Commun. math. Phys. 10, 69—92 (1968)

On Current-Density Algebras, Gradient Terms and Saturation *

(Conserved Currents)

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Received February 21, 1968

Abstract. The equal time limit of commutator matrix elements of conserved currents is rigorously calculated by means of structures which follow from general principles of relativistic quantum field theory and current conservation. We prove: (a) In general derivatives of δ-functions occur (gradient terms). — (b) The proper (non-gradient) part of the equal time limit is exactly given by the divergence-free causal one particle structures constructed from those intermediate one particle states which have the same main quantum numbers (mass, total spin and total isospin) as one of the external states (saturation by two one particles states!). — (c) All the other intermediate discrete one particle states drop out completely and the continuous many particle states contribute at most to gradient terms. — (d) The gradient terms emerging from the remaining two discrete intermediate one particle states can be removed without any restrictions on the form factors. — (e) From current algebras of conserved currents in the form proposed and used in the literature one cannot deduce any predictions for form factors beyond the algebraic conditions for coupling constants which already follow from the algebra of the charges.

I. Introduction

In recent years we have witnessed an ever increasing interest in the field theoretical aspects of current operators. Generalizing group theoretical features of conserved charges to non-conserved charges [1] proved to be very successful in deriving sum rules of various types [2—6]. Encouraged by these results physicists have conjectured algebraic structures also for the current densities. Whereas numerous articles, and lecture notes [7][1] have been published on the applications of these current density algebras, considerably less effort has been devoted to the problem of consistency of the density algebras with quantum field theory.

In addition the combination of the density algebras with simple saturation assumptions (one particle saturation) lead to kinematical inconsistencies of the results [4]. Without looking for possible diseases

* Supported in part by the U.S. Atomic Energy Commission under Contract AT (30-1)-3829.

[1] For a general information on current algebras see [7] and the bibliography given there.

of the assumptions, the removal of these inconsistencies combined with the claim for infinite sets of one particle intermediate states has been used as a dynamical recipe for the determination of form factors [8, 9].

In a series of papers we will investigate the consistency of current density algebras without gradient terms and their saturation by a finite number of intermediate one particle states with general principles of relativistic quantum field theory. In the present first article we restrict ourselves to the case of two conserved currents.

Let us consider for example the following simple situation of the internal symmetry group $SU(2)$ and the π-system: From the group theoretical aspects of isospin one obtains the following commutation relations between total iso-charges and isospin currents:

$$\begin{aligned} \langle \pi k_1 | \, [Q_\alpha, j^\mu_\beta(x)] \, | \pi k_2 \rangle^T &= i \, \varepsilon_{\alpha\beta\gamma} \, \langle \pi k_1 | \, j^\mu_\gamma(x) \, | \pi k_2 \rangle \\ \langle \pi k_1 | \, [j^\mu_\alpha(x), Q_\beta] \, | \pi k_2 \rangle^T &= i \, \varepsilon_{\alpha\beta\gamma} \, \langle \pi k_1 | \, j^\mu_\gamma(x) \, | \pi k_2 \rangle \, . \end{aligned} \tag{1}$$

The most general form of the truncated charge density commutator matrix elements consistent with (1) is[2]:

$$\begin{aligned} \langle \pi k_1 | \, [j^0_\alpha(x), j^0_\beta(y)] \, | \pi k_2 \rangle^T_{x_0 = y_0 = 0} &= \delta(\boldsymbol{x} - \boldsymbol{v}) \, e^{-i(\boldsymbol{k}_1 - \boldsymbol{k}_2)\boldsymbol{x}} \, A^0_{\alpha\beta}(k_1, k_2) \\ &- S^{0r}_{\alpha\beta}(k_1, k_2) \, (\partial_{(x)_r} - \partial_{(y)_r}) \, [e^{-i(\boldsymbol{k}_1 \boldsymbol{x} - \boldsymbol{k}_2 \boldsymbol{y})} \, \delta(\boldsymbol{x} - \boldsymbol{y})] \\ &+ A^{0kl}_{\alpha\beta}(k_1, k_2) \, [\partial_{(x)_k} \partial_{(y)_l} + \partial_{(x)_l} \partial_{(y)_k}] \, [e^{-i(\boldsymbol{k}_1 \boldsymbol{x} - \boldsymbol{k}_2 \boldsymbol{y})} \, \delta(\boldsymbol{x} - \boldsymbol{y})] \\ &- \bar{S}^{0r}_{\alpha\beta}(k_1, k_2) \, \boldsymbol{\nabla}_{(x)} \boldsymbol{\nabla}_{(y)} \, (\partial_{(x)_r} - \partial_{(y)_r}) \, [e^{-i(\boldsymbol{k}_1 \boldsymbol{x} - \boldsymbol{k}_2 \boldsymbol{y})} \, \delta(\boldsymbol{x} - \boldsymbol{y})] \\ &+ \text{higher terms.} \end{aligned} \tag{2}$$

with

$$(k_1 - k_2)_r \, S^{0r}_{\alpha\beta}(k_1, k_2) = 0 \, . \tag{3}$$

The proposal of a current algebra is that all the gradient terms vanish and that $A^0_{\alpha\beta}$ can be identified with $i \, \varepsilon_{\alpha\beta\gamma} \, \langle \pi k_1 | \, j^0_\gamma(0) \, | k_2 \pi \rangle$. Furthermore the one particle saturation in its simplest version means to introduce a complete set of intermediate states in the commutator on the left hand side of (2), keep from this infinite set only the discrete one particle states and drop all the others. It is obvious that such a procedure if it is consistent with general principles leads to relations between form factors. Unfortunately it destroys locality and because locality is strongly connected with Lorentz covariance this procedure shows the well known kinematical inconsistencies in the equal time limit.

In this article we will prove from general principles of relativistic quantum field theory, current conservation and some technical high energy assumptions the following six statements:

I. The equal time limit of the density commutators is exactly given by the right hand side of (2).

[2] Latin indices run from 1 to 3, Greek indices from 0 to 3 with the metric $(+, -, -, -)$.

II. In one particle approximations which satisfy locality and current conservation (such approximations will be called from now on DCOP-structures) the equal time limit does not show any inconsistencies.

III. From all DCOP-structures only those two which correspond to the intermediate one particle states with the same main quantum numbers as one of the external states contribute to the equal time limit (in the example at hand this is the one pion state). All the others drop out.

IV. All contributions to the gradient terms emerging from the remaining two DCOP-structures can be removed without any condition on the form factors.

V. The proper (non-gradient) part of the charge density algebra does not lead to any new dynamical restrictions apart from the algebraic relations for the coupling constants which also follow from the algebra of the charges. $A^0_{\alpha\beta}$ is *exactly* given by products of coupling constants and form factors which automatically satisfy the algebra without any restrictions on the form factors for momentum transfer $t \neq 0$.

VI. All continuous many particle intermediate states contribute at most to gradient terms. Sum rules can only emerge from structural assumptions on these gradient terms, for example their vanishing.

From these six statements it follows that the current density algebras for conserved currents in the form, in which they are proposed and extensively used in applications, are dynamically empty as far as the calculation of form factors is concerned. The only possible dynamical content is contained in the assumption that no gradient terms do occur (small distance behaviour of matrix elements). But in order to extract this content one has to calculate directly the three coefficients $S^{0r}_{\alpha\beta}$, $A^{0kl}_{\alpha\beta}$ and $\bar{S}^{0r}_{\alpha\beta}$ on the right hand side of (2) in terms of physical quantities and to check what the vanishing of the three coefficients means for these quantities.

In the proof of these statements we rely heavily on the field theoretical causal one particle structure of the commutator matrix elements. Such causal one particle structure properties in the case of Green's functions have been studied by ZIMMERMANN [10, 11], SYMANZIK [12, 13] and STREATER [14]. They are a consequence of locality, and the existence of non zero mass one particle states below a continuous spectrum. For special Wightman functions and the case at hand one of the present authors (A. H. VÖLKEL) has investigated the one particle structure [15, 16]. It was shown that the connected matrix element of the commutator of two currents can be written:

$$\langle k_1| \, [j^\mu(x), g^\nu(y)] \, |k_2\rangle^T = F^{\mu\nu}(x, y)^I + F^{\mu\nu}(x, y)^{II} \tag{4}$$

where $F^{\mu\nu}(\ )^I$ is a product of (retarded) three point functions and $F^{\mu\nu}(\ )^{II}$ contains only contributions from the continuous energy momentum spectrum. Every part is local for itself. However the occurence of retarded three point functions in $F^{\mu\nu}(\;)^I$ destroys current conservation for each part separately on the right hand side of (4). It was shown by two of the present authors (Uta Völkel and A. H. Völkel) [16, 17] that by changing the decomposition one can rewrite the right hand side of (4) as:

$$F^{\mu\nu}(x, y)^{I\,G} + F^{\mu\nu}(x, y)^{II\,G} \tag{5}$$

where both terms separately satisfy current conservation. Both are separately local. Furthermore $F^{\mu\nu}(\;)^{I\,G}$ contains all the one particle contributions and incorporates only the three point functions information of the theory.

This decomposition together with current conservation exposes enough structure of the commutator matrix elements so that the statements above can be rigorously proven. This structure is however not strong enough to show whether *all* gradient terms vanish trivially, give rise to non-trivial sum rules, or cannot be removed in principle.

We want to mention that all forthcoming high energy assumptions are sufficient assumptions which can be weakened with the *only* effect that further gradient terms arise.

We restrict ourselves to the case of spin zero particles. All our statements remain unchanged for particles with spin. It is a simple matter of introducing spin indixes to generalize our calculations to the case of particles with spin. The case of spin 1/2 particles will be treated in the second article of this series, where we investigate the $SU(2) \times SU(2)$ algebra of vector and axial vector currents.

