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# Quasi-Free States of the C.C.R. — Algebra and Bogoliubov Transformations*

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**Abstract.** We give a complete characterization of quasi-free states (generalized free states) of the C.C.R. algebra. We prove that the pure quasi-free states areall Fock states and that any two Fock states are related through a symplectic automorphism (Bogoliubov transformation). We make an explicit construction of these representations which correspond to primary quasi-free states.

## I. Introduction

In this work we study the set of quasi-free states on the C.C.R. algebra. The notion of quasi-free states is introduced by D. W. ROBINSON [1] in his study of the ground state of the Bose gas. Until now, one was not able to construct exactly solvable physical models, whose solutions do not belong to the set of quasi-free states. It is interesting to study this set of states in order to derive its most general properties hoping that their general properties may throw some light on the problem of construction of non-trivial models.

From a technical point of view, we start with a symplectic space $(H, \sigma)$ and consider the C.C.R. $C^*$-algebra $\overline{\Delta(H, \sigma)}$ [2] built on it. We prove that the pure quasi-free states are all Fock states and that any pure quasi-free state can be obtained from another pure quasi-free state by acting on it through an automorphism of the algebra induced by a symplectic operator on $(H, \sigma)$. The converse statement is well known by physicists as Bogoliubov transformations. Explicit representations induced by quasi-free states of C.C.R. are given. Amongst all representations we characterize the primary ones. The last property turned to be important to characterize physical systems in statistical mechanics [3]. This property was outlined by ARAKI and WOODS [4] for the temperature states of the free Bose gas which are quasi-free states.

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In section II we recall the definition and some properties of our basic $C^*$-algebra $\overline{\Delta(H,\sigma)}$, and collect the mathematical tools we need in section III for the treatment of quasi-free states.

## II. Mathematical Preliminaries

### *II.1 The C.C.R. algebra* $\overline{\Delta(H,\sigma)}$

For completeness we recall the definition of the C.C.R. $C^*$-algebra $\overline{\Delta(H,\sigma)}$. More details can be found in ref. [2].

Let $H$ be a real vector space and $\sigma$ a symplectic form on $H$ (i.e. $\sigma$ is a bilinear, antisymmetric, regular mapping from $H \times H$ into $R$). We denote by $\delta_\psi$ the real function on $H$ defined by $\delta_\psi(\varphi) = 0$ if $\psi \neq \varphi$ and $\delta_\psi(\psi) = 1$. The product of $\delta_\psi$ with $\delta_\varphi$ is defined by

$$\delta_\psi \delta_\varphi = e^{-i\sigma(\psi,\varphi)} \delta_{\psi+\varphi}$$

and we consider the complex algebra $\Delta(H,\sigma)$ generated by the $\delta_\psi$'s for all $\psi \in H$; equiped with the involution $\delta_\psi \to (\delta_\psi)^* = \delta_{-\psi}$ the algebra $\Delta(H,\sigma)$ becomes a $*$-algebra.

The set $\mathscr{R}(H,\sigma)$ of representations of the C.C.R. is the set of representations $\pi$ of $\Delta(H,\sigma)$ such that the mapping $\lambda \in R \to \pi(\delta_{\lambda\psi})$ is a weakly continous mapping from $R$ into $\mathscr{L}(\mathscr{H}_\pi)$ for all $\psi \in H$. All these representations induce the same norm $\|\cdot\|$ on $\Delta(H,\sigma)$ (i.e. $\forall\, a \in \Delta(H,\sigma) : \|a\| = \|\pi(a)\|$). The completion of $\Delta(H,\sigma)$ with respect to this norm is the $C^*$-algebra $\overline{\Delta(H,\sigma)}$, isomorphic with the $C^*$-algebra generated by the Weyl operators $e^{iB(\psi)}$ where $B(\psi)$ are the field operators.

Let $\mathscr{I}$ be the set of all functions $f$ mapping $H$ into $C$ and satisfying the condition $\sum_{k,j=1}^{n} \bar{a}_k a_j e^{i\sigma(\psi_k,\psi_j)} f(\psi_j - \psi_k) \geqq 0$ for all $a_k \in C$, $\psi_k \in H$, $k \in \{1, \ldots, n\}$ and $n \in N$.

