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Representations of Anticommutation Relations and Bogolioubov Transformations

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Abstract. A description of the quasi-free states on a Clifford algebra and their representations is given, and we prove that the pure quasi-free states are Fock States.

I. Introduction

In this paper we complete the study of quasi-free states on a Clifford algebra started in ref. [1], where essentially the translation invariant states were treated. Here we use however a different method which turned out to be more powerful to derive general properties of the set of quasi-free states. The relation with ref. [1] is established in appendix A.

Our starting point is a C^*-Clifford algebra $\overline{\mathfrak{A}(H, s)}$ built on an euclidean space (H, s) (i. e. H is a real vector space on which a bilinear, symmetric, positive definite form s is defined). Without loss of generality we suppose that H is complete. For more details we refer to ref. [2]. Let B be the canonical mapping of H into $\overline{\mathfrak{A}(H, s)}$ such that

$$[B(\psi), B(\varphi)]_+ = 2s(\psi, \varphi) \quad \text{for} \quad \psi, \varphi \in H . \tag{1}$$

Let T be an element of the group $\mathcal{O}(H, s)$ of orthogonal operators on (H, s) and $\alpha(\overline{\mathfrak{A}(H, s)})$ the group of automorphisms of $\overline{\mathfrak{A}(H, s)}$, then the mapping $B(\psi) \to B(T\psi)$ can be extended to an automorphism τ_T of $\overline{\mathfrak{A}(H, s)}$. Furthermore the operator $\tau : T \to \tau_T \in \alpha(\overline{\mathfrak{A}(H, s)})$ is a monomorphism. In theorem 1 we prove that any two Fock states are related by such an automorphism. We also remark that such an automorphism corresponds to a generalized Bogoliubov transformation (see appendix A).

Furthermore we explicitly construct all representations induced by quasi-free states and give a criterium under which they are irreducible.

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In section IV we give a more detailed treatment of translation-invariant quasi-free states. Such states can be parametrized by pairs of functions $\tilde{a}_1, \tilde{a}_2 \in L^\infty(R^n)$ which permit a fruitful application in the study of physical models.

In appendix B the Clifford algebra $\overline{\mathfrak{A}(H, s)}$ is constructed as an infinite tensor product of finite dimensional C^*-algebras.

II. Quasi Free States

Quasi-free states [1] on $\overline{\mathfrak{A}(H, s)}$ are completely determined by their values on the subspace $\mathfrak{A}_2$ of $\overline{\mathfrak{A}(H, s)}$ generated by the set $\{B(\psi)\, B(\varphi) \mid \psi, \varphi \in H\}$. They can also be characterized as follows.

Proposition 1. *Let ω be a quasi-free state on $\overline{\mathfrak{A}(H, s)}$; ω determines a bounded operator A on H, defined by*

$$\omega(1) = 1 \tag{2}$$

$$\omega(B(\psi)\, B(\varphi)) = s(\psi, \varphi) + i\, s(A\, \psi, \varphi)\,; \quad \psi, \varphi \in H \tag{3}$$

satisfying $A^+ = -A$[1] *and $\|A\| \leqq 1$. Conversely every such operator A determines by* (2) *and* (3) *a quasi-free state ω_A.*

Proof. Let ω be quasi-free state, then it is determined by (2) and

$$\omega(B(\psi)\, B(\varphi)) = s(\psi, \varphi) + i\, \sigma(\psi, \varphi) \tag{4}$$

where σ is a bilinear, antisymmetric, real form on H. A necessary condition for the positivity is

$$\omega([B(\psi) + i\, B(\varphi)]\, [B(\psi) - i\, B(\varphi)]) \geqq 0 \quad \text{for all} \quad \psi, \varphi \in H$$

yielding $\|\sigma\| \leqq 1$, therefore σ is a continuous bilinear form on H and the completeness of H ensures the existence of an operator A on H such that $\sigma = s \circ A$. The property $A^+ = -A$ follows from (1) and (4). The positivity of the state requires $\|\sigma\| \leqq 1$ and therefore $\|A\| \leqq 1$. Sufficiency follows from theorem 2 below.