II. The Causal, Divergence-Free one Particle Structure (DCOPS)

The commutator matrix element (4) can be written as a sum of the one particle intermediate state contribution and the continuous rest[3]:

$$\begin{aligned}\langle \boldsymbol{k}_1 M_1|\,[j^\mu(x), g^\nu(y)]\,|M_2 \boldsymbol{k}_2\rangle^T &= \int \frac{d^3q}{2q^0}\{\langle \boldsymbol{k}_1 M_1|\,j^\mu(x)\,|m\boldsymbol{q}\,\gamma\rangle\,\langle\gamma\,\boldsymbol{q}\,m|\,g^\nu(y)\,|M_2\boldsymbol{k}_2\rangle \\ &\quad - \langle \boldsymbol{k}_1 M_1|\,g^\nu(y)\,|m\boldsymbol{q}\,\gamma\rangle\,\langle\gamma\,\boldsymbol{q}\,m|\,j^\mu(x)\,|M_2\boldsymbol{k}_2\rangle\} \\ &\quad + \text{continuous rest}\,.\end{aligned} \tag{6}$$

But in this decomposition the one particle contribution and the rest separately do not have all the linear properties of commutators. They do not vanish for space like distances. An unpleasant side effect of this

[3] The internal quantum number γ denotes always the pair $(\Gamma, \bar{\Gamma})$ with summation over $\bar{\Gamma}$. Γ denotes the representation of the corresponding symmetry group and $\bar{\Gamma}$ the states within this representation.

decomposition would be the explicit $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ dependence of each separate contribution to the equal time commutator. If one for example omits the continuous contribution then one has to introduce ad hoc devices ($\boldsymbol{k} \to \infty$ method) to get kinematical consistency. The way to repair this situation is to change the decomposition (6) to [10–17]:

$$\langle \boldsymbol{k}_1 M_1 | \, [j^\mu(x), g^\nu(y)] \, | M_2, \boldsymbol{k}_2 \rangle^T = F^{\mu\nu}(x, y)^I + F^{\mu\nu}(x, y)^{II} \tag{7}$$

with:

$$\begin{aligned} F^{\mu\nu}(x, y)^I &= 2\pi i \int d^4q \, \{ e^{-i(k_2 - q)x + i(k_1 - q)y} \\ &\quad \cdot [\langle k_1 | \, R[g^\nu(0), \tilde{B}_{\bar{\gamma}}(-q)] \, |0\rangle \, \overleftrightarrow{\Delta}'_{\mathrm{ret}}(q) \, \langle 0 | \, R[\tilde{B}_\gamma(q), j^\mu(0)] \, |k_2\rangle \\ &\quad - \langle k_1 | \, R[\tilde{B}_{\bar{\gamma}}(-q), g^\nu(0)] \, |0\rangle \, \overleftrightarrow{\Delta}'_{\mathrm{av}}(q) \, \langle 0 | \, R[j^\mu(0), \tilde{B}_\gamma(q)] \, |k_2\rangle] \\ &\quad - e^{i(k_1 - q)x - i(k_2 - q)y} \\ &\quad \cdot [\langle k_1 | \, R[j^\mu(0), \tilde{B}_{\bar{\gamma}}(-q)] \, |0\rangle \, \overleftrightarrow{\Delta}'_{\mathrm{ret}}(q) \, \langle 0 | \, R[\tilde{B}_\gamma(q), g^\nu(0)] \, |k_2\rangle \\ &\quad - \langle k_1 | \, R[\tilde{B}_{\bar{\gamma}}(-q), j^\mu(0)] \, |0\rangle \, \overleftrightarrow{\Delta}'_{\mathrm{av}}(q) \, \langle 0 | \, R[g^\nu(0), \tilde{B}_\gamma(q)] \, |k_2\rangle] \} \, . \end{aligned} \tag{8}$$

Here $\tilde{B}_\gamma(q)$ $[\tilde{B}_{\bar{\gamma}}(q) =: \tilde{B}_\gamma(-q)^*]$ is the interpolating field of the intermediate one particle state with mass m and internal quantum numbers γ. $R[\ ]$ denotes the retarded product. The kernel

$$\overleftrightarrow{\Delta}'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q) =: \Delta'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q)^{-1} \, \Delta'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q) \, \Delta'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q)^{-1} \tag{9}$$

is given by the two point functions of the interacting field $B_\gamma(x)$. They can be represented by [12]:

$$\Delta'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q) = \frac{1}{m^2 - (q \pm i\varepsilon)^2} + \int_{4m^2}^{\infty} \frac{ds \, \varrho(s, \gamma)}{s - (q \pm i\varepsilon)^2} \tag{10}$$

$$\Delta'_{\substack{\mathrm{ret} \\ \mathrm{av}}}(q)^{-1} = (m^2 - q^2) \left\{ A_\gamma + \int_{4m^2}^{\infty} \frac{ds \, \sigma(s, \gamma)}{[s - (q \pm i\varepsilon)^2] \, [s - m^2]} \right\} \tag{11}$$

$$\varrho(s, \gamma) \geqq 0 \, , \quad \sigma(s, \gamma) \geqq 0 \, , \quad A_\gamma \geqq 0 \, ,$$

$$\left[A_\gamma + \int_{4m^2}^{\infty} \frac{ds \, \sigma(s, \gamma)}{[s - m^2]^2} \right] = 1 \, . \tag{12}$$

The expression (8) has on the mass shell of the intermediate momentum the same structure as (6). But it is in addition local and is given solely in terms of three point functions with one particle on the mass shell.

Of course such a "causal one particle" expression is not unique. Without changing the normalisation of the intermediate one particle structure one can take any definition of retarded products or any other propagator with the same one particle poles and residua and the same starting point of the continuum. As long as one does not make any approximation the choice of retarded products and propagators is a pure matter of bookkeeping in the decomposition (7).

The next step in our argument is the realisation that the decomposition (7) is still not satisfactory. For a conserved current $\partial_\mu j^\mu(x) = 0$ we would like to redefine our causal one particle term in such a way that current conservation is separately fulfilled for each term of our decomposition. In other words, what we call a "divergence-free causal one particle structure" (DCOP-structure), is an expression which on the mass shell agrees with (6), is local, satisfies the above mentioned divergence condition and is a bilinear functional of the three point functions. Two of the present authors (Uta Völkel and A. H. Völkel) have explicitely constructed such DCOP-structures [16, 17] in terms of Jost-Lehmann-Dyson representations [18–21]. They are given by[4]:

$$F^{\mu\nu}(x,y)^{IG} = : F^{\mu\nu}(x,y)^I - e^{i(k_1 x - k_2 y)} \cdot [K^{\mu\nu}(x-y)_1 + H^{\mu\nu}(x-y)_1] + e^{-i(k_2 x - k_1 y)} [K^{\mu\nu}(x-y)_2 + H^{\mu\nu}(x-y)_2] \tag{13}$$

with

$$\begin{Bmatrix} K^{\mu\nu} \\ H^{\mu\nu} \end{Bmatrix} (x)_{\substack{1\\2}} = \frac{1}{(2\pi)^{3/2}} \int d^4 q\, e^{\mp i q x} \begin{Bmatrix} \tilde{K}^{\mu\nu} \\ \tilde{H}^{\mu\nu} \end{Bmatrix} (q)_{\substack{1\\2}} \tag{14}$$

$$\begin{aligned}
\tilde{K}^{\mu\nu}(q)_r = & - i(-1)^r \int d^4 u\, ds\, \varepsilon(q^0 - u^0)\, \delta((q-u)^2 - s) \\
& \left\{ \frac{(q + k_r - 2u)^\mu}{s - (u - k_r)^2} \frac{(q + k_{3-r} - 2u)^\nu}{s - (u - k_{3-r})^2} \right. \\
& \left[\pi(u,s)_r + (q - k_r)_\lambda \left(\phi_j^\lambda(u,s)_r + \frac{(k_r - k_{3-r})^\lambda}{(k_1 - k_2)^2} \psi_j(u,s)_r \right) \right. \\
& \left. - (q - k_{3-r})_\lambda \left(\phi_g^\lambda(u,s)_r + \frac{(k_r - k_{3-r})^\lambda}{(k_1 - k_2)^2} \psi_g(u,s)_r \right) \right] \\
& - \frac{(q + k_{3-r} - 2u)^\nu}{(s - (u - k_{3-r})^2)} \left[\phi_j^\mu(u,s)_r - \frac{(k_r - k_{3-r})^\mu}{(k_1 - k_2)^2} (\psi_j(u,s)_r + \psi_g(u,s)_r) \right. \\
& \left. - (q + k_r - 2u)^\mu\, \delta(s - (u - k_r)^2)\, e_j(u,s)_r \right] \\
& + \frac{(q + k_r - 2u)^\mu}{s - (u - k_r)^2} \left[\phi_g^\nu(u,s)_r + \frac{(k_r - k_{3-r})^\nu}{(k_1 - k_2)^2} (\psi_j(u,s)_r + \psi_g(u,s)_r) \right. \\
& \left. \left. - (q + k_{3-r} - 2u)^\nu\, \delta(s - (u - k_{3-r})^2)\, e_g(u,s)_r \right] \right\} .
\end{aligned} \tag{15}$$

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[4] The following Eq. (15) differs from Eq. (40) in [17] by the absence of the terms containing the spectral functions ϕ, w^μ, ϱ and E, which do not occur in the case of conserved currents.

Here $\{\phi_j^\mu, \psi_j, e_j\}\,(u, s)_r$ are the Dyson spectral functions of the following matrix elements:

$$\tilde{I}_j^\mu(q)_1 =: i(2\pi)^{5/2}$$
$$\cdot \{\langle k_1|\, R[j^\mu(0), \tilde{B}_{\bar{\gamma}}(-q)]\,|0\rangle\, \overleftrightarrow{\Delta}'_{\text{ret}}(q) - \langle k_1|\, R[\tilde{B}_{\bar{\gamma}}(-q), j^\mu(0)]\,|0\rangle\, \overleftrightarrow{\Delta}'_{\text{av}}(q)\}$$
$$\cdot P_g(\gamma, q, k_2)_2\,, \tag{16}$$

$$\tilde{I}_j^\mu(q)_2 = i(2\pi)^{5/2}\, P_g(\gamma, q, k_1)_1$$
$$\cdot \{\overleftrightarrow{\Delta}'_{\text{ret}}(q)\,\langle 0|\, R[\tilde{B}_\gamma(q), j^\mu(0)]\,|k_2\rangle - \overleftrightarrow{\Delta}'_{\text{av}}(q)\,\langle 0|\, R[j^\mu(0), \tilde{B}_\gamma(q)]\,|k_2\rangle\} \tag{17}$$

with the abbreviations:

$$\begin{aligned} P_g(\gamma, q, k_1)_1 &=: (k_1 - q)_\mu\,\langle k_1|\, R[g^\mu(0), \tilde{B}_{\bar{\gamma}}(-q)]\,|0\rangle \\ P_g(\gamma, q, k_2)_2 &=: (k_2 - q)_\mu\,\langle 0|\, R[\tilde{B}_\gamma(q), g^\mu(0)]\,|k_2\rangle\,. \end{aligned} \tag{18}$$

Locality implies the functions $P_g(\gamma, q, k_r)_r$ to be polynomials in q.

Similarly $\{\phi_g^\mu, \psi_g, e_g\}\,(u, s)_r$ are the Dyson spectral functions of the matrix elements $\tilde{I}_g^\mu(q)_r$ which are obtained from $\tilde{I}_j(q)_{3-r}$ by the replacements $j^\mu \to g^\mu$ and $P_g(\gamma, q, k_r)_r \to P_j(\gamma, q, k_r)_r$ on the right hand sides of (16), (17). $P_j(\gamma, q, k_r)_r$ is given by (18) with g replaced by j. All these spectral functions are introduced in such a way that the following divergence conditions are automatically fulfilled by the Dyson representations:

$$\left.\begin{aligned} &(q - k_r)_\mu\, \tilde{I}_j^\mu(q)_r \\ &(q - k_{3-r})_\mu\, \tilde{I}_g^\mu(q)_r \end{aligned}\right\} = -\, i(2\pi)^{5/2}\, P_j(\gamma, q, k_r)_r \tag{19}$$
$$[\Delta'_{\text{ret}}(q)^{-1} - \Delta'_{\text{av}}(q)^{-1}]\; P_g(\gamma, q, k_{3-r})_{3-r}\,.$$

These equations follow immediately from current conservation and the definition of the matrix elements $\tilde{I}^\mu_{j\,(g)}(q)_r$.