**Proposition 1.** *$\omega$ is a positive linear form on $\Delta(H,\sigma)$ if and only if the function $f$, defined by $f(\psi) = \omega(\delta_\psi)$ for all $\psi \in H$, belongs to $\mathscr{I}$.*

*Proof.* See ([2], 3.2.1.).

Under these conditions $\omega$ is denoted by $\omega_f$ and the representation induced by $\omega_f$ through the construction of Gelfand-Naimark is denoted by $\pi_f$ or $\pi_{\omega_f}$.

**Proposition 2.** *Let $f \in \mathscr{I}$, it is necessary and sufficient, in order that $\pi_f \in \mathscr{R}(H,\sigma)$ that the mapping $\lambda \in R \to f(\lambda\psi + \varphi)$ be continous for all $\psi, \varphi \in H$.*

*Proof.* See ([2], 3.2.2).

We denote by $\mathscr{I}_0$ the set of all elements $f \in \mathscr{I}$ such that $\pi_f \in \mathscr{R}(H,\sigma)$.

A symplectic operator $T$ on $(H,\sigma)$ is an operator from $H$ onto $H$ satisfying $\sigma(T\psi, T\varphi) = \sigma(\psi,\varphi)$ for $\psi, \varphi \in H$; let $S(H,\sigma)$ be the group of symplectic operators on $(H,\sigma)$.

**Proposition 3.** *For every $T \in S(H, \sigma)$, the mapping $\tau_T: \delta_\psi \to \delta_{T\psi}$ can be extended to a unique automorphism of $\overline{\Delta(H, \sigma)}$.*

*Proof.* See ([2], 4.1.1).

**Proposition 4.** *Let $H^*$ be the algebraic dual of $H$. For every $\chi \in H^*$ the mapping $\zeta_\chi: \delta_\psi \to e^{i\chi(\psi)}\delta_\psi$ can be extended to a unique automorphism of $\overline{\Delta(H, \sigma)}$.*

*Proof.* See ([2], 4.4.1).

**Proposition 5.** *Let $\pi$ be a cyclic element of $\mathscr{R}(H, \sigma)$, and $s$ a seal scalar product on $H$, such that $\sigma$ is $s$-norm continous and has a continous regular extension $\sigma'$ to the $s$-norm closure $\overline{H}^s$ of $H$. If the mapping $\psi \in H \to \pi(\delta_\psi)$ is $s$-weakly continous, $\pi$ has a unique continous cyclic extension $\pi'$ to $\overline{\Delta(\overline{H}^s, \sigma')}$ and for every $\psi \in \overline{H}^s$, $\pi'(\delta_\psi) \in \pi(\Delta(H, \sigma))''$.*

The proof is an immediate extension of the proof of ([2], 3.3.4).

**Corollary.** *With the same notations as in prop. 5, $\pi$ is irreducible (primary) if and only if $\pi'$ is irreducible (primary).*

### *II.2 Real Scalar Products on $(H, \sigma)$*

We consider the set $\mathscr{S}$ of bilinear, symmetric, positive forms $s$ mapping $H \times H$ into $R$ such that

a) $|\sigma(\psi, \varphi)| \leqq s(\psi, \psi)^{1/2} s(\varphi, \varphi)^{1/2}$ (implying that $s$ is a scalar product).

b) The norm continous extension $\sigma'$ of $\sigma$ to $\overline{H}^s$ is non degenerate.

Each $s \in \mathscr{S}$ induces on $H$ the $s(H)$-weak topology (the weak dual is denoted by $s(H) = \{s_\psi | \psi \in H, s_\psi(\varphi) = s(\psi, \varphi)\}$) and the $s$-norm topology (the norm dual is denoted $H'_s$).