Moreover, if the operator A satisfies $A^+ = -A$ and $A^2 = -1$, then A defines a complex structure on (H, s)[2] and the corresponding state ω_A is called a Fock state; $A = 0$ defines the central state ω_0 on $\overline{\mathfrak{A}(H, s)}$.

Lemma 1. *Let A_i $(i = 1, 2)$ be operators on H satisfying $A_i^+ = -A_i$ and $A_i^2 = -1$ then there exists an operator $T \in \mathcal{O}(H, s)$ such that $A_1 = T^+ A_2\, T$.*

Proof. Let $\{\varepsilon_k^i, \varphi_k^i\}$ be an orthonormal basis of H such that $A_i\, \varepsilon_k^i = \varphi_k^i$ and $A_i\, \varphi_k^i = -\varepsilon_k^i$ for $i = 1, 2$ and all k; then the operator T of the Lemma is the linear orthogonal operator on H defined by $T\, \varepsilon_k^1 = \varepsilon_k^2$ and $T\, \varphi_k^1 = \varphi_k^2$ for any k. One verifies

$$T\, A_1 = A_2\, T \quad \text{and} \quad T^+\, T = 1\,.$$

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[1] A^+ denotes the adjoint of A relative to the bilinear scalar product s.

[2] Setting $(\alpha + i\, \beta)\, \psi = \alpha\, \psi + \beta\, A\, \psi$, α and β being real numbers.

Theorem 1. *If ω_{A_1} and ω_{A_2} are Fock states on $\overline{\mathfrak{A}(H, s)}$, then there exists an element $T \in \mathcal{O}(H, s)$ such that $\omega_{A_1} = \omega_{A_2} \circ \tau_T$.*

The proof is straightforward by remarking that T is the operator defined in Lemma 1 and

$$\begin{aligned}\omega_2 \circ \tau_T(B(\psi)\, B(\varphi)) &= \omega_2(B(T\,\psi)\, B(T\,\varphi)) = s(T\,\psi, T\,\varphi) \\ &\quad + i\, s(A_2\, T\,\psi, T\,\varphi) \\ &= s(\psi, \varphi) + i\, s(T^+ A_2\, T\,\psi, \varphi) = \omega_1(B(\psi)\, B(\varphi))\,.\end{aligned}$$

It follows from theorem 1 that if ω_A is a Fock state, all other Fock states are obtained by combining ω_A with all elements of $\alpha(\overline{\mathfrak{A}(H, s)})$ induced by $\mathcal{O}(H, s)$.

Let ω_J be a Fock state. The creation and annihilation operators are defined as

$$B^{\pm}(\psi) = \frac{1}{2}\{B(\psi) \mp i\, B(J\,\psi)\}\,.$$

One easily checks that $B^+(\psi)$ is c-linear [i. e. $B^+(J\,\psi) = i\, B^+(\psi)$] and $B^-(\psi)$ is c-antilinear [i. e. $B^-(J\,\psi) = -i\, B^-(\psi)$]. The Fock representation, induced by ω_J, is denoted as π_J and the representation space as $\mathcal{H}_J$. The Fock space $\mathcal{H}_J$ contains the cyclic vector Ω_J of the representation satisfying

$$\pi_J(B^-(\psi))\, \Omega_J = \pi_{-J}(B^+(\psi))\, \Omega_{-J} = 0 \quad \text{for any} \quad \psi \in H$$

where π_{-J} is the Fock representation induced by ω_{-J}.

From now on we choose a particular operator J such that $J^+ = -J$, $J^2 = -1$. By theorem 1, for every state ω_A with $A^2 = -1$ an operator $T \in \mathcal{O}(H, s)$ can be found such that $\omega_A = \omega_J \circ \tau_T$. This proves that the representation induced by ω_A is completely described in terms of the Fock representation induced by ω_J; as a consequence it is also irreducible.

III. Representations

Now we consider the general case. We look for cyclic representations π_A induced by quasi-free states ω_A, satisfying $A^+ = -A$ and $\|A\| \leqq 1$.