Now the Dyson representation of $\tilde{I}^\mu$ in terms of $\{\phi^\mu, \psi, e\}$ are simply obtained by the most general solutions of the Eq. (19). If $-\,i(-1)^r\pi(u,s)_r$ are the Dyson spectral functions[5] of the right hand side of (19) then these solutions are given by [17]:

$$\begin{aligned} \tilde{I}_j^\mu(q)_r = &-i(-1)^r \int d^4u\, ds\, \varepsilon(q^0 - u^0)\, \delta((q-u)^2 - s) \\ &\cdot \left\{ \frac{(q + k_r - 2u)^\mu}{s - (u - k_r)^2}\, [\pi(u,s)_r + \psi_j(u,s)_r + (q - k_r)_\lambda\, \phi_j^\lambda(u,s)_r] \right. \\ &\left. -\, \phi_j^\mu(u,s)_r + (q + k_r - 2u)^\mu\, \delta(s - (u - k_r)^2)\, e_j(u,s)_r \right\} \end{aligned} \tag{20}$$

$$\begin{aligned} \tilde{I}_g^\nu(q)_r = &-i(-1)^r \int d^4u\, ds\, \varepsilon(q^0 - u^0)\, \delta((q-u)^2 - s) \\ &\cdot \left\{ \frac{(q + k_{3-r} - 2u)^\nu}{s - (u - k_{3-r})^2}\, [\pi(u,s)_r - \psi_g(u,s)_r - (q - k_{3-r})_\lambda\, \phi_g^\lambda(u,s)_r] \right. \\ &\left. +\, \phi_g^\nu(u,s)_r - (q + k_{3-r} - 2u)^\nu\, \delta(s - (u - k_{3-r})^2)\, e_g(u,s)_r \right\} \end{aligned} \tag{21}$$

[5] From the polynomial character of $P_{j(g)}$ and Eq. (11) it follows that the support of $\pi(u, s)_r$ is concentrated in the point $u = 0$.

with

$$\int d^4u\, ds\, \varepsilon(q^0 - u^0)\, \delta((q-u)^2 - s)\, \psi_{j(g)}(u, s)_r = 0$$

and

$$\begin{aligned} \frac{\phi_j^\lambda(u,s)_r}{s-(u-k_r)^2} &=: \frac{\partial}{\partial s} \frac{Z_j^\lambda(u,s)_r}{s-(u-k_r)^2} \\ \frac{\phi_g^\lambda(u,s)_r}{s-(u-k_{3-r})^2} &=: \frac{\partial}{\partial s} \frac{Z_g^\lambda(u,s)_r}{s-(u-k_{3-r})^2}\,. \end{aligned} \tag{23}$$

All the spectral functions have certain support properties which follow in the well known manner [18–21] from the support properties of the commutator matrix elements $\tilde{I}^\mu_{j(g)}(q)_r$ in momentum space.

As we mentioned earlier the DCOP-structure is not unique. In part this non-uniqueness is expressed in the occurrence of the functions $H^{\mu\nu}(x)_r$ in Eq. (13). Their fourier transforms are arbitrary local solutions of the divergence equations:

$$\begin{aligned} (q-k_r)_\mu \tilde{H}^{\mu\nu}(q)_r &= 0 \\ (q-k_{3-r})_\nu \tilde{H}^{\mu\nu}(q)_r &= 0 \end{aligned} \qquad r = 1, 2 \tag{24}$$

with their support contained in the support of $\tilde{K}^{\mu\nu}(q)_r$. Solutions of this boundary value problem are given by [17]:

$$\begin{aligned} \tilde{H}^{\mu\nu}(q)_r = i(-1)^r \int & d^4u\, ds\, \varepsilon(q^0-u^0)\, \delta((q-u)^2 - s) \\ & \Big\{(q+k_r-2u)^\mu (q+k_{3-r}-2u)^\nu \\ & \cdot \Big[\frac{(q-k_r)_\lambda (q-k_{3-r})_\varkappa}{[s-(u-k_1^2]\,[s-(u-k_2)^2]} Z^{\lambda\varkappa}(u,s)_r + \frac{\delta(s-(u-k_{3-r})^2)}{s-(u-k_r)^2} \\ & \cdot (q-k_r)_\lambda E_I^\lambda(u,s)_r + \frac{\delta(s-(u-k_r)^2)}{s-(u-k_{3-r})^2}(q-k_{3-r})_\varkappa E_{II}^\varkappa(u,s)_r \\ & + \delta(s-(u-k_r)^2)\,\delta(s-(u-k_{3-r})^2)\, E(u,s)_r\Big] \\ & + Z^{\mu\nu}(u,s)_r - \frac{(q+k_r-2u)^\mu}{s-(u-k_r)^2}(q-k_r)_\lambda Z^{\lambda\nu}(u,s)_r \\ & - \frac{(q+k_{3-r}-2u)^\nu}{s-(u-k_{3-r})^2}(q-k_r)_\varkappa Z^{\mu\varkappa}(u,s)_r \\ & - (q+k_r-2u)^\mu\, \delta(s-(u-k_r)^2)\, E_{II}^\nu(u,s)_r \\ & - (q+k_{3-r}-2u)^\nu\, \delta(s-(u-k_{3-r})^2)\, E_I^\mu(u,s)_r\Big\}\,. \end{aligned} \tag{25}$$

The spectral functions $Z^{\mu\nu}(u, s)_r$ and $E^{(\mu)}_{(I,II)}(u, s)_r$ are arbitrary except the condition that their support is contained in the support of $\phi_j^\mu(u, s)_r + \phi_g^\mu(u, s)_{3-r}$ and $e_j(u, s)_r + e_g(u, s)_{3-r}$ respectively.

As we will see in section IV the arbitrariness of all these spectral functions is strongly restricted, if we demand that the equal time limit does not contain derivatives of δ-functions.

III. The Equal Time Limit of the DCOP-Structure

In order to calculate the equal time limit we have to introduce Dyson representations for the retarded and advanced matrix elements and to make certain high energy assumptions. Essentially these are assumptions on the "large s-behaviour" of the Dyson spectral functions.

First of all we assume the validity of Ward's identity for the three point matrix elements. By this assumption it follows from (18) that the polynomials $P_g(\gamma, q, k_r)_r$ are independent of q[6]. Therefore we obtain by means of the asymptotic condition:

$$\begin{aligned} P_g(\gamma, q, k_1)_1 \equiv P_g(\gamma, k_1)_1 &= \lim_{q \to k_1} (k_1 - q)_\mu \langle k_1 | R [g^\mu(0), \tilde{B}_{\bar{\gamma}}(-q)] |0\rangle \\ &= -\frac{\sqrt{2\pi}}{2k_1^0} \langle \boldsymbol{k}_1 M_1 | g^0(0) | M_1 \boldsymbol{k}_1 \gamma\rangle \langle \gamma \boldsymbol{k}_1 M_1 | B_{\bar{\gamma}}(0) |0\rangle \end{aligned} \tag{26}$$

and similar:

$$\begin{aligned} P_g(\gamma, q, k_2)_2 &\equiv P_g(\gamma, k_2)_2 \\ &= -\frac{\sqrt{2\pi}}{2k_2^0} \langle 0 | B_\gamma(0) | M_2 \boldsymbol{k}_2 \gamma\rangle \langle \gamma \boldsymbol{k}_2 M_2 | g^0(0) | M_2 \boldsymbol{k}_2\rangle . \end{aligned} \tag{27}$$

From these equations we deduce that $P_g(\gamma, k_r)_r$ is unequal to zero only if the mass m of the intermediate state is equal to the external mass M_r. Otherwise the matrix elements of B vanish (B is the interpolating field of the intermediate particle with mass m). $P_g(\gamma, k_r)_r$ is also diagonal in the total isospin because the space integral over g^0 is a generator of the isospin group. If we had included particles with spin the same statement would be true for the total spins, because the generators of the isospin group commute with the generators of the Lorentz group.

On the "large- s-behaviour" of the spectral functions we make the following assumptions:

$$\begin{aligned} &\int d^4u \int_{a>0}^{\infty} ds\, h(u)\, s^{-1} \{\phi^\mu, \psi\} (u, s)_r \quad < \infty \\ &\int d^4u \int_{a>0}^{\infty} ds\, h(u)\, s^{-3/2} Z^{\mu\nu}(u, s)_r \quad < \infty \end{aligned} \tag{28}$$

for $h(u) \in C^\infty$

$$\int_{4m^2}^{\infty} ds\, \varrho(\gamma, s) < \infty . \tag{29}$$

The assumptions (28) and (29) are sufficient conditions in order that the equal time limits of the DCOP-structures and the so called continuous rest exist separately.

[6] In case the ETC between the currents and the intermediate Heisenbergfield like $\langle k_1 | [g^0(x), B(y)] 10\rangle$ contain gradient terms, one would have instead of a constant P_g a polynomial in q. The nonconstant part of P_g would then give additional gradient contributions to the ET-structure of the 4-point function.

From the first inequality of (28) and the condition (22) it follows:

$$\int d^4u\, ds \frac{\psi(u, s)_r}{s - (u - q)^2} = 0 . \tag{30}$$

Furthermore we define the retarded and advanced parts of the matrix elements $\tilde{I}^{\mu}_{j(g)}$ (16)–(21) by:

$$\left.\begin{array}{l} \langle k_1|\, R\,[j^\mu(0), \widetilde{B}_{\bar{\gamma}}(-q)]\, |0\rangle\, \Delta'_{\mathrm{ret}}(q)^{-1}\, P_g(\gamma, k_2)_2 \\ \langle k_1|\, R\,[\widetilde{B}_{\bar{\gamma}}(-q), j^\mu(0)]\, |0\rangle\, \Delta'_{\mathrm{av}}(q)^{-1}\, P_g(\gamma, k_2)_2 \end{array}\right\}$$
$$\begin{aligned} &= -\frac{i}{(2\pi)^{7/2}} \int d^4u\, ds \Bigg[\frac{(q + k_1 - 2u)^\mu}{s - (q \pm i\varepsilon - u)^2} \\ &\quad\cdot \left\{\frac{1}{s - (u - k_1)^2}\,[\pi(u, s)_1 + \psi_j(u, s)_1 + (q - k_1)_\lambda\, \phi^\lambda_j(u, s)_1]\right. \\ &\quad \left. + \delta(s - (u - k_1)^2)\, e_j(u, s)_1\right\} \\ &\quad - \frac{(q - u)^2 - (u - k_1)^2}{[s - (u - k_1)^2]\,[s - (q \pm i\varepsilon - u)^2]}\, \phi^\mu_j(u, s)_1\Bigg] \\ &\quad + Q^\mu_j(\gamma, q/k_1)_1 . \end{aligned} \tag{31}$$

$$\left.\begin{array}{l} P_g(\gamma, k_1)_1\, \Delta'_{\mathrm{ret}}(q)^{-1}\, \langle 0|\, R\,[\widetilde{B}_\gamma(q), j^\mu(0)]\, |k_2\rangle \\ P_g(\gamma, k_1)_1\, \Delta'_{\mathrm{av}}(q)^{-1}\, \langle 0|\, R\,[j^\mu(0), \widetilde{B}_\gamma(q)]\, |k_2\rangle \end{array}\right\}$$
$$\begin{aligned} &= +\frac{i}{(2\pi)^{7/2}} \int d^4u\, ds \Bigg[\frac{(q + k_2 - 2u)^\mu}{s - (q \pm i\varepsilon - u)^2} \\ &\quad\cdot \left\{\frac{1}{s - (u - k_2)^2}\,[\pi(u, s)_2 + \psi_j(u, s)_2 + (q - k_2)_\lambda\, \phi^\lambda_j(u, s)_2]\right. \\ &\quad \left. + \delta(s - (u - k_2)^2)\, e_j(u, s)_2\right\} \\ &\quad - \frac{(q - u)^2 - (u - k_2)^2}{[s - (u - k_2)^2]\,[s - (q \pm i\varepsilon - u)^2]}\, \phi^\mu_j(u, s)_2\Bigg] \\ &\quad + Q^\mu_j(\gamma, q/k_2)_2 . \end{aligned} \tag{32}$$

Here $Q^\mu_j(\gamma, q/p)$ are polynomials in q.