**Proposition 6.** *If $s \in \mathscr{S}$ and $H$ is $s(H)$-quasi-complete, then $H$ is $s$-norm complete (we have $H = \overline{H}^s$ or $(H, s)$ is a real Hilbert space).*

*Proof.* By definition $H \subset \overline{H}^s$. We show that $\overline{H}^s \subset H$. If $\psi_0 \in \overline{H}^s$, then there exists a Cauchy sequence $(\psi_n)_n$ in $H$, norm converging to $\psi_0$. Using Schwartzs inequality $(\psi_n)_n$ converges weakly in $\overline{H}^s$ to $\psi_0$:

$$s(\psi_n, \varphi) \to s(\psi_0, \varphi) \quad \text{for all } \varphi \in \overline{H}^s .$$

As $H$ is quasi-complete, there exists a $\varphi_0 \in H$ such that

$$s(\psi_n, \varphi) \to s(\varphi_0, \varphi) \quad \text{for all } \varphi \in H .$$

Therefore $s(\varphi_0 - \psi_0, \varphi) = 0$ for all $\varphi \in H$. Noting that $H$ is strongly dense in $\overline{H}^s$ we have that $\psi_0 = \varphi_0 \in H$.

We denote by $\sigma(H)$ the set $\{\sigma_\psi | \psi \in H, \sigma_\psi(\varphi) = \sigma(\psi, \varphi)\}$. The vector space $H$ equipped with the $\sigma(H)$-weak topology is a Haussdorff topological vector space (the weak dual is $\sigma(H)$), because $\sigma$ is supposed to be non degenerate.

Let $A$ be a $\sigma(H)$-weakly continous linear operator of $H$, then there exists a unique linear continuous operator $A^+$ of $H$ such that $\sigma(\psi, A\varphi) = \sigma(A^+\psi, \varphi)$ for all $\psi, \varphi \in H$ ([5], p. 419); $A^+$ is called the adjoint of $A$ with respect to $\sigma$.

**Proposition 7.** *Let $s \in \mathcal{S}$ and let $D_s$ be the linear operator $s^{-1}\sigma$ mapping $\sigma^{-1}(\sigma(H) \cap s(H))$ onto $s^{-1}(\sigma(H) \cap s(H))$ then*

1⁰) $\|D_s\|_s \leqq \|\sigma\|_s \leqq 1 \left(\|\sigma\|_s = \sup_{\|\varphi\|_s = \|\psi\|_s = 1} |\sigma(\psi, \varphi)|\right)$.

2⁰) If $\overline{H}^s = H$ we have $\sigma(H) \subset s(H)$ and $\begin{cases} \overline{D_s H^s} = H \\ \|D_s\|_s = \|\sigma\|_s \\ D_s^+ = -D_s \end{cases}$

3⁰) *If $\overline{H}^s = H$ and $\sigma(H) = s(H)$ we have that $A_s = -D_s^{-1}$ is bounded and $A_s^+ A_s \geqq 1$.*

*Proof.* Noting that $D_s$ is a mapping from a normed vector space into another one and that

$$\frac{\|D_s\psi\|_s}{\|\psi\|_s} = \frac{s(D_s\psi, D_s\psi)}{\|\psi\|_s\|D_s\psi\|_s} = \frac{|\sigma(\psi, D_s\psi)|}{\|\psi\|_s\|D_s\psi\|_s} \leqq \|\sigma\|_s$$

for all $\psi \in H$, we prove 1⁰).

It follows from the fact that $D_s$ is injective, normal $(\sigma(\psi, D_s\psi) = -\sigma(D_s\psi_s, \varphi))$ and every where defined on $H$ that $\overline{D_s H^s} = H$ and $\sigma = s \circ D_s$.

Therefore

$$\|\sigma\|_s = \sup_{\psi, \psi'} \frac{|s(D_s\psi, \psi')|}{\|\psi\|_s\|\psi'\|_s} \leqq \sup_{\psi \in H} \frac{\|D_s\psi\|_s}{\|\psi\|_s} = \|D_s\|_s$$

which proves 2⁰).

