N(X) = N(Y)}} J(X;\, Y) \prod_{x \in X} a_x^+ \prod_{y \in Y} a_y$$

where $J(X;\, Y)$ is a translationally invariant function over pairs of finite sets of Z^ν. Although we have not discovered a technique for handling such general Hamiltonians the methods of this paper can be apparently generalized to cover the following case

$$H_\Lambda = \sum_{x, y \in \Lambda} K(x-y)\, a_x^+ a_y + \sum_{X \subset \Lambda} J(X) \prod_{x \in X} a_x^+ a_x$$

where we assume $\sum_{x \neq 0} |K(x)| < +\infty$, $\sum_{0 \in X} |J(X)| < +\infty$. This should follow from the combination of the present techniques and those of [5, 7].

[2] Due to a misprint this number was given as 0.4 in [7].

Acknowledgements. The authors are grateful to Professor L. MOTCHANE for his kind hospitality at the I.H.E.S. They are also indebted to Dr. J. GINIBRE for a pre-publication copy of his work.

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