The corresponding representation for the retarded and advanced matrix elements of the current g^μ are obtained from (31) and (32) replacing $\{\phi^\mu_j, \psi_j, e_j, \pi\}\,(u, s)_r$ by $\{\phi^\mu_g, \psi_g, e_g, -\pi\}\,(u, s)_{3-r}$ and $Q^\mu_j(\gamma, q/k_r)_r$ by $Q^\mu_g(\gamma, q/k_r)_{3-r}$ on the right hand sides.

In order to calculate the polynomials Q^μ, we multiply for instance the Eq. (31) by $(k_1 - q)_\mu$. By means of Eq. (18) we obtain:

$$\begin{aligned} P_j(\gamma, k_1)_1\, \Delta'_{\mathrm{ret}}(q)^{-1}\, P_g(\gamma, k_2)_2 &= (k_1 - q)_\mu\, Q^\mu_j(\gamma, q/k_1)_1 - \frac{i}{(2\pi)^{7/2}} \int d^4u\, ds \\ &\left[\frac{(k_1 - u)^2 - (q - u)^2}{[s - (q \pm i\varepsilon - u)^2]\,[s - (u - k_1)^2]}\, \pi(u, s)_1 + \delta(s - (u - k_1)^2)\, e_j(u, s_1)\right] . \end{aligned} \tag{33}$$

Taking the limit $q \to k_1$ we deduce from Eq. (11), the diagonality of P_j in the masses and the polynomial character of Q^μ_j that the δ-function term in (33) must vanish.

$$\begin{aligned} &\int d^4u\, ds\, \delta(s - (u - k_r)^2)\, e_j(u, s)_r = 0 \\ &\int d^4u\, ds\, \delta(s - (u - k_{3-r})^2)\, e_g(u, s)_r = 0\,. \end{aligned} \tag{34}$$

If we use the fact that $\pi(u, s)$ is the Dyson spectral function of the right hand side of (19), and therefore apart from trivial factors equal to the spectral function $\sigma(\gamma, s)$ of the inverse two point functions, we get by means of some elementary calculations from the Eqs. (11), (33) and (34)[7]:

$$\begin{aligned} Q_j^\mu(\gamma, q/k_r)_r &= (q + k_r)^\mu A_\gamma P_j(\gamma, k_r)_r\, P_g(\gamma, k_{3-r})_{3-r} \\ Q_g^\mu(\gamma, q/k_r)_{3-r} &= (q + k_r)^\mu A_\gamma P_g(\gamma, k_r)_r\, P_j(\gamma, k_{3-r})_{3-r}\,. \end{aligned} \tag{35}$$

The retarded and advanced matrix elements defined in (31), (32) and (35) are needed for the calculation of the equal time limit of the terms $K^{\mu\nu}$ in (13). In order to calculate the equal time limit of $F^{\mu\nu}(x, y)^I$ defined in Eq. (8) we need similar representations for the retarded and advanced matrix elements without the (constant) factors $P_g\,[P_j]$ occurring in (31), (32). These can be constructed in exactly the same manner as above because we never used any properties of these factors. Therefore the result is simply obtained by dropping these factors in the Eq. (31)–(35).

With these preparations it is straight forward to calculate the equal time limit by means of the Gauchy integral formula and the occurring δ-functions respectively, if we assume, that the limit $y^0 \to x^0$ can be interchanged with the u, s and q^0 integrals. We obtain:

$$\begin{aligned} \lim_{y^0 \to x^0} F^{00}(x, y)^{IG} &= (2\pi)^5\, e^{i(k_1 - k_2)x}\, \delta(\boldsymbol{x} - \boldsymbol{y}) \\ &\Big\{\frac{1}{4} P_j(\gamma, k_1)_1\, [\Delta'_{\mathrm{ret}}(k_1)^{-1} \langle 0|\, R\,[\tilde{B}_\gamma(k_1), g^0(0)]\, |k_2\rangle \\ &+ \Delta'_{\mathrm{av}}(k_1)^{-1} \langle 0|\, R\,[g^0(0), \tilde{B}_{\bar{\gamma}}(k_1)]\, k_2\rangle] \\ &+ \frac{1}{4} [\langle k_1|\, R\,[j^0(0), \tilde{B}_{\bar{\gamma}}(-k_2)]\, |0\rangle\, \Delta'_{\mathrm{ret}}(k_2)^{-1} \\ &+ \langle k_1|\, R\,[\tilde{B}_{\bar{\gamma}}(-k_2), j^0(0)]\, |0\rangle\, \Delta'_{\mathrm{av}}(k_2)^{-1}]\, P_g(\gamma, k_2)_2 \\ &- (j \leftrightarrow g) + \bar{A}^0_{jg}(k_1, k_2/\gamma)\Big\} \\ &+ e^{i(k_1^0 - k_2^0)x^0}\, \{- S^{0:r}_{jg}(k_1, k_2/\gamma)\, (\partial_{(x)\,r} - \partial_{(y)\,r}) \\ &+ A^{0:rl}_{jg}(k_1, k_2/\gamma)\, (\partial_{(x)\,r}\, \partial_{(y)\,l} + \partial_{(x)\,l}\, \partial_{(y)\,r}) \\ &- \bar{S}^{0:r}_{jg}(k_1, k_2/\gamma)\, (\partial_{(x)\,r} - \partial_{(y)\,r})\, \partial_x \partial_y\}\, [e^{-i(\boldsymbol{k}_1\boldsymbol{x} - \boldsymbol{k}_2\boldsymbol{y})}\, \delta(\boldsymbol{x} - \boldsymbol{y})]\,. \end{aligned} \tag{36}$$

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[7] The solution (35) is unique if we limit the polynomials to be of degree one. In the opposite case, we can add a higher polynomial of the form:

$$\bar{Q}^\mu_r = [(k_r - q)^\mu\, (q^2 - k_r^2) + (k_r + q)^\mu\, (k_r - q)^2]\, p(q, k)\,,$$

where p is an invariant polynomial. We drop such polynomial as part of our high energy assumptions.

In order to give the explicit expressions for the four coefficients A, $\bar{A}$, S and $\bar{S}$ we introduce the following abbreviations:

$$\Phi^{\mu}_{\pm}(u,s)_r =: (\phi^{\mu}_j(u,s)_r \pm \phi^{\mu}_g(u,s)_{3-r}) - \frac{k^{\mu}_r - k^{\mu}_{3-r}}{(k_1-k_2)^2}(\psi_j(u,s)_r \pm \psi_g(u,s)_{3-r}), \tag{37}$$

$$Z^{\mu\nu}_{\pm}(u,s)_r =: \frac{1}{2}(Z^{\mu\nu}(u,s)_r \pm Z^{\nu\mu}(u,s)_r). \tag{38}$$

In terms of these abbreviations the coefficients are given by:

$$\begin{aligned}\bar{A}^0_{jg}(k_1,k_2/\gamma) = -\frac{1}{2}\frac{i}{(2\pi)^{7/2}}\int d^4u\,ds \Big\{&\frac{\delta(s-(u-k_1)^2)}{s-(u-k_2)^2}[(k_1+k_2-2u)^0(e_j(u,s)_1 - e_g(u,s)_2)\\ &+(k_1-k_2)_\lambda\{(k_1+k_2-2u)^0(E^\lambda_I(u,s)_2 + E^\lambda_{II}(u,s)_1)\\ &-(k_1+k_2-2u)^\lambda(E^0_I(u,s)_2 + E^0_{II}(u,s)_1)\}]\\ &+2(k_1+k_2-2u)^0\,\delta(s-(u-k_1)^2)\,\delta(s-(u-k_2)^2)\\ &\cdot E(u,s)_1 + (1\leftrightarrow 2)\Big\}.\end{aligned} \tag{39}$$

$S^{0:r}$ and $\bar{S}^{0:r}$ are the $(0, r)$ – components $(r = 1, 2, 3)$ of the following tensors:

$$\begin{aligned}S^{\mu:\nu}_{jg}(k_1,k_2/\gamma) = -\frac{1}{2}(2\pi)^{3/2}\int d^4u\,ds\\ \cdot\Big\{&\frac{1}{[s-(u-k_1)^2][s-(u-k_2)^2]}[(k_1+k_2-2u)^\mu\,\Phi^\nu_+(u,s)_1\\ &-(k_1+k_2-2u)^\nu\,\Phi^\mu_+(u,s)_1 + (k_1-k_2)_\lambda\\ &\cdot((k_1+k_2-2u)^\mu Z^{\nu\lambda}_-(u,s)_1 - (k_1+k_2-2u)^\nu Z^{\mu\lambda}_-(u,s)_1\\ &+(k_1+k_2-2u)^\lambda Z^{\mu\nu}_-(u,s)_1)]\\ &-\frac{\delta(s-(u-k_1)^2)}{s-(u-k_2)^2}[(k_1+k_2-2u)^\mu(E^\nu_I(u,s)_2 - E^\nu_{II}(u,s)_1)\\ &-(k_1+k_2-2u)^\nu(E^\mu_I(u,s)_2 - E^\mu_{II}(u,s)_1)] - (1\leftrightarrow 2)\Big\}.\end{aligned} \tag{40}$$

$$\begin{aligned}\bar{S}^{\mu:\nu}_{jg}(k_1,k_2/\gamma) = -(2\pi)^{3/2}\int d^4u\,ds\,\frac{1}{[s-(u-k_1)^2][s-(u-k_2)^2]}\\ \cdot(Z^{\mu\nu}_+(u,s)_1 - Z^{\mu\nu}_+(u,s)_2),\end{aligned} \tag{41}$$

$$\begin{aligned}A^{0;kl}_{jg}(k_1,k_2/\gamma) = -\frac{i}{2}(2\pi)^{3/2}\int d^4u\,ds\\ \cdot\Big[&\frac{1}{[s-(u-k_1)^2][s-(u-k_2)^2]}\{\delta^l_k(\Phi^0_-(u,s)_1 + (k_1-k_2)_\lambda Z^{0\lambda}_-(u,s)_1)\\ &+(k_1+k_2-2u)^0\cdot Z^{kl}_+(u,s)_1 - 2(k_1+k_2-2u)^k Z^{0l}_+(u,s)_1\}\\ &+\frac{\delta(s-(u-k_1)^2)}{s-(u-k_2)^2}\delta^l_k(E^0_{II}(u,s)_1 + E^0_I(u,s)_2) + (1\leftrightarrow 2)\Big].\end{aligned} \tag{42}$$

Now the first question is if all the gradient terms in (36) can be removed. As will be seen in the next section this can be achieved by (strong) restrictions on the arbitrary spectral functions $Z^{\mu\nu}$, $E^{\mu}_{I,II}$. In other words the condition that no gradient terms occur in the density algebras fixes in part the approximations which are consistent with these algebras.