If $\overline{H}^s = H$ and $\sigma(H) = s(H)$ then $D_s$ is a one-to-one mapping from $H$ onto $H$. Consequently $A_s = -D_s^{-1}$ is a bounded operator and $A_s^+ A_s \geqq 1$ ([6], § 4, th. VI).

A complex Hilbert structure ([8], p. 28—29) on $(H, \sigma)$ is given by an operator $J$ on $H$ satisfying $J^2 = -1$, $J^+ = -J$ and $s_J = -\sigma \circ J \geqq 0$. It follows from prop. 6 that $s_J \in \mathcal{S}$ and that $(H, s_J + i\sigma)$ is a complex Hilbert space.

Consider an element $s \in \mathcal{S}$ and suppose that $\overline{H}^s = H$, it follows from prop 7 that $A_s = D_s^{-1}$ is a normal operator defined on a dense domain of $H$. Let $D_s = J|D_s|$ be the polar decomposition of $D_s$ then $[J, |D_s|]_- = 0$, $J^+ = -J$, $J^2 = -1$ ([5], part II, p. 935). The operator $J$ defines a complex structure on $(H, \sigma)$, because the range of $|D_s|$ is dense in $H$ and

$$s_J(|D_s|\psi, |D_s|\psi) = -\sigma(J|D_s|\psi, |D_s|\psi) = -\sigma(D_s\psi, |D_s|\psi)$$
$$= s(D_s\psi, |D_s|D_s\psi) \geqq 0 \quad \text{and} \quad \|s_J\|_s = \|\sigma\|_s .$$

From the polar decomposition of $D_s$ and $A_s = -D_s^{-1}$ and the uniqueness of the polar decomposition ([5], part II, sect XII, 7) it follows that $A_s = J|A_s|$ where $|A_s| = |D_s|^{-1} \geqq 0$.

For every Hilbert space, in particular for $(\overline{H}^{s_J}, s_J + i\sigma)$, there exists at least one conjugation $\Lambda$ (i.e. $[\Lambda, J]_+ = 0, \Lambda^2 = 1$) ([7], prof. A1). Now we prove.

**Proposition 8.** *For all* $s \in \mathcal{S}$ *verifying* $H = \overline{H}^s$, *a conjugation* $\Lambda$ *can be found on* $(\overline{H}^{s_J}, s_J + i\sigma)$ *such that* $[\Lambda, |A_s|]_- = 0$.

*Proof.* Let us firstly notice that:

$$\|D_s\|_{s_J} = \sup_{\psi \in H} \frac{s_J(D_s\psi, D_s\psi)}{s_J(\psi, \psi)} = \sup_{\psi \in H} \frac{s(D_s|D_s|^{1/2}\psi, D_s|D_s|^{1/2}\psi)}{s(|D_s|^{1/2}\psi, |D_s|^{1/2}\psi)} = \|D_s\|_s .$$

The last equality follows, since the range of $D_s$ is dense in $H$. Thus $D_s$ has a unique continuous extension (denoted as $D_s$) to $\overline{H}^{s_J}$. Furthermore, for any $\psi \in H$, $\|\psi\|_{s_J} = \|\, |D_s|^{1/2}\psi\|_s \leqq \|\psi\|_s$, and the conditions to get a Friedricks extension ([9], n° 124), are satisfied. The range of $D_s$ is known to be included in $H$. The following formula is satisfied in $\overline{H}^{s_J}$,

$$|D_s| = \int_0^{\|D_s\|} \lambda dE(\lambda) .$$

We get $\overline{H}^{s_J} = \int_{\oplus 0}^{\|D_s\|} H_\lambda d\lambda$, so, let $\Lambda_\lambda$ any conjugation in $H_\lambda$, then it readily follows that $\Lambda = \int_\oplus \Lambda_\lambda d\lambda$ is a conjugation in $(\overline{H}^{s_J}, s_J + i\sigma)$, commuting with $|D_s|$; consequently $\Lambda$ commutes with $|A_s|$ also.

Remark that the operators $A$ and $B$ are said to commute if they commute on their common domain. It follows from prop. 8 that $\Lambda$ commutes with $(|A_s| \pm 1)^{1/2}$ which exists because $|A_s| \geqq 1$.