We make the ansatz

$$\pi_A(B(\psi)) = \frac{1}{\sqrt{2}}\{\pi_J(B(T_1\,\psi) \otimes 1 + \theta \otimes \pi_{-J}(B(T_2\,\psi))\}\,, \quad \psi \in H \tag{5}$$

on $\mathcal{H}_A = \mathcal{H}_J \otimes \mathcal{H}_{-J}$ with cyclic vector $\Omega_J \otimes \Omega_{-J}$; θ is an operator anticommuting with any $\pi_J(B(\psi))$, $\psi \in H$ and such that $\theta\,\Omega_J = \Omega_J$; T_1 and T_2 are linear operators on H. It is easy to check that π_A is a representation induced by ω_A, if and only if T_1 and T_2 satisfy

$$T_1^+\, T_1 + T_2^+\, T_2 = 2\,, \tag{6}$$

$$T_1^+\, J\, T_1 - T_2^+\, J\, T_2 = 2A\,. \tag{7}$$

If we can find a solution for T_1 and T_2 in equations (6) and (7), and prove cyclicity, we proved our ansatz. The fact that π_A in (5) induces a quasi-free state follows from the analogous property of Fock states.

a) Construction of the Representation

First we look for a solution of the equations (6) and (7) and consider independently the cases when the kernel $\mathfrak{N}_A$ of A is even or infinite dimensional and $\mathfrak{N}_A$ is odd dimensional.

1°) $\mathfrak{N}_A$ is even or infinite dimensional.

Let $A = U|A|$[3] be the polar decomposition of A with respect to the real Hilbert space (H, s); U is a partial isometry and $0 \leqq |A| \leqq 1$. The operator A is normal, therefore we can choose U unitary, commuting with $|A|$ and with any operator commuting with A[3]. If $\dim \mathfrak{N}_A$ is even or infinite, U can furthermore be taken anti-hermitean, i. e. $U^+ = -U$, because $A^+ = -A$ and $U|A| = |A|\, U$. The operator U is therefore a particular complexification of the space (H, s) and there exists an operator $T \in \mathcal{O}(H, s)$ such that $U = T^+ J\, T$. The equations (6) and (7) become now

$$T_1'^+ T_1' + T_2'^+ T_2' = 2 \tag{6'}$$

$$T_1'^+ U\, T_1' - T_2'^+ U\, T_2' = 2\, U\, |A| \tag{7'}$$

where $T_1' = T^+ T_1$ and $T_2' = T^+ T_2$.

Now we can write down a solution of (6′) and (7′)

$$T_1' = (1 + |A|)^{1/2}\,, \quad T_2' = (1 - |A|)^{1/2}$$

and therefore

$$T_1 = T(1 + |A|)^{1/2}\,; \quad T_2 = T(1 - |A|)^{1/2}$$

is a solution of (6) and (7).

2°) $\mathfrak{N}_A$ is odd dimensional.

We add one dimension to $\mathfrak{N}_A$ and consider the space (H', s') where $H' = \mathbb{R}\,\eta \oplus H$ and s' a symmetric, bilinear, real form such that: $s'(\eta, \eta) = 1$; $s'(\eta, \psi) = 0$ and $s'(\psi, \varphi) = s(\psi, \varphi)$ for $\psi, \varphi \in H$. We consider the algebra $\mathfrak{A}(H', s')$, containing $\mathfrak{A}(H, s)$ as a subalgebra, and the quasi-free state $\omega'_{A'}$, which is defined by

$$\omega'_{A'}(B(\psi')\, B(\varphi')) = s'(\psi', \varphi') + i\, s'(A'\, \psi', \varphi')\,; \quad \psi', \varphi' \in H'$$

where A' satisfies: $A'^+ = -A'$, $\|A'\| \leq 1$, $A'\,\eta = 0$ and $A'\, H = A\, H$. The restriction of a positive form to a subalgebra remains a positive form, here in fact $\|A'\|' = \|A\| \leqq 1$ and the restriction of the state $\omega'_{A'}$ to the subalgebra $\mathfrak{A}(H, s)$ is the quasi-free state ω_A:

Since the spaces H' and H have the same dimension, there exists an isomorphism T' mapping H' onto $H: T'^+ T' = 1_{H'}$, $T'\, T'^+ = 1_H$, $s(T'\, \psi', T'\, \varphi') = s'(\psi', \varphi')$, $s'(T'^+ \psi, T'^+ \varphi) = s(\psi, \varphi)$. The complex

[3] There is a polar decomposition in any real Hilbert space, because the existance of the square root of any positive oparetor is only needed.

structure J on H defines on H' a complex structure J' defined by: $J' = T'^{+} J\, T'$. The isomorphism T' defines an isomorphic mapping $\tau_{T'}: B(\psi') \to B(T'\, \psi')$ of $\overline{\mathfrak{A}(H', s')}$ onto $\overline{\mathfrak{A}(H, s)}$ and we have the following relation between the Fock representations $\pi'_{J'}$ and π_J:

$$\pi'_{J'} = \pi_J \circ \tau_{T'}\,; \quad \Omega_J = \Omega_{J'}\,. \tag{8}$$

Now $\mathfrak{N}_{A'}$ is even dimensional and the method of 1°) can be used to get a representation $\pi'_{A'}$ on $\mathscr{H}'_J \otimes \mathscr{H}'_{-J}$ induced by $\omega'_{A'}$ or to find operators T'_1 and T'_2 satisfying the equations

$$T_1'^{+}\, T_1' + T_2'^{+}\, T_2' = 2\, 1_{H'}$$

$$T_1'^{+}\, J'\, T_1' - T_2'^{+}\, J'\, T_2' = 2 A'$$

and $\pi'_{A'}(B(\psi'))$ for $\psi' \in H'$ is of the form (5) on $\mathscr{H}'_J \otimes \mathscr{H}'_{-J}$. Using (8) and the fact that T' induces also an isomorphism between $\mathscr{H}'_J \otimes \mathscr{H}'_{-J}$ and $\mathscr{H}_J \otimes \mathscr{H}_{-J}$, we can write down a representation $\pi'_{A'}$ of $\omega'_{A'}$ on $\mathscr{H}_J \otimes \mathscr{H}_{-J}$

$$\pi_{A'}(B(\psi') = \frac{1}{\sqrt{2}} \{\pi_J(B(T_1\, \psi')) \otimes 1 + \theta \otimes \pi_{-J}(B(T_2\, \psi')\}\,; \quad \psi' \in H' \tag{9}$$

where $T_1 = T'\, T'_1$ and $T_2 = T'\, T'_2$ are known to be linear mappings of H' into H and their restrictions T_1^H and T_2^H to H map H into H and satisfy equations (6) and (7). Therefore the restriction

$$\pi_A(B(\psi)) = \frac{1}{\sqrt{2}} \{\pi_J(B(T_1^H\, \psi)) \otimes 1 + \theta \otimes \pi_{-J}(B(T_2^H\, \psi))\} \text{ for } \psi \in H \tag{10}$$

of the representation (9) is a representation on $\mathscr{H}_J \otimes \mathscr{H}_{-J}$ induced by ω_A.

b) Cyclicity

To obtain cyclic representations with $\Omega_J \otimes \Omega_{-J}$ as cyclic vector we consider again the following cases

1°) $\mathfrak{N}_A$ is even or infinite dimensional.

First remark that T_1 is always invertible; if T_2 is invertible then an argument analogous to that of ref. [4], by using the creation and annihilation operators associated with J, shows that π_A is cyclic; if $\mathfrak{N}_{T_2}$ is not trivial then the representation of π_A on the closure of $\mathscr{H}_J \otimes \overline{\mathfrak{A}(H \ominus \mathfrak{N}_{T_2}, s)}\, \Omega_{-J}$ is cyclic.

2°) $\mathfrak{N}_A$ is odd dimensional.

The operator T_1^H is always invertible on $T'\, H$; if T_2^H is invertible on $T'\, H$ then, because of the same argument as in 1°) the subrepresentation of π_A on the closure of $\overline{\mathfrak{A}(T'\, H, s)}\, \Omega_J \otimes \overline{\mathfrak{A}(T'\, H, s)}\, \Omega_{-J}$ is cyclic. If T_2^H is not invertible on $T'\, H$, we must repeat once more the procedure of 1°) to obtain a cyclic representation.