Furthermore it will be shown there, that the coefficient $\bar{A}^{0}_{jg}$ must vanish in order that the commutation relations (1) between one charge and one current density are satisfied. This in turn leads to a further restriction on the spectral function $E^{\lambda}_{I,II}(u, s)$. This condition must be fulfilled because the only contribution to the non gradient terms in the equal time limit comes from the DCOP-structure as will be shown in section V.

IV. Consistency of DCOP-Saturation of Current Algebras

We first discuss the removal of the gradient terms and prove the following theorem:

Theorem. *Necessary and sufficient for the vanishing of all gradient terms:*

$$S^{0:r}_{jg}(k_1, k_2/\gamma) = \bar{S}^{0:r}_{jg}(k_1, k_2/\gamma) = A^{0:rl}_{jg}(k_1, k_2) = 0 \tag{43}$$

is the condition:

$$(k_1 - k_2)_\mu \, S^{\nu:\mu}_{jg}(k_1, k_2/\gamma) = 0 \,. \tag{44}$$

Proof. We define antisymmetric tensors by:

$$\begin{aligned} B^{\mu\nu}_{-}(k_1, k_2) =: -\frac{1}{2}(2\pi)^{3/2}\int d^4u\, ds \Big\{ & \frac{1}{[s-(u-k_1)^2]\,[s-(u-k_2)^2]} \\ & \cdot [(k_1+k_2-2u)^\mu \, \Phi^{\nu}_{+}(u,s)_1 - (k_1+k_2-2u)^\nu \, \Phi^{\mu}_{+}(u,s)_1] \\ & - \frac{\delta(s-(u-k_1)^2)}{s-(u-k_2)^2} \, [(k_1+k_2-2u)^\mu \, (E^{\nu}_{I}(u,s)_2 - E^{\nu}_{II}(u,s)_1) \\ & - (k_1+k_2-2u)^\nu \, (E^{\mu}_{I}(u,s)_2 - E^{\mu}_{II}(u,s)_1)] - (1 \leftrightarrow 2)\Big\}, \end{aligned} \tag{45}$$

$$\begin{aligned} t^{\mu:\nu\lambda}(k_1, k_2) =: \frac{1}{2}(2\pi)^{3/2}\int d^4u\, ds & \frac{1}{[s-(u-k_1)^2]\,[s-(u-k_2)^2]} \\ & \cdot \{(k_1+k_2-2u)^\mu \, [Z^{\nu\lambda}_{-}(u,s)_1 + Z^{\nu\lambda}_{-}(u,s)_2]\}\,, \end{aligned} \tag{46}$$

$$\Gamma^{\mu\nu\lambda}(k_1, k_2) =: \{t^{\mu:\nu\lambda} - t^{\nu:\mu\lambda} + t^{\lambda:\mu\nu}\}(k_1, k_2)\,. \tag{47}$$

The tensors $Z^{\mu\nu}_{-}$, $B^{\mu\nu}_{-}$ and $t^{\lambda:\mu\nu}$ are antisymmetric in μ and ν. $\Gamma^{\mu\nu\lambda}$ is antisymmetric in all three indices. With these definitions $S^{0:r}$ are the $(0, r)$ – components of the expression:

$$S^{\mu:\nu}(k_1, k_2/\gamma) = B^{\mu\nu}_{-}(k_1, k_2) - (k_1 - k_2)_\lambda \, \Gamma^{\mu\nu\lambda}(k_1, k_2)\,. \tag{48}$$

In order that $S^{\mu:\nu}$ vanishes we must find a totally antisymmetric tensor $\Gamma^{\mu\nu\lambda}$ such that $(k_1 - k_2)_\lambda \, \Gamma^{\mu\nu\lambda}$ is equal to the first term of (48)

$$(k_1 - k_2)_\lambda \, \Gamma^{\mu\nu\lambda}(k_1, k_2) = B^{\mu\nu}_{-}(k_1, k_2)\,. \tag{49}$$

The antisymmetry of both tensors in this equation implies the necessary condition:

$$(k_1 - k_2)_\mu \, B^{\nu\mu}(k_1, k_2) = 0 \tag{50}$$

which is equivalent to (44).

If this condition is satisfied then the most general solution of Eq. (49) is given by:

$$\Gamma^{\mu\nu\lambda} = \frac{1}{(k_1 - k_2)^2} \{(k_1 - k_2)^\lambda \, B_-^{\mu\nu} - (k_1 - k_2)^\mu \, B_-^{\lambda\nu} + (k_1 - k_2)^\nu \, B_-^{\lambda\mu}\} + h^{\mu\nu\lambda} \tag{51}$$

with

$$(k_1 - k_2)_\lambda \, h^{\mu\nu\lambda} = 0 \,. \tag{52}$$

For our purpose it is sufficient to take only special solutions of the homogenous Eq. (52), which can be represented in the form:

$$h^{\mu\nu\lambda} = \hat{t}^{\mu:\nu\lambda} - \hat{t}^{\nu:\mu\lambda} + \hat{t}^{\lambda:\mu\nu} \tag{53}$$

$$\hat{t}^{\mu:\nu\lambda} = \frac{1}{2} (2\pi)^{3/2} \int d^4u \, ds \, \frac{1}{[s - (u - k_1)^2] \, [s - (u - k_2)^2]} \cdot (k_1 + k_2 - 2u)^\mu \, [\hat{Z}_-^{\nu\lambda}(u, s)_1 + \hat{Z}_-^{\nu\lambda}(u, s)_2] \,. \tag{54}$$

In order to satisfy condition (52) it is sufficient that $\hat{Z}$ has the form:

$$\hat{Z}_-^{\nu\lambda}(u, s)_r = (k_1 + k_2 - 2u)^\nu \, V^\lambda(u, s)_r - (\nu \leftrightarrow \lambda) \,. \tag{55}$$

Solving the Eqs. (46), (47) and (51)–(54) for $Z_-^{\nu\lambda}$ we obtain by means of the Eq. (45):

$$\begin{aligned}
\frac{Z_-^{\nu\lambda}(u, s)_r}{[s - (u - k_1)^2] \, [s - (u - k_2)^2]} &= \frac{1}{(k_1 - k_2)^2} \\
&\cdot \left\{ \frac{1}{[s - (u - k_1)^2] \, [s - (u - k_2)^2]} \, [(k_r - k_{3-r})^\nu \, \Phi_+^\lambda(u, s)_r - (\lambda \leftrightarrow \nu)] \right. \\
&\left. - \frac{\delta(s - (u - k_r)^2)}{s - (u - k_{3-r})^2} \, [(k_r - k_{3-r})^\nu \, (E_I^\lambda(u, s)_{3-r} - E_{II}^\lambda(u, s)_r) - (\lambda \leftrightarrow \nu)] \right\} \\
&+ \frac{\hat{Z}^{\nu\lambda}(u, s)_r}{[s - (u - k_1)^2] \, [s - (u - k_2)^2]} \,.
\end{aligned} \tag{56}$$

Inserting this expression into Eq. (42) we get for $A_{jg}^{0:kl}(k_1, k_2/\gamma)$:

$$\begin{aligned}
A_{jg}^{0:kl}(k_1, k_2/\gamma) = &- \frac{i}{2} (2\pi)^{3/2} \int d^4u \, ds \left\{ \frac{1}{[s - (u - k_1)^2] \, [s - (u - k_2)^2]} \right. \\
&\cdot \left[\delta_l^k \left(\Phi_-^0(u, s)_1 - \Phi_+^0(u, s)_1 + \frac{(k_1 - k_2)^0}{(k_1 - k_2)^2} (k_1 - k_2)_\lambda \, \Phi_+^\lambda(u, s)_1 \right. \right. \\
&\left. + (k_1 - k_2)_\lambda \, \hat{Z}_-^{0\lambda}(u, s)_1 \right) + (k_1 + k_2 - 2u)^0 \, Z_+^{kl}(u, s)_1 \\
&\left. - 2(k_1 + k_2 - 2u)^k \, Z_+^{0l}(u, s)_1 \right] + \delta_l^k \, \frac{\delta(s - (u - k_1)^2)}{s - (u - k_2)^2} \\
&\cdot \left[2E_I^0(u, s)_2 - \frac{(k_1 - k_2)^0}{(k_1 - k_2)^2} (k_1 - k_2)_\lambda \, (E_I^\lambda(u, s)_2 - E_{II}^\lambda(u, s)_1) \right] \\
&\left. + (1 \leftrightarrow 2) \right\} .
\end{aligned} \tag{57}$$

It is easy to see that $A^{0:kl}_{jg}$ vanishes with the following choice of Z_+ and $\hat{Z}_-$:

$$Z^{\mu\nu}_{+}(u, s)_r =: g^{\mu\nu} \frac{1}{(k_r - u)^2 - (k_{3-r} - u)^2}$$

$$\cdot \left\{ \frac{1}{[s - (u - k_1)^2][s - (u - k_2)^2]} \cdot (k_r - k_{3-r})_\lambda \, \Phi^\lambda_-(u, s)_r \right.$$

$$\left. + \frac{\delta(s - (u - k_r)^2)}{s - (u - k_{3-r})^2} + (k_r - k_{3-r})_\lambda \, [E^\lambda_{II}(u, s)_r - E^\lambda_I(u, s)_{3-r}] \right\}. \quad (58)$$

$$\frac{\hat{Z}^{\mu\nu}_-(u, s)_r}{[s - (u - k_1)^2][s - (u - k_2)^2]} =: \frac{1}{(k_r - u)^2 - (k_{3-r} - u)^2}$$

$$\cdot \left\{ \frac{1}{[s - (u + k_1)^2][s - (u - k_2)^2]} \left[(k_1 + k_2 - 2u)^\mu \left(\Phi^\nu_-(u, s)_r \right. \right. \right.$$

$$\left. \left. - \Phi^\nu_+(u, s)_r + \frac{(k_1 - k_2)^\nu}{(k_1 - k_2)^2} (k_1 - k_2)_\lambda \, \Phi^\lambda_+(u, s)_r \right) - (\mu \leftrightarrow \nu) \right]$$

$$+ \frac{\delta(s - (u - k_r)^2)}{s - (u - k_{3-r})^2} \left[(k_1 + k_2 - 2u)^\mu \left\{ 2 E^\nu_I(u, s)_{3-r} \right. \right.$$

$$\left. \left. \left. - \frac{(k_1 - k_2)^\nu}{(k_1 - k_2)^2} (k_1 - k_2)_\lambda \left(E^\lambda_I(u, s)_{3-r} - E^\lambda_{II}(u, s)_r \right) \right\} - (\mu \leftrightarrow \nu) \right] \right\}. \quad (59)$$

The tensor $\hat{Z}^{\mu\nu}_-$ as defined by (59) has the structure (55) and therefore satisfies the condition (52). Furthermore with the choice (58) for $Z^{\mu\nu}_+$ the gradient term $\bar{S}^{0:r}_{jg}(k_1, k_2)$ also vanishes because $Z^{\mu\nu}_+$ is diagonal in μ and ν. ∎

Next we have to investigate the condition (44) of the theorem. Before we plung into the discussion of this condition we need a little generalization of our formalism.