**Proposition 9.** *Let* $H$ *be* $\sigma(H)$*-quasi-complete and* $J_1$, $J_2$ *be operators on* $H$ *defining a complex Hilbert structure on* $(H, \sigma)$, *then there exists an operator* $T \in S(H, \sigma)$ *such that* $J_1 = T^+ J_2 T$.

The proof of this proposition is completely analogous to that of ([7], lemma 1).

## III. Quasi-free States

### *III.1 Definitions*

Let $f$ be a mapping from $H$ into $C$ such that $f(0) = 1$, then $f$ is called quasi-free if

$$f(\psi) = \exp\left\{f_0'(\psi) + \frac{1}{2} f_T''(\psi, \psi)\right\}$$

where

$$f_T''(\psi, \psi) = f_0''(\psi, \psi) - f_0'(\psi)^2$$

$$f_\varphi'(\psi) = \lim_{\lambda \to 0} \frac{f(\varphi + \lambda\psi) - f(\varphi)}{\lambda}$$

and consider the mapping $\varphi \in H \to f_\varphi'(\psi) \in C$ then

$$f_\varphi''(\psi_1, \psi_2) = \lim_{\lambda \to 0} \frac{f_{\varphi+\lambda\psi_2}'(\psi_1) - f_\varphi'(\psi_1)}{\lambda} .$$

A quasi-free mapping is therefore at least twice differentiable in the above sense.

Suppose that $f$ is a quasi-free mapping, we define a linear form $\omega_f$ on $\overline{\Delta(H,\sigma)}$ by $\omega_f(\delta_\psi) = f(\psi)$. From proposition 1 it follows that $\omega_f$ is a state if and only if $f \in \mathscr{J}$. Remark that for quasi-free mappings $f \in \mathscr{J}$ implies $f \in \mathscr{J}_0$. Under these conditions $\omega_f$ is called *a quasi-free state* and we denote by $Q$ the set of quasi-free states. Let us remark that this definition of quasi-free states coincides with that of D. W. ROBINSON [1], $f_0'$ and $f_0''$ are in a trivial way related to the one-point and two-point Wightman functions respectively.

For any $\omega_f \in Q$, by Stones Theorem $\pi_f(\delta_\psi) = e^{iB_f(\psi)}$ where $B_f(\psi)$ is a hermitean, unbounded operator on the representation space, $B_f$ is linear and

$$\omega_f(\delta_\psi) = \langle \xi_f | \, e^{iB_f(\psi)} | \xi_f \rangle \tag{1}$$

where $\xi_f$ is the cyclic vector of $\pi_f$. From (1) it follows that

$$f_0'(\psi) = i(\xi_f | \, B_f(\psi) \, | \xi_f)$$

and

$$- i f_0' \in H^* .$$

In what follows we suppose that $f_0' = 0$, because if

$$g(\psi) = \exp\left\{\frac{1}{2} f_T''(\psi, \psi)\right\} .$$

Then

$$\omega_f = \omega_g \circ \zeta_{-\mathrm{i}f_0'} \quad \text{and} \quad \pi_f = \pi_g \circ \zeta_{-\mathrm{i}f_0'}$$

where $\zeta_{-\mathrm{i}f_0'}$ is a gauge automorphism (proposition 4).

Denote by $Q_0$ the set $\{\omega_f \in Q | f_0' = 0\}$. If $\omega_f \in Q_0$ then

$$f_T''(\psi_1, \psi_2) = i\sigma(\psi_1, \psi_2) - (\xi_f | \, B_f(\psi_1) \, B_f(\psi_2) \, | \xi_f)$$

and

$$f_T''(\psi, \psi) = - \, (\xi_f | \, B_f(\psi)^2 | \xi_f) \equiv - \, s(\psi, \psi) .$$

Therefore any $\omega_f \in Q_0$ can be written as

$$\omega_f(\delta_\psi) \equiv \omega_s(\delta_\psi) = \exp\left\{-\frac{1}{2} s(\psi, \psi)\right\}$$

where $s$ is a bilinear, symmetric form on $H$.