We summarize our results of a) and b) in

Theorem 2. *All quasi-free states* ω_A *induce cyclic representations or subrepresentations* π_A *of the form* (5) *on the space* $\mathscr{H}_J \otimes \mathscr{H}_{-J}$ *with cyclic vector* $\Omega_J \otimes \Omega_{-J}$.

Finally we prove

Theorem 3. *In order that a quasi-free state* ω_A *with* $\mathfrak{N}_A$ *of even or infinite dimension be pure it is necessary and sufficient that* A *satisfies* $A^2 = -1$.

Proof. The sufficiency follows from Theorem 1. We prove that it is also necessary. Because $\dim \mathfrak{N}_A$ is even or infinite there exists a complexification U, commuting with A (see the proof of theorem 2, a, 1°); suppose $A^2 \neq -1$, then there exists a vector ψ such that

$$K^2 \psi = \frac{1}{4} (1 - A^+ A)^{1/2} \psi \neq 0 , \quad s(\psi, \psi) = 1$$

and a two-dimensional projection operator E defined by

$$E \varphi = s(K \psi, \varphi) K \psi + s(U K \psi, \varphi) U K \psi \quad \text{for any} \quad \varphi \in H .$$

One verifies that $[E, J]_- = 0$. We define the operators A_1 and A_2: $A_1 = A + U E$ and $A_2 = A - U E$ satisfying

$$A_1^+ = -A_1 ; \quad A_2^+ = -A_2 \tag{11}$$

$$A_1 + A_2 = 2A \tag{12}$$

and one easily checks that

$$A_1^+ A_1 \leqq 1 ; \quad A_2^+ A_2 \leqq 1 . \tag{13}$$

The properties (11) and (13) enable us to define the quasi-free states ω_{A_1} and ω_{A_2} such that

$$\omega_{A_1}(B(\varphi_1) B(\varphi_2)) = s(\varphi_1, \varphi_2) + i s(A_1 \varphi_1, \varphi_2)$$

$$\omega_{A_2}(B(\varphi_1) B(\varphi_2)) = s(\varphi_1, \varphi_2) + i s(A_2 \varphi_1, \varphi_2)$$

$$\omega_{A_1}(1) = \omega_{A_2}(1) = 1 .$$

Let $\{\psi_1, \psi_2\}$ be an orthonormal basis of the subspace $E H$ of H then

$$\omega_{A_1}(B(\psi_1) B(\psi_2)) \neq \omega_{A_2}(B(\psi_1) B(\psi_2)) \tag{14}$$

and it follows from (12) that

$$\omega_{A_1}(B(\psi_1) B(\psi_2)) + \omega_{A_2}(B(\psi_1) B(\psi_2)) = 2 \omega_A(B(\psi_1) B(\psi_2)) . \tag{15}$$

Furthermore

$$\omega_{A_1}(B(\varphi_1) B(\varphi_2)) = \omega_{A_2}(B(\varphi_1) B(\varphi_2)) = \omega_A(B(\varphi_1) B(\varphi_2)) \tag{16}$$

if φ_1 or (and) φ_2 belong to the orthogonal complement of $E H$ in H. It follows from (15) and (16) that on $\mathfrak{A}_2$

$$\omega_A = \frac{1}{2} \omega_{A_1} + \frac{1}{2} \omega_{A_2} . \tag{17}$$

A straightforward calculation shows that (17) holds on all the elements of the form $B(\psi_1)\, B(\psi_2)\, B(\varphi_1) \ldots B(\varphi_{2n})$ and $B(\varphi_1) \ldots B(\varphi_{2n})$ which form a basis of $\overline{\mathfrak{A}(H, s)}$, and therefore (17) holds on the whole algebra. This shows that ω_A is not pure.

Remark. The condition that $\mathfrak{N}_A$ is even or infinite dimensional is always satisfied in the case of translation or gauge invariant states (prop. 3).

IV. Invariant States

We take $H = L^2(R^n)$ with the real inner product of $\varphi = \varphi_1 + i\,\varphi_2$ and $\psi = \psi_1 + i\,\psi_2$ defined by

$$s(\varphi, \psi) = \int_{R^n} (\varphi_1(x)\, \psi_1(x) + \varphi_2(x)\, \psi_2(x))\, dx\,.$$

Let T be an orthogonal operator on (H, s). The state ω_A is said to be T-invariant, if

$$\omega_A(\tau_T\, a) = \omega_A(a) \quad \text{for every} \quad a \in \overline{\mathfrak{A}(H, s)}\,.$$

The condition on A in order that ω_A be T-invariant is given by

Lemma 2. *The quasi-free state ω_A is T-invariant if and only if*

$$[A, T]_- = 0\,.$$

Proof. The state ω_A is T-invariant if and only if

$$\omega_A(B(T\,\varphi)\, B(T\,\psi)) = \omega_A(B(\varphi)\, B(\psi))\,, \quad \varphi, \psi \in H\,.$$

Or

$$s(T\,\varphi, T\,\psi) + i\, s(A\,T\,\varphi, T\,\psi) = s(\varphi, \psi) + i\, s(A\,\varphi, \psi)\,.$$

Since T is orthogonal, this is equivalent to

$$[A, T]_- = 0\,.$$

In particular, ω_A is translation-invariant if and only if

$$[A, T_k]_- = 0 \quad \text{for every} \quad k \in R^n\,,$$

where T_k is defined by $T_k\, f(x) = f(x - k)$, i. e. if A is a translation-invariant operator.

The state ω_A is J — gauge-invariant if and only if

$$[A, e^{J\alpha}]_- = 0\,, \quad -\infty < \alpha < \infty$$

or, equivalently

$$[A, J]_- = 0\,.$$

Let us consider

$$A_1 = -\frac{J}{2}\,[J, A]_+\,, \quad A_2 = -\Lambda\,\frac{J}{2}\,[J, A]_- \tag{18}$$

(where Λ is defined in appendix A, prop. A 1), so that

$$A = A_1 + \Lambda\, A_2\,, \tag{19}$$

and

$$[A_1, J]_- = [A_2, J]_- = 0 .$$

Now we choose $J = i\,I$, which is a translation-invariant operator. Then it follows from Lemma 2, that the state ω_A is translation-invariant if and only if A_1 and A_2 are translation-invariant operators, and ω_A is gauge-invariant if and only if $A_2 = 0$.

Let now ω_A be a translation-invariant state. Then A_1 and A_2 are complex-linear, translation-invariant operators, and it is well known (cf. [8]) that there exist tempered distributions a_i on R^n, $i = 1, 2$ such that $\tilde{a}_i \in L^\infty(R^n)$, and for $\varphi \in H$

$$A_i\,\varphi = a_i * \varphi \tag{20}$$

where the functions $\tilde{a}_i$ are the Fourier transforms of a_i, and $*$ denotes convolution. Via the Fourier transformation the operators A_i become

$$\tilde{A}_i\,\tilde{\varphi} = \tilde{a}_i \cdot \tilde{\varphi} . \tag{21}$$

The Fourier transform of the operator A is defined by

$$\tilde{A}\,\tilde{\varphi}(\xi) = \tilde{a}_1(\xi)\,\tilde{\varphi}(\xi) + \bar{\tilde{a}}_2(-\xi)\,\bar{\tilde{\varphi}}(-\xi) . \tag{22}$$

We notice that $\tilde{A}$ is a simple multiplication operator if and only if ω_A is gauge-invariant. In many problems it is an advantage to work with A_1 and A_2 in the simple form (18) rather than with the operator A. This is the case for instance in the treatment of models by variational procedures. For this it is of interest to give the following explicit characterization of the pairs of functions $(\tilde{a}_1, \tilde{a}_2)$ which define a quasi-free state ω_A via the operator A defined by (19), (20) and (21).

Proposition 2. *The pair of functions* $(\tilde{a}_1, \tilde{a}_2)$ *define a quasi-free state* ω_A *if and only if for any* ξ

(i) $\tilde{a}_1(\xi)$ *is purely imaginary,*

(ii) $\tilde{a}