So far we have only considered the DCOP-structure of one single discrete intermediate one particle state. If there are several discrete one particle states with the internal quantum numbers γ_i and masses m_i ($i = 1, \ldots, N$) then we have simply to replace γ by γ_i and m by m_i in all our expressions and to take the sum over i [16, 22]. This will be done explicitely from now on. Furthermore we introduce $SU(2)$ quantum numbers $\delta_1 = (\Delta_1, \bar{\Delta}_1)$ and $\delta_2 = (\Delta_2, \bar{\Delta}_2)$ for the external states. Here again Δ_r denotes the representation of $SU(2)$ and $\bar{\Delta}_r$ the states within this representation.

By means of the Eq. (31), (32), (35), (37) and (40) the condition (44) can be rewritten as:

$$\frac{i}{(2\pi)^5} (k_1 - k_2)_\mu \, S^{\nu:\mu}_{jg}(k_1, k_2) = D^\nu_I(k_1, k_2) + D^\nu_{II}(k_1, k_2) = 0\,, \quad (60)$$

where we have introduced the abbreviations:

$$D_I^\mu(k_1, k_2) =: -\frac{1}{4}\Big\{\sum_i [(\langle\delta_1 k_1|\, R[j^\mu(0), \tilde{B}_{\bar{\gamma}_i}(-k_2)]\,|0\rangle\, \Delta'_{\text{ret}}(k_2)_i^{-1}$$
$$+ \langle\delta_1 k_1|\, R[\tilde{B}_{\bar{\gamma}_i}(-k_2), j^\mu(0)]\,|0\rangle\, \Delta'_{\text{av}}(k_2)_i^{-1})\, P_g(\gamma_i, k_2\delta_2)_2$$
$$- P_j(\gamma_i, k_1\delta_1)_1\, (\Delta'_{\text{ret}}(k_1)_i^{-1}\, \langle 0|\, R[\tilde{B}_{\gamma_i}(k_1), g^\mu(0)]\,|k_2\delta_2\rangle$$
$$+ \Delta'_{\text{av}}(k_1)_i^{-1}\, \langle 0|\, R[g^\mu(0), \tilde{B}_{\gamma_i}(k_1)]\,|k_2\delta_2\rangle)] + (g \leftrightarrow j)\Big\}, \quad (61)$$

$$D_{II}^\mu(k_1, k_2) =: -\frac{i}{2}\frac{1}{(2\pi)^{7/2}}\Big\{\int d^4u\, ds\, \frac{\delta(s-(u-k_1)^2)}{s-(u-k_2)^2}$$
$$\cdot \Big[\sum_i (k_1 + k_2 - 2u)^\mu\, (e_j(u, s/\gamma_i)_1 + e_g(u, s/\gamma_i)_2)$$
$$- (k_1 - k_2)^\lambda\, \{(k_1 + k_2 - 2u)^\mu\, (E_I^\lambda(u, s)_2 - E_{II}^\lambda(u, s)_1)$$
$$- (k_1 + k_2 - 2u)^\lambda\, (E_I^\mu(u, s)_2 - E_{II}^\mu(u, s)_1)\}\Big] + (1 \leftrightarrow 2)\Big\}. \quad (62)$$

Let us first consider D_I^μ. From the Eqs. (26), (27) we obtain in the usual normalisation of the matrix elements of B:

$$P_j(\gamma_i/k_1\delta_1)_1 = -\frac{1}{4\pi k_1^0}\left\langle \begin{matrix}\Delta_1 & M_1\\ \bar{\Delta}_1 & \boldsymbol{k}_1\end{matrix}\right|\, j^0(0)\, \left|\begin{matrix}M_1 & \Gamma_i\\ \boldsymbol{k}_1 & \bar{\Gamma}_i\end{matrix}\right\rangle \delta_{M_1, m_i}$$
$$P_j(\gamma_i/k_2\delta_2)_2 = -\frac{1}{4\pi k_2^0}\left\langle \begin{matrix}\Gamma_i & M_2\\ \bar{\Gamma}_i & \boldsymbol{k}_2\end{matrix}\right|\, j^0(0)\, \left|\begin{matrix}M_2 & \Delta_2\\ \boldsymbol{k}_2 & \bar{\Delta}_2\end{matrix}\right\rangle \delta_{M_2, m_i}. \quad (63)$$

As we already remarked in connection with the Eqs. (26), (27) the matrix elements on the right hand side are also diagonal in the spin quantum numbers. Furthermore if we use the fact that $j = j_\alpha$ and $g = j_\beta$ are members of the regular representation of $SU(2)$ then from the connection of j_α with the group generators, and translation invariance it follows that these matrix elements are also diagonal in the (main) quantum numbers Δ_r, Γ_i. Introducing the coupling constants $G_\Delta(M_r)$ via the Wigner-Eckart theorem [23] we obtain:

$$P_{j_\alpha}(\gamma_i, k_1\delta_1)_1 = \frac{1}{2\pi}\,\delta_{M_1, m_i}\,\delta_{\Delta_1, \Gamma_i}\, G_{\Delta_1}(M_1)\, (-1)^{\Delta_1 + \bar{\Delta}_1}\begin{pmatrix}\Delta_1 & 1 & \Gamma_i\\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma}_i\end{pmatrix}$$
$$P_{j_\alpha}(\gamma_i, k_2\delta_2)_2 = \frac{1}{2\pi}\,\delta_{M_2, m_i}\,\delta_{\Delta_2, \Gamma_i}\, G_{\Delta_2}(M_2)\, (-1)^{\Delta_2 + \bar{\Gamma}_i}\begin{pmatrix}\Gamma_i & 1 & \Delta_2\\ -\bar{\Gamma}_i & \alpha & \bar{\Delta}_2\end{pmatrix}. \quad (64)$$

By means of the L-S-Z-reduction formalism it follows:

$$\langle\delta_1 k_1|\, R[j_\alpha^\mu(0), \tilde{B}_{\bar{\gamma}_i}(-q)]\,|0\rangle\, \Delta'_{\text{ret}}(q)_i^{-1}\Big|_{\substack{q = k_2\\ m_i = M_2}}$$
$$= \langle\delta_1 k_1|\, R[\tilde{B}_{\bar{\gamma}_i}(-q), j_\alpha^\mu(0)]\,|0\rangle\, \Delta'_{\text{av}}(q)_i^{-1}\Big|_{\substack{q = k_2\\ m_i = M_2}}$$
$$= -\frac{1}{2\pi}\langle\delta_1 k_1|\, j_\alpha^\mu(0)\,|k_2\gamma_i\rangle. \quad (65)$$

Similar relations follow for the other retarded matrix elements in Eq. (61). If we define form factors by:

$$\left\langle \begin{matrix} \Delta_1 & M_1 \\ \bar{\Delta}_1 & \boldsymbol{k}_1 \end{matrix} \middle| j^\mu_\alpha(0) \middle| \begin{matrix} M_2 & \Gamma \\ \boldsymbol{k}_2 & \bar{\Gamma} \end{matrix} \right\rangle = (-1)^{\Delta_1 + \bar{\Delta}_1 + 1} \begin{pmatrix} \Delta_1 & 1 & \Gamma \\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma} \end{pmatrix} \tag{66}$$

$$\{(k_1 + k_2)^\mu F_{\Delta_1 \Gamma}(M_1, t, M_2)_I + (k_1 - k_2)^\mu F_{\Delta_1 \Gamma}(M_1, t, M_2)_{II}\}$$

with $t =: (k_1 - k_2)^2$,

then we finally get for $D^\mu_I(k_1, k_2)$:

$$D^\mu_I(k_1, k_2) = \frac{1}{8\pi^2} (-1)^{\Delta_1 + \bar{\Delta}_1} \{(k_1 + k_2)^\mu F_{\Delta_1 \Delta_2}(M_1, t, M_2)_I$$
$$+ (k_1 - k_2)^\mu F_{\Delta_1 \Delta_2}(M_1, t, M_2)_{II}\} \tag{67}$$
$$\cdot \sum_{\bar{\Gamma}} \left\{ G_{\Delta_1}(M_1) (-1)^{\Delta_1 + \bar{\Gamma}} \left[\begin{pmatrix} \Delta_1 & 1 & \Delta_1 \\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma} \end{pmatrix} \begin{pmatrix} \Delta_1 & 1 & \Delta_2 \\ -\bar{\Gamma} & \beta & \bar{\Delta}_2 \end{pmatrix} + (\alpha \leftrightarrow \beta) \right] \right.$$
$$\left. - G_{\Delta_2}(M_2) (-1)^{\Delta_2 + \bar{\Gamma}} \left[\begin{pmatrix} \Delta_1 & 1 & \Delta_2 \\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma} \end{pmatrix} \begin{pmatrix} \Delta_2 & 1 & \Delta_2 \\ -\bar{\Gamma} & \beta & \bar{\Delta}_2 \end{pmatrix} + (\alpha \leftrightarrow \beta) \right] \right\} .$$

However the commutator algebra (1) between iso-charges and current-densities implies that the sum over Γ in (75) vanishes. Furthermore from these equations it follows:

$$G_\Delta(M) = \frac{1}{(2\pi)^3} \sqrt{\Delta(\Delta + 1)(2\Delta + 1)} . \tag{68}$$

In most cases of physical interest D^μ_{II} vanishes identically because of the support properties of the Dyson integrand. For instance $\delta(s - (u - k_1)^2) \cdot e_j(u, s, \gamma)_1$ is a part of the Dyson spectral funciton of a commutator matrix element of the type $(q^2 - m^2) \langle \boldsymbol{k}_1 M_1 | [j(0), B_{\bar{\gamma}}(-q)] | 0 \rangle$. Let μ_1 and μ_2 be the masses of the lowest intermediate states with

$$(q^2 - m^2) \langle M_1 | j^\mu(0) | \mu_1 \rangle \langle \mu_1 | B_\gamma(-q) | 0 \rangle \neq 0$$
$$(q^2 - m^2) \langle M_1 | B_\gamma(-q) | \mu_2 \rangle \langle \mu_2 | j(0) | 0 \rangle \neq 0 .$$