**Proposition 10.** *The linear form $\omega_s$ belongs to $Q_0$ if and only if*

$$|\sigma(\psi, \varphi)|^2 \leqq s(\psi, \psi) \, s(\varphi, \varphi) \quad \text{for all} \quad \psi, \varphi \in H . \tag{2}$$

*Proof.* If $\omega_s \in Q_0$ then one has

$$(\xi_s | \, B_s(\psi_1) \, B_s(\psi_2) \, | \xi_s) = s(\psi_1, \psi_2) + i\sigma(\psi_1, \psi_2) .$$

A necessary condition for the positivity of $\omega_s$ is

$$(\xi_f | \, [B_s(\psi) + iB_s(\varphi)] \, [B_s(\psi) - iB_s(\varphi)] \, | \xi_f) \geqq 0 \quad \text{for all} \quad \psi, \varphi \in H .$$

This implies

$$|\sigma(\psi, \varphi)| \leqq s(\psi, \psi)^{1/2} \, s(\varphi, \varphi)^{1/2} .$$

The converse statement follows from theorem 2 and 3 below.

Proposition 10 shows that a one-to-one mapping can be found from the set $Q_0$ onto the set of all elements $s$ satisfying (2).

Now we only consider those quasi-free states $\omega_s$ such that $s \in \mathscr{S}$. The remaining quasi-free states will be discussed at the end. Without loss of generality we suppose that for any $s \in \mathscr{S}$, $H$ is complete for the $s$-norm topology (prop. 5 and corollary).

### *III.2 Pure States*

The Fock states on $\overline{\Delta(H, \sigma)}$ are the elements $\omega_s \in Q_0$ such that $A_s^2 = -1$. The operators $A_s$ define then a complex structure on $(H, \sigma)$. The corresponding creation and annihilation operators are

$$B_s^{\pm}(\psi) = \frac{1}{2} \{B_s(\psi) \mp i B_s(A_s \psi)\} \quad \text{for all} \quad \psi \in H .$$

The cyclic vector $\Omega_s$ of the Fock representation $\pi_s$, induced by $\omega_s$, satisfies

$$B_s^-(\psi)\, \Omega_s = 0 \quad \text{for all} \quad \psi \in H .$$

Fock representations are irreducible, therefore the Fock states are pure.

Since we supposed $H$ $s$-norm complete, it follows from the fact that $A_s$ is a bijection of $H$, that $\sigma(H) = s(H)$, and that $H$ is also $\sigma(H)$-quasi-complete. Proposition 6 insures that $H$ remains complete for the norm topology induced by any other complex structure. The proof of the following theorem is now a direct consequence of proposition 9.

**Theorem 1.** *If $\omega_1$ and $\omega_2$ are both Fock states, then an operator $T \in S(H, \sigma)$ can be found such that $\omega_1 = \omega_2 \circ \tau_T$.*

$\tau_T$ is the Bogoliubov transformation; see ([7], appendix A).

## IV. Representations

In this section we construct all representations $\pi_s$ induced by quasi-free states $\omega_s \in Q_0$, $s \in \mathscr{S}$. It follows from proposition 7 that any state $\omega_s \in Q_0$ in uniquely determined by an operator $A_s$ on $H$ such that $A_s^+ = -A_s$ and $A_s^+ A_s \geqq 1$. In section II we found the polar decomposition of $A_s : A_s = J|A_s|$ where $|A_s| \geqq 1$ and $|A_s|$ defined on a dense domain, say $H_0$ of $H$. Furthermore $[J, |A_s|]_- = 0$.

