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On the Converse of the Reeh-Schlieder Theorem

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Abstract. It will be shown that the weak additivity-property is not only sufficient but also necessary for the derivation of the Reeh-Schlieder theorem.

I. Introduction and Results

Many results in quantum field theory which have been derived so far are based directly or indirectly on the Reeh-Schlieder theorem [1]. The central role of this theorem makes it worthwhile to give it separate consideration.

The problem can be formulated as follows. Let $\mathscr{A}$ be a v. Neumann-algebra acting in a Hilbert space $\mathscr{H}$ and $\mathscr{B} \subset \mathscr{A}$ a sub v. Neumann-algebra. Do there exists vectors $x \in \mathscr{H}$ such that $\overline{\mathscr{B} x} = \overline{\mathscr{A} x}$? If yes, how can one characterize such vectors? This problem will hardly be soluable in full generality but there exists a class of v. Neumann algebras for which the answer is partly known. This class is of particular interest for physics.

Here we have a v. Neumann algebra $\mathscr{A}$ together with a n-parametric group G of normal automorphisms which are implemented by a strongly continuous unitary representation of G having its spectrum in a proper closed cone.

We consider G as the additive group of $\boldsymbol{R}^n$ and denote by $U(g)$ its representation in $\mathscr{H}$. A vector $x \in \mathscr{H}$ is called analytic for if $U(g)x$ has an analytic extension into a full neighbourhood of the origin in $\boldsymbol{C}^n$.

Let $\mathscr{N}$ be any open neighbourhood of the origin in $\boldsymbol{R}^n$ and $\mathscr{B}$ a subalgebra of $\mathscr{A}$ then we denote by $(\mathscr{B}, \mathscr{N})$ the v. Neumann algebra generated by $\{U(g)\, \mathscr{B}\, U(g^{-1});\, g \in \mathscr{N}\}$.

With these notations we get:

1. Theorem (Reeh-Schlieder). *With $\mathscr{A}$, G and $U(g)$, $g \in G$ as described above, assume $\mathscr{B} \subset \mathscr{A}$ is a sub v. Neumann algebra of $\mathscr{A}$ such that*

$$(\mathscr{B}, G) = \Big\{\bigcup_{g \in G} U(g)\, \mathscr{B}\, U(g^{-1})\Big\}'' = \mathscr{A}\,.$$

Then for any vector $x \in \mathscr{H}$ which is analytic for $U(g)$, and for any open neighbourhood $\mathscr{N}$ of the origin in G, we have the relation

$$\overline{(\mathscr{B}, \mathscr{N})x} = \overline{\mathscr{A} x}\,.$$

A special case of this result was discovered by REEH and SCHLIEDER [1]. This general version can be found in [2].

2. Remarks. The known examples of quantum field theory give nontrivial examples for the Reeh-Schlieder theorem such that the rings $(\mathscr{B}, \mathscr{N})$ are proper subrings of $\mathscr{A}$. But it is essential for the existence of these examples that the spectrum of $U(g)$, the automorphism-group, is unbounded. In the case where the spectrum is bounded we get:

2.1 Corollary. Preserving the notations and assumptions of theorem 1., assume in addition that the spectrum of $U(g)$ is bounded. Then

$$(\mathscr{B}, \mathscr{N}) = \mathscr{A} .$$

This follows easily from theorem 1 since every vector $x \in \mathscr{H}$ is now analytic for $U(g)$.

A simple reformulation of this last corollary yields an observation due to R. KADISON [3] namely that the spectrum of the translation group in local quantum field theory must be unbounded.

2.2 Corollary. Preserve the assumptions and nations of theorem 1. If for some $\mathscr{N}$ the v. Neumann algebra $(\mathscr{B}, \mathscr{N})$ is a proper subalgebra of $\mathscr{A}$ it follows that the spectrum of $U(g)$ is unbounded.

We are now coming to the converse of the Reeh-Schlieder theorem:

3. Theorem. *Let $\mathscr{A}$, G and $U(g)$ $g \in G$ be defined as before. Assume $\mathscr{B} \subset \mathscr{A}$ is a sub v. Neumann algebra of $\mathscr{A}$. If for any vector $x \in \mathscr{H}$ which is analytic for $U(g)$ the relation*

$$\overline{\mathscr{B} x} = \overline{\mathscr{A} x} \quad \textit{holds} ,$$

then we have

$$(\mathscr{B}, G) = \mathscr{A} .$$

The statement of this theorem deserves the following clarifying.

4. Remark.[1] Assume $U(g)$ and $V(g)$ are each continuous representations of G with spectrum in a proper cone, and that they implement the same automorphism group of $\mathscr{A}$. Then $\overline{\mathscr{B} x} = \overline{\mathscr{A} x}$ for each analytic vector of $U(g)$ if and only if $\overline{\mathscr{B} x} = \overline{\mathscr{A} x}$ for each analytic vector of $V(g)$.

This is a pleasant fact which serves to make the conditions of theorem 3. independent of the group representation, and simplifies its proof by allowing use of the most convenient representation.

5. Example. Theorem 1. and theorem 3. are not exactly the converse of each other. To show that the strict converse to theorem 3. is not possible we will construct an example in which $(\mathscr{B}, G) = \mathscr{A}$ but for which there exists a vector x analytic for $U(g)$ with

$$\overline{\mathscr{B} x} \neq \overline{\mathscr{A} x} .$$

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[1] I am owing this remark to J. IRWIN, private communication.

Let l_2 be the Hilbert space of square summable sequences $\{q_n\}_{n=0}^{+\infty}$. Define the one parameter unitary $U(\lambda)$ by

$$U(\lambda)\{a_n\} = \{e^{i\lambda n} a_n\}\ .$$

Then $U(\lambda)$ has semibounded spectrum. Let x_0 be the vector $\{e^{-n}\}$. It is analytic for $U(\lambda)$. Let E_{x_0} be the projection onto the one dimensional subspace $\{\mu x_0\}$ and $\mathscr{B}$ the algebra

$$\mathscr{B} = \{\mu_1 1 + \mu_2 E_{x_0}\}\ , \qquad (\mu, \mu_1, \mu_2 \in \boldsymbol{C})\ .$$

It is easy to see, that $(\mathscr{B}, G) = \mathscr{L}(\mathscr{H})$ are all bounded operators. But we have

$$\overline{\mathscr{B} x_0} = E_{x_0} \mathscr{H} \quad \text{and} \quad \overline{\mathscr{A} x_0} = \mathscr{H}$$

and these two spaces are unequal by construction.

Finally we will prove the following results which are closely related to the Reeh-Schlieder property. ($\overline{\mathscr{B} x} = \overline{\mathscr{A} x}$ for analytic vectors.) Part (a) is a generalization of theorem 3., part (b) is another consequence of this property.

6. Theorem. Let $\mathscr{A}$, G, $U(g)$ and $\mathscr{B}$ be as in theorem 3. Let $E \subset \mathscr{A}$ be a projection commuting with all $U(g)$. Denote by $\{\mathscr{B}, E\}$ the v. Neumann algebra generated by $\mathscr{B}$ and E and by $U(g)_E$ the restriction of $U(g)$ to $E\mathscr{H}$ then we have:

a) $(\{\mathscr{B}, E\}_E, G) = \mathscr{A}_E$.

b) If the central carrier of E is equal to 1 then $\{\mathscr{B} \cup E \mathscr{A} E\}'' = \mathscr{A}$.

Special cases of this result have been used in field theory, namely, when $U(g)$ has invariant vectors and E is the projection onto the subspace spanned by these vectors [1, 4].

II. Proofs

Since our notation here is slightly more general then that of ref. [2] we will also include a

Proof of Theorem 1. Let $x \in \mathscr{H}$ be analytic for $U(g)$ then $U(g)x$ is analytic in some complex neighbourhood $\mathscr{M}$ of the origin. Since now G is an abelian group. $U(g + g_1)x$ is also analytic in the same neighbourhood $\mathscr{M}$ for arbitrary $g_1 \in G$. Hence $U(g)\,x$ is analytic in a tube $\{g; \mathscr{I}_m\, g \in \mathscr{M}'\}$ where $\mathscr{M}'$ is some convex neighbourhood of the origin. On the other hand, since the spectrum $U(g)$ is contained in some proper convex cone $\mathscr{C}$, it follows that $U(g)$ is the boundary value (in the strong operator topology) of an analytic operator-valued function, holomorphic in the tube $\{g; \mathscr{I}_m\, g \in \mathscr{C}'\}$ where $\mathscr{C}'$ denotes the dual-cone of $\mathscr{C}$. Suppose y is orthogonal to $(\mathscr{B}, \mathscr{N})\,x$. Taking arbitrary $B_i \in \mathscr{B}$, $i = 1, 2, \ldots, r$ and writing $B(g) = U(g)\,B\,U(g^{-1})$ this means

$$(y, B_1(g_1)\,B_2(g_2)\ldots B_r(g_r)\,x) = 0 \quad \text{for} \quad g_i \in \mathscr{N}\ , \quad i = 1, 2, \ldots, r \quad (*)$$

On the other hand

$$(y, B_1(g_1) \dots B_r(g_r)\, x) = (y, U(g_1)\, B_1 U(g_2 - g_1) \dots B_r U(-g_r)\, x)$$

is the boundary value of an analytic function holomorphic in the tube,

$$\{g_1, \dots, g_r; \mathscr{I}_m\, g_1 \in \mathscr{C}', \mathscr{I}_m(g_2 - g_1) \in \mathscr{C}', \dots, \mathscr{I}_m(g_r - g_{r-1}) \in \mathscr{C}'$$
$$\mathscr{I}_m(-g_r) \in \mathscr{M}' + \mathscr{C}'\}\,.$$

Since $\mathscr{M}'$ is an open neighbourhood of the origin the tube is not empty and, (*) must vanish identically for all $g_i \in G$ (by analytic continuation). This implies y is orthogonal to $\overline{(\mathscr{B}, G)\, x} = \overline{\mathscr{A} x}$ which proves the theorem. Now the

Proof of corollary 2.1. Since $U(g)$ has bounded spectrum, every vector $x \in \mathscr{H}$ is analytic for $U(g)$. Hence for every $x \in \mathscr{H}$ the projector onto $\overline{(\mathscr{B}, \mathscr{N})\, x}$ is an element of $\mathscr{A}'$. Since these projectors generate $(\mathscr{B}, \mathscr{N})'$ follows $(\mathscr{B}, \mathscr{N})' = \mathscr{A}'$ or $(\mathscr{B}, \mathscr{N}) = \mathscr{A}$.

Proof of Remark 4. We first prove this remark assuming the representations W and V commute with one another. Then $W(g) = U(g)\, V(g^{-1})$ is also a continuous representation of G. Assume $\overline{\mathscr{B} x} = \overline{\mathscr{A} x}$ for all x analytic for U, and consider a vector, y, analytic for V. For all projectors E, associated to bounded subsets of the spectrum of $W(g)$ one has Ey is analytic for U. (This is easily seen by writing

$$U(g)\, Ey = W(g)\, E\, V(g)\, y$$

and noting that $W(g)\, E$ is an analytic operator valued function of g.) Hence it follows that $\overline{\mathscr{B} E y} = \overline{\mathscr{A} E y}$ and since the $W(g) \in \mathscr{A}'$, $E\, \overline{\mathscr{B} y} = E\, \overline{\mathscr{A} y}$. Now there are such projectors E in each strong neighbourhood of 1, hence $\overline{\mathscr{B} y} = \overline{\mathscr{A} y}$.

According to the results of [5], we may write $U(g) = U_1(g)\, U_2(g)$ and $V(g) = V_1(g)\, V_2(g)$ with $U_1(g), V_1(g) \in \mathscr{A}$ and $U_2(g), V_2(g) \in \mathscr{A}'$. Since V and V_1, V_1 and U_1, and U_1 and U commute with one another, remark 4 is established by the considerations of the proceeding paragraph.

Proof of Theorem 3. Since the spectrum of $U(g)$ is contained in some proper closed cone we have, according to [5], that for any v. Neumann algebra $\mathscr{R}$ which is invariant under $U(g)$, i.e. $U(g)\, \mathscr{R}\, U(g^{-1}) = \mathscr{R}\ \forall\, g \in G$, $U(g)$ defines an inner automorphism of $\mathscr{R}$. In particular we can assume $U(g) \in \mathscr{A}$. Now $(\mathscr{B}, G)$ is also an invariant algebra. Hence we can write $U(g) = U_1(g) \cdot U_2(g)$ with $U_1(g) \in (\mathscr{B}, G)$ and $U_2(g) \in (\mathscr{B}, G)' \cap \mathscr{A}$. Let E be any spectral projection of $U_2(g)$, then E commutes with $U(g)$ and $(\mathscr{B}, G)$. Let $x \in \mathscr{H}$ be analytic for $U(g)$. Then it follows that Ex is also analytic for $U(g)$.
Thus we obtain

$$\overline{\mathscr{A} E x} = \overline{\mathscr{B} E x} = E\, \overline{\mathscr{B} x} = E\, \overline{\mathscr{A} x}\,.$$

Since the set of vectors $\{Ex; x \text{ analytic}\}$ is dense in $E\mathscr{H}$, this equation implies $E\mathscr{H}$ is invariant under $\mathscr{A}$, or $E \in \mathscr{A}'$. Thus $U_2(g) \in \mathscr{A}' \cap \mathscr{A} = \mathfrak{Z}$ and $U_1(g)\, A\, U_1(g^{-1}) = U(g)\, A\, U(g^{-1})$ for all $A \in \mathscr{A}$. In other words, we may assume $U(g) \in (\mathscr{B}, G)$.

Now take $E \in (\mathscr{B}, G_1)'$ and x analytic. Ex must again be analytic, and the identical argument establishes that $E \in \mathscr{A}'$, or equivalently $(\mathscr{B}, G)' \subset \mathscr{A}'$. We know that $(\mathscr{B}, G) \subset \mathscr{A}$, hence $(\mathscr{B}, G) = \mathscr{A}$.

Proof of Theorem 6. Let $E \in \mathscr{A}$ commute with all $U(g)$ and $x \in E\mathscr{H}$ be analytic for $U(g)$. If we denote by $F \in \mathscr{A}'$ the projection onto $\mathscr{A}x$ then we find $\overline{E\mathscr{A}Ex} = \overline{E\mathscr{A}x} = EF\mathscr{H}$. Since x is analytic for $U(g)$ we get $EF\mathscr{H} = \overline{E\mathscr{A}Ex} = \overline{E\mathscr{B}Ex} \subset \overline{E\{\mathscr{B}, E\}Ex} \subset \overline{E\mathscr{A}Ex} = EF\mathscr{H}$. Hence we have for any vector $x \in E\mathscr{H}$ which is analytic for $U(g)$, $\overline{\mathscr{A}_E x} = \overline{\{\mathscr{B}, E\}_E x}$ and hence by theorem 3 $(\{\mathscr{B}, E\}_E, G) = \mathscr{A}_E$. This proves statement a). Denote by $\mathscr{D}$ the v. Neumann algebra generated by $\mathscr{B}$ and $E\mathscr{A}E$ then $\mathscr{D}_E = \mathscr{A}_E$ hence $\mathscr{D}'_E = \mathscr{A}'_E$. Since E has central support 1 in $\mathscr{A}$ follows that the map $\mathscr{A}' \to \mathscr{A}'_E$ is one to one. Hence for each operator $T \in \mathscr{D}'$ there exists a unique $S \in \mathscr{A}'$ such that $ET = ES$ or $(T - S)E = 0$. Since T and S are both in $\mathscr{B}'$, $(T - S)\,\overline{\mathscr{B}E\mathscr{H}} = 0$. Since the vectors $x \in E\mathscr{H}$ which are analytic for $U(g)$ are dense in $E\mathscr{H}$, $\overline{\mathscr{B}E\mathscr{H}} = \overline{\mathscr{A}E\mathscr{H}} = \mathscr{H}$, where the last relation follows from the assumption that E has central support 1. Hence $T - S = 0$ and $T \in \mathscr{A}'$. This implies $\mathscr{A}' \supset \mathscr{D}'$ and since on the other hand $\mathscr{D} \subset \mathscr{A}$ follows $\mathscr{D} = \mathscr{A}$. This proves the theorem.

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