If $\mu_1 > M_1$ or $\mu_2 > 2M_1$ then $\delta(s - (u - k_1)^2) e_j(u, s | \gamma)_1$ vanishes identically. This statement is a special case of a Lemma to be formulated in section V and proved in appendix I. Similar statements hold for all the other terms of D^μ_{II}. In the opposite case where D^μ_{II} is not a priori equal to zero we obtain by means of the Eq. (34) the following simple solution of $D^\mu_{II}(k_1, k_2) = 0$:

$$E^\lambda_I(u, s)_{3-r} - E^\lambda_{II}(u, s)_r = \frac{(k_r - k_{3-r})^\lambda}{(k_1 - k_2)^2}$$
$$\cdot \sum_i (e_j(u, s/\gamma_i)_r + e_g(u, s/\gamma_i)_{3-r}) . \tag{69}$$

After we have removed all the gradient terms we may now discuss the remaining proper part of the equal time commutator. Inserting the Eqs. (64)–(66) into (36) we obtain for the contributions of *all* discrete intermediate one particle states to the equal time limit:

$$\lim_{y^0 \to x^0} F^{00}(x, y)^{IG} = (2\pi)^3\, e^{i(k_1-k_2)x}\, \delta(\boldsymbol{x} - \boldsymbol{y})$$
$$\cdot \left\{ \frac{1}{2} [(k_1 + k_2)^0 F_{\Delta_1 \Delta_2}(M_1, t, M_2)_I + (k_1 - k_2)^0 F_{\Delta_1 \Delta_2}(M_1, t, M_2)_{II}] \right.$$
$$\cdot (-1)^{\Delta_1 + \bar{\Delta}_1} \sum_{\bar{\Gamma}} \left[G_{\Delta_1}(M_1)\, (-1)^{\Delta_1 + \bar{\Gamma}} \begin{pmatrix} \Delta_1 & 1 & \Delta_1 \\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma} \end{pmatrix} \cdot \begin{pmatrix} \Delta_1 & 1 & \Delta_2 \\ -\bar{\Gamma} & \beta & \bar{\Delta}_2 \end{pmatrix} \right. \tag{70}$$
$$\left. + G_{\Delta_2}(M_2)\, (-1)^{\Delta_2 + \bar{\Gamma}} \begin{pmatrix} \Delta_1 & 1 & \Delta_2 \\ -\bar{\Delta}_1 & \alpha & \bar{\Gamma} \end{pmatrix} \cdot \begin{pmatrix} \Delta_2 & 1 & \Delta_2 \\ -\bar{\Gamma} & \beta & \bar{\Delta}_2 \end{pmatrix} - (\alpha \leftrightarrow \beta) \right]$$
$$\left. + (2\pi)^2 \sum_i \bar{A}^0_{j\alpha j\beta}(k_1, k_2, \gamma_i) \right\} = i\, \varepsilon^{\alpha\beta\gamma} \left\langle \begin{matrix} \Delta_1 & M_1 \\ \bar{\Delta}_1 & \boldsymbol{k}_1 \end{matrix} \right| j^0_\gamma(0) \left| \begin{matrix} M_2 & \Delta_2 \\ \boldsymbol{k}_2 & \bar{\Delta}_2 \end{matrix} \right\rangle e^{i(k_1-k_2)x}\, \delta(\boldsymbol{x} - \boldsymbol{y}) \,.$$

If we integrate this equation over the space coordinates x or y then consistency with the algebra of one charge and one density (1) requires the condition (68) on the coupling constants and:

$$\sum_i \bar{A}^0_{j\alpha j\beta}(k_1, k_2/\gamma_i) = 0 \,. \tag{71}$$

According to Eq. (39) $\bar{A}^0$ has the same structure as D^μ_{II}. Therefore all the remarks above are also valid for $\bar{A}^0$. Exactly the same argument applied above for the solution of $D^\mu_{II} = 0$ leads to the following solution of (71):

$$\frac{E^\lambda_I(u, s)_{3-r} + E^\lambda_{II}(u, s)_r}{s - (u - k_{3-r})^2} = - \frac{(k_r - k_{3-r})^\lambda}{(k_1 - k_2)^2}$$
$$\cdot \left\{ \frac{1}{s - (u - k_{3-r})^2} \sum_i (e_{j\alpha}(u, s/\gamma_i)_r - e_{j\beta}(u, s/\gamma_i)_{3-r}) \right. \tag{72}$$
$$\left. + \delta(s - (u - k_{3-r})^2)\, (E(u, s)_1 + E(u, s)_2) \right\} .$$

The equation together with (69) specifies the spectral functions $E^\lambda_{I,II}(u, s)_r$ completely.

Summing up the results obtained so far we may make the following statements in the case of (0, 0)-*components of conserved* currents:

Equal time density commutation relations without gradient terms together with saturation by a finite number of intermediate one particle structures are completely consistent with general principles of local quantum field theory, if the one particle structures are constructed in such a way that they satisfy themselves all these principles (DCOP-structures). Therefore we may conclude that all kinematical difficulties occuring in other approaches to this problem, for instance the k-dependence in the naive one particle approximation [8], simply reflect the fact that one of these principles (locality!) is destroyed by the approximation.

Furthermore from all *discrete* intermediate one particle states the equal time limiting procedure picks out only those states which have the same main quantum numbers (mass, tolat isospin, total spin) as the external states. This is due to the fact that the main contributions to the equal time limit come from the terms which must be added to the local one particle structure F^I in order to save current conservation. These terms are proportional to the matrix elements $P_j(\gamma_i, k_r \delta_r)_r$ (18). By Ward's identity these matrix elements are diagonal in the main quantum numbers.

However there exist physical examples [24] (external states are π's and intermediate states ω, ϕ) where these terms do not occur because the local one particle structure F^I itself is divergence free. In the next section we will prove that *all* intermediate states, discrete one particle states as well as continuous many particle states, with masses larger than both masses of the external states can at most contribute to the gradient terms of the equal time limit. Therefore the ω- and ϕ-contributions to the commutator matrix elements between pion states drop out of the proper non gradient term of the equal time limit[8]. This is in complete agreement with the results obtained by Bardakci, Halpern and Segré [25] from perturbation theory and in disagreement with the dispersion methods [24, 25].

A further consequence of this pick-out-mechanism is that the density commutator algebras do *not* lead to any predictions beyond those which can be already obtained from the algebra of one charge and one density. The density algebras for conserved currents are only consistency requirements restricting at most the structure of the separately local and divergence free remainder of the matrix element.

For instance the arbitrary spectral functions $Z^{\mu\nu}$ which we left open at the beginning of our calculations are determined in part by the removal of the gradient terms. Similarly the spectral functions $E^{\lambda}_{I,II}$ are fixed in part by this. The remaining part of $E^{\lambda}_{I,II}$ is then determined by the consistency requirement for the proper part of the equal time limit with the algebra of one charge and one density (1).

V. The Equal Time Limit of the Rest

We have split the truncated current commutator matrix elements into two parts:

$$\langle k_1 M_1| \, [j^\mu(x), g^\nu(y)] \, |M_2 k_2\rangle^T = F^{\mu\nu}(x, y)^{I\,G} + F^{\mu\nu}(x, y)^{II\,G} \qquad (73)$$

where both parts have all the properties which follow from general principles of quantum field theory and current conservation. Furthermore $F^{\mu\nu}(x, y)^{I\,G}$ is explicitely given by products of three point functions (DCOP-structures).

In this section we adopt the following bookkeeping:

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[8] Even if the mass of the ω and ϕ would be smaller or equal to the mass of the π this remains true. This follows from the general structure of the local one particle term F^I corresponding to these intermediate states.

$F^{\mu\nu}(x, y)^{IG}$ contains all DCOP-structures constructed from one particle intermediate states with masses smaller or equal to the maximum of M_1 and M_2.

$F^{\mu\nu}(x, y)^{IIG}$ contains all other discrete one particle and continuous many particle contributions to the commutator matrix elements.

We want to prove from current conservation that $F^{00}(x, y)^{IIG}$ *does not contribute to the proper non gradient terms* of the equal time limit provided the threshold of the continuous many particle intermediate states is larger than the maximum of M_1 and M_2. In these cases $F^{00}(x, y)^{IIG}$ can at most contribute to the gradient terms.

By translation invariance $F^{\mu\nu}(x, y)^{IIG}$ is of the form:

$$F^{\mu\nu}(x, y)^{IIG} = e^{i\Delta(x+y)} \cdot \frac{1}{(2\pi)^{3/2}} \int d^4 q \, e^{-iq(x-y)} \tilde{F}^{\mu\nu}(\Delta, q, P)^{IIG}, \tag{74}$$

$$\Delta =: \frac{k_1 - k_2}{2}, \quad P =: \frac{k_1 + k_2}{2}. \tag{75}$$

From current conservation we obtain the two conditions:

$$(q - \Delta)_\mu \tilde{F}^{\mu\nu}(\Delta, q, p)^{IIG} = 0, \tag{76a}$$

$$(q + \Delta)_\nu \tilde{F}^{\mu\nu}(\Delta, q, p)^{IIG} = 0. \tag{76b}$$

As it was shown in [17] the most general solution of (76a) is:

$$\begin{aligned} \tilde{F}^{\mu\nu}(\Delta, q, P)^{IIG} = \int d^4u \, ds \, \varepsilon(q^0 - u^0) \, \delta((q-u)^2 - s) \\ \Big\{(q + \Delta - 2u)^\mu \Big[\frac{\varrho^\nu(u, s)}{s - (u - \Delta)^2} + \delta(s - (u - \Delta)^2) \, E^\nu(u, s) \\ + (q - \Delta)_\lambda \frac{\partial}{\partial s} \frac{\psi^{\lambda\nu}(u, s)}{s - (u - \Delta)^2}\Big] - (s - (u - \Delta)^2) \frac{\partial}{\partial s} \frac{\psi^{\mu\nu}(u, s)}{s - (u - \Delta)^2}\Big\} \end{aligned} \tag{77}$$

$\varrho^\nu(u, s)$ satisfies condition (22).