We consider the following operators on $H_0$

$$T_1 = \frac{1}{\sqrt{2}} (|A_s| + 1)^{1/2} \tag{3}$$

$$T_2 = \frac{A}{\sqrt{2}} (|A_s| - 1)^{1/2} \tag{4}$$

where $\Lambda$ is a conjugation commuting with $|A_s|$ (prop. 8), and we consider the algebra $\overline{\Delta(H_0,\sigma)}$; remark that the restriction of $\sigma$ to $H_0$ remains regular, that $\overline{\Delta(H_0,\sigma)} \subset \overline{\Delta(H,\sigma)}$ and $\pi_s(\overline{\Delta(H_0,\sigma)})'' = \pi_s(\overline{\Delta(H,\sigma)})''$ (prop. 5). The restriction of the state $\omega_s$ on $\overline{\Delta(H,\sigma)}$ to $\overline{\Delta(H_0,\sigma)}$ is again a quasi-free state, we denote it by the same symbol.

**Theorem 2.** *Let $s \in \mathscr{S}$, $T_1$ and $T_2$ defined as in* (3) *and* (4), *then $\pi_s$ is a subrepresentation of the representation*

$$\pi(\delta_\psi) = \pi_J(\delta_{T_1\psi}) \otimes \pi_J(\delta_{T_2\psi}) \quad \text{for all} \quad \psi \in H_0 \text{ of } \Delta(H_0,\sigma), \tag{5}$$

*on $\mathscr{H}_J \otimes \mathscr{H}_J (\mathscr{H}_J =$ Fock space associated with $\pi_J)$ with cyclic vector $\Omega_J \otimes \Omega_J$ and reproducing the quasi-free state $\omega_s$.*

The proof of this theorem is only a matter of verification by noting that the domains of $T_1$ and $T_2$ contain $H_0$.

**Proposition 11.** *All representations $\pi$ induced by the states $\omega_s \in Q_0$ such that $s \in \mathscr{S}$ are primary.*

*Proof.* One readily verifies that $\pi'$ defined by

$$\pi'(\delta_\psi) = \pi_J(\delta_{T_2\psi}) \otimes \pi_J(\delta_{T_1\psi}), \qquad \psi \in H_0,$$

is a representation of $\overline{\Delta(H_0,\sigma)}$ commuting with the representation $\pi$ defined in (5).

Let $L$ be the von Neumann algebra generated by the representation $\pi$(5), then

$$\{\pi'(\delta_\psi) \mid \psi \in H_0\}'' \subset L'.$$

Remark that

$$\pi(\delta_{T_1\psi})\,\pi'(\delta_{T_2\psi}) \text{ is equal to } \pi_J(\delta_\psi) \otimes 1 \text{ up to a scalar}$$

and that

$$\pi(\delta_{T_2\psi})\,\pi'(\delta_{T_1\psi}) \text{ is equal to } 1 \otimes \pi_J(\delta_\psi) \text{ up to a scalar.}$$

Therefore

$$\mathscr{L}(\mathscr{H}_J) \otimes 1 \subset \{L \cup L'\}''$$

and

$$1 \otimes \mathscr{L}(\mathscr{H}_J) \subset \{L \cup L'\}''.$$

The set of operators $P \otimes Q$ on $\mathscr{H}_J \otimes \mathscr{H}_J$ is dense in $\mathscr{L}(\mathscr{H}_J \otimes \mathscr{H}_J)$ and every operator of this form commuting with $\{L \cup L'\}''$ must be a multiple of the identity. Consequently

$$\{L \cup L'\}' = L \cap L' = C1.$$

q.e.d.

**Proposition 12.** *A state $\omega_s \in Q_0$ is pure if and only if $A_s^+ A_s = 1$.*

*Proof.* If $A_s^+ A_s = 1$ then $\omega_s$ is a Fock state and therefore pure. On the other hand if $\omega_s$ is pure, we prove that $A_s^+ A_s = 1$. Suppose that $A_s^+ A_s \neq 1$ then a vector $\psi \in H$ can be found such that $K^2\psi = ((A_s^+ A_s)^{1/2} - 1)\,\psi \neq 0$. We define the operator $E$ by

$$E\varphi = \frac{s_J(K\psi,\varphi)}{s_J(K\psi,K\psi)^{1/2}}\,K\psi, \qquad s_J(\psi,\psi) = 1 \quad \text{for all} \quad \varphi \in H_0,$$

where $J$, is the unitary part of the polar decomposition of $A_s$. We define the bilinear, symmetric form $s^E$ on $H$ by

$$s^E(\varphi_1, \varphi_2) = s(\varphi_1, \varphi_2) - s_J(\varphi_1, E\psi)\, s_J(E\psi, \varphi_2)$$

satisfying

$$s^E(\varphi, \varphi) \geqq s_J(\varphi, \varphi)$$

and therefore

$$|\sigma(\varphi_1, \varphi_2)|^2 \leqq s^E(\varphi_1, \varphi_1)\, s^E(\varphi_2, \varphi_2)\,.$$

Consequently the linear forms $\omega_\lambda$ on the C.C.R. algebra defined by

$$\omega_\lambda(\delta_\varphi) = \exp\left\{i\lambda s_J(E\psi, \varphi) - \frac{1}{2} s^E(\varphi, \varphi)\right\}, \ \varphi \in H\,,$$

belong to the set $Q$. One readily verifies that

$$\omega_s = \int_{-\infty}^{\infty} \frac{d\lambda}{(2\pi)^{1/2}}\, e^{-\lambda^2/2}\, \omega_\lambda$$

which proves that $\omega_s$ is not pure, in contradiction with the assumption.

Finally we discuss the quasi-free states $\omega_s$ such that $s \notin \mathscr{S}$. This means that $s$ satisfies the formula (2) but the sympletic form $\sigma$ has not a continous, regular extension to $\overline{H}^s$.

**Theorem 3.** *Let $s$ be a bilinear, symmetric, positive definite form on $H$, such that $|\sigma(\psi, \varphi)| \leqq s(\psi, \psi)^{1/2} s(\varphi, \varphi)^{1/2}$ for all $\psi$, $\varphi \in H$, and such that the continous extension $\sigma'$ of $\sigma$ to $\overline{H}^s$ is not regular, then $\omega_s$ is a quasi-free state. The representation $\pi_s$ induced by $\omega_s$ is not primary.*

*Proof.* Let $\psi$ in $\overline{H}^s$ such that $\sigma(\psi, \varphi) = 0$ for any $\varphi \in \overline{H}^s$, and let $(\psi_n)_{n\in N}$ any sequence in $H$ wich converges to $\psi$. The sequence of unitary operators $(\pi_s(\delta_{\psi_n}))_{n\in N}$ converges in strong sense to an operator $U$: for any $p \in N$ and any $\varphi \in H$,

$$\|[\pi_s(\delta_{\psi_n}) - \pi_s(\delta_{\psi_{n+p}})]\,\hat{\delta}_\varphi\|^2 = 2\,[1 - \mathscr{R}e\exp\{i(\sigma(\psi_n, \psi_{n+p}) - \sigma(\varphi, \psi_n) - \sigma(\psi_{n+p}, \varphi)) - 1/2\,\|\psi_n - \psi_{n+p}\|^2\}]\,,$$

vanishes when $n$ goes to infinity. From the corollary of lemma 2.2 in [4], we know that $U$ is unitary. $U$ is commuting with $\pi_s(\delta_\varphi)$ for any $\varphi \in H$, from strong continuity of the mapping $S \to ST$ and $S \to TS$, together with the relation

$$\pi_s(\delta_{\psi_n})\, \pi_s(\delta_\varphi) = e^{2i\sigma(\psi_n, \varphi)}\, \pi_s(\delta_\varphi)\, \pi_s(\delta_{\psi_n})\,.$$

Consequently $U \in \pi_s(\overline{\Delta(H, \sigma)})'' \cap \pi_s(\overline{\Delta(H, \sigma)})'$. Nevertheless $U$ is not a scalar operator because if $U = \lambda I$, it would follow from unitarity of $U$, $|\lambda| = 1$, and this would contradict:

$$|\langle\hat{\delta}_0|U\hat{\delta}_0\rangle| = e^{-\frac{1}{2}\|\psi\|_s^2} < 1\,.$$

Therefore $\pi_s$ is not a primary representation.

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