Furthermore all the spectral functions have the well known support properties [20–21] uniquely given by the support properties of $F^{\mu\nu}(x, y)^{IIG}$ in momentum space. From these support properties we prove in appendix I the following lemma:

Lemma. *If the mass of the lowest discrete or continuous state in the commutator $F^{\mu\nu}(x, y)^{IIG}$ is larger than the maximum of the masses M_1, M_2 of the external states then the intersection of the support of $E(u, s)$ and the δ-functions $\delta(s - (u \pm \Delta)^2)$ is empty.*

Therefore according to our bookkeeping the δ-function term in (77) vanishes. The remaining part of (77) can be rewritten in the following way:

$$\begin{aligned} \tilde{F}^{\mu\nu}(\Delta, q, P)^{IIG} = \int d^4u \, ds \, \varepsilon(q^0 - u^0) \, \delta((q-u)^2 - s) \cdot \Big\{\frac{(q + \Delta - 2u)^\mu}{s - (u - \Delta)^2} \varrho^\nu(u, s) \\ + (q - \Delta)_\lambda \Big[(q + \Delta - 2u)^\mu \frac{\partial}{\partial s} \frac{\psi^{\lambda\nu}(u, s)}{s - (u - \Delta)^2} - (\lambda \leftrightarrow \mu)\Big]\Big\}. \end{aligned} \tag{78}$$

The first term in this equation occurs only because the Dyson representations are non unique[9]. From condition (22) it follows that it vanishes in the equal time limit. According to the antisymmetry of the cornered bracket in λ and μ we obtain from (74) and (78) for the equal time limit:

$$\lim_{y^0 \to x^0} F^{00}(x, y)^{II\,G} = \frac{1}{(2\pi)^{3/2}}\, e^{2i\Delta^0 x^0} \int d^3q\,(q - \Delta)_k\, e^{+i(\boldsymbol{q}-\boldsymbol{\Delta})\boldsymbol{x} - i(\boldsymbol{q}+\boldsymbol{\Delta})\boldsymbol{y}}$$

$$\cdot \lim_{y^0 \to x^0} \int dq^0\, e^{-i(x^0 - y^0)q^0} f^{k;00}(\Delta, q, P)\,. \tag{79}$$

In exactly the same manner we obtain from the most general solution of (76b):

$$\lim_{y_0 \to x^0} F^{00}(x, y)^{II\,G} = \frac{1}{(2\pi)^{3/2}}\, e^{2i\Delta^0 x^0} \cdot \int d^3q\,(q + \Delta)_l\, e^{i(\boldsymbol{q}-\boldsymbol{\Delta})\boldsymbol{x} - i(\boldsymbol{q}+\boldsymbol{\Delta})\boldsymbol{y}}$$

$$\cdot \lim_{y^0 \to x^0} \int dq^0\, e^{-i(x^0 - y^0)q^0} g^{l;00}(\Delta, q, P)\,. \tag{80}$$

Each of the limits occuring on the right hand sides of (79) and (80) is because of locality at most a polynomial in $\boldsymbol{q}$. Hence, from these two equations we deduce:

$$\lim_{y^0 \to x^0} F^{00}(x, y)^{II\,G} = \frac{1}{(2\pi)^{3/2}}\, e^{2i\Delta^0 x^0} \Big\{ (\partial_{(x)\,r} - \partial_{(y)\,r})\, h^{0,r}(\Delta, P) \tag{81}$$

$$e^{-i\Delta(\boldsymbol{x}+\boldsymbol{y})}\, \delta(\boldsymbol{x} - \boldsymbol{y}) + \partial_{(x)\,k}\, \partial_{(y)\,l} \Big[e^{-i\Delta(\boldsymbol{x}+\boldsymbol{y})} \lim_{y^0 \to x^0} H^{k,l;00}(\Delta, x - y, P) \Big] \Big\}$$

with

$$\Delta_r h^{0,r}(\Delta, P) = 0\,. \tag{82}$$

By locality the limit on the right hand side of (81) is a finite sum of δ-functions and their derivatives[10]. We want to point that we did not need for our proof any other high energy assumptions than the existence of the limit[9].

An interesting question is if these gradient terms really do occur or not. It is easy to construct sufficient conditions for their absence in form of high energy assumptions or special structure properties for the explicit solutions of the Eq. (76a–b) [17] similar to the considerations of section IV. However, in contrast to the one particle structure, we do not know enough about the many particle structures [12–13] in order to see if these conditions are consistent or not. Therefore, we do not write them down here.

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[9] In most cases the support in momentum space is symmetric after splitting off the one particle contributions. Then we can use the unique JOST-LEHMANN representation [19] and the term containing ϱ does not appear [16].

[10] In writing down explicitly the most general gradient term in (81) one should make use of the identity $[i\Delta_r + 1/2\,(\partial_{(x)r} + \partial_{(y)r})]\, e^{-i\Delta(\boldsymbol{x}+\boldsymbol{y})}\, \delta(\boldsymbol{x} - \boldsymbol{y}) = 0$. This allows to reduce many of the higher gradient terms to lower ones.

VI. Final Remarks

In the proof of the statement that the proper (non gradient) part of the equal time limit is exactly given by the contributions of the DCOP-structure we had only to assume that the limits of this structure and the so called rest exist seperately[9]. No assumptions on the definite behaviour at high energies or an interchanges of integrals and limits were needed. On the other hand for the explicit calculation of the equal time limit of the DCOP-structures we needed further (sufficient) assumptions (section III) which at a first glance seem to be very strong. One can probably weaken these conditions with the effect that new gradient terms emerge. The relevant discussion should be similar to the discussion on the connection of large s behaviour and occurrence of additional gradient terms as given by MEYER and SUURA [26] and STICHEL and SCHROER [27].

It would be interesting to see if all our results for the time-time-components remain true for space-time-or even space-space-components of conserved currents. A further interesting problem is the investigation of algebras containing non conserved currents (for instance $SU(2) \times SU(2)$) by our methods. Work on these problems is in progress.

Acknowledgement. We want to thank Professor P. STICHEL for his constant encouragement, for helpful discussions and critical remarks.

Appendix I

Proof of the lemma of section V:
From the support in momentum space we have [20–21]

$$\operatorname{supp}(E(u,s)) = \{(u,s): (p+u) \in V_+,\ (p-u) \in V_+,\ \sqrt{s} = \max(0, \mu_1 - \sqrt{(p+u)^2}, \mu_2 - \sqrt{(p-u)^2}\}, \tag{A1}$$

(μ_1, μ_2 are the masses of the lowest intermediate states in the commutator $F^{\mu\nu}(x,y)^{IIG}$).

We look for necessary conditions that the support of $\delta(s-(u \pm \Delta)^2) \cdot E(u,s)$ be not empty, that is for a solution u of the conditions

$$p + u \in V_+ \wedge p - u \in V_+ \wedge (u \pm \Delta)^2 \geqq 0, \tag{A2}$$

$$\wedge\ (u \pm \Delta)^2 \geqq \max\left(0, \mu_1 - \sqrt{(p+u)^2}, \mu_2 - \sqrt{(p-u)^2}\right). \tag{A3}$$

Let us first consider the positive sign $(u + \Delta)$:

We have to distinguish three cases according to the different shapes of relation (A3):

$$1.\ \sqrt{(u+\Delta)^2} \geqq 0 \wedge \mu_1 \leqq \sqrt{(p+u)^2} \wedge \mu_2 \leqq \sqrt{(p-u)^2} \tag{A4}$$

$$2.\ \sqrt{(u+\Delta)^2} \geqq \mu_1 - \sqrt{(p+u)^2} \geqq 0 \wedge \mu_1 - \sqrt{(p+u)^2} \geqq \mu_2 - \sqrt{(p-u)^2} \tag{A5}$$

$$3.\ \sqrt{(u+\Delta)^2} \geqq \mu_2 - \sqrt{(p-u)^2} \geqq 0 \wedge \mu_2 - \sqrt{(p-u)^2} \geqq \mu_1 - \sqrt{(p+u)^2} \tag{A6}$$

where $p = \frac{k_1 + k_2}{2}$, $\Delta = \frac{k_1 - k_2}{2}$, $k_r^2 = M_r^2$, $r = 1, 2$.

Let us consider case 1.:

1.a) $$(u + \Delta)_0 \geqq 0 .$$

Then from (A4) it follows

$$\Delta_0 + u_0 \geqq |\boldsymbol{\Delta} + \boldsymbol{u}| \wedge \mu_2^2 + |\boldsymbol{p} - \boldsymbol{u}|^2 \leqq (p_0 - u_0)^2 ,$$

or

$$\mu_2^2 + |\boldsymbol{p} - \boldsymbol{u}|^2 \leqq (p_0 + \Delta_0 - |\boldsymbol{\Delta} + \boldsymbol{u}|)^2 ,$$

and after some calculation

$$\mu_2^2 \leqq M_1^2 - 2|\boldsymbol{\Delta} + \boldsymbol{u}| \times (k_1^0 - \cos\alpha\, |\boldsymbol{k}_1|) , \tag{A7}$$

where

$$\cos\alpha = \frac{\boldsymbol{k}_1(\boldsymbol{\Delta} + \boldsymbol{u})}{|\boldsymbol{k}_1|\, |\boldsymbol{\Delta} + \boldsymbol{u}|} .$$

From (A7) and $k_1^0 - \cos\alpha\, |\boldsymbol{k}_1| \geqq 0$ we have

$$\mu_2^2 \leqq M_1^2 . \tag{A8}$$

For 1.b) $(u + \Delta)_0 \leqq 0$ we get from an analogue calculation

$$\mu_1^2 \leqq M_2^2 . \tag{A9}$$

Therefore we have the necessary condition for a solution of (A4)

$$\mu_1 \leqq M_2 \vee \mu_2 \leqq M_1 . \tag{A10}$$

Now we study case 2.:

From (A5) it follows

$$\mu_1 \leqq \sqrt{(p + u)^2} + \sqrt{(\Delta + u)^2} \vee \mu_2 \leqq \sqrt{(p - u)^2} + \sqrt{(\Delta + u)^2} . \tag{A11}$$

We define the two functions

$$f_\pm(u) =: \sqrt{(p \pm u)^2} + \sqrt{(\Delta + u)^2} . \tag{A12}$$

An investigation of these functions shows, that

$$f_+(u) \leqq M_2 \quad \text{for } (u + \Delta)_0 \leqq 0 \tag{A13}$$
$$f_-(u) \leqq M_1 \quad \text{for } (u + \Delta)_0 \geqq 0 .$$

From (A11) and (A13) we get the necessary condition for a solution of (A5)

$$\mu_1 \leqq M_2 \vee \mu_2 \leqq M_1 . \tag{A14}$$

As for case 3. we also get relation (A11) from (A6) and therewith (A14). The case $(u - \Delta)$ in (A2), (A3) can be reduced to the above considered case by changing $\Delta \to -\Delta$, that means $k_1 \leftrightarrow k_2$ and $M_1 \leftrightarrow M_2$.

We conclude:

A necessary condition for a solution of (A2), (A3) is

$$\mu_1 \leqq \underset{1}{M_2} \vee \mu_2 \leqq \underset{2}{M_1} .$$

In other words:

$\mathrm{supp}\big(\delta(s-(u+\Delta)^2)\,E(u,s)\big)$ and $\mathrm{supp}\big(\delta(s-(u-\Delta)^2)\,E(u,s)\big)$ are both empty, if

$$\mu_1 > \max(M_1, M_2) \quad \text{and} \quad \mu_2 > \max(M_1, M_2)\,.$$

Note added in proof. After this work was completed, we learned that DIETZ and KUPSCH [28] have proved the analogue of our result in Section V for the special case of zero momentum transfer ($\Delta = 0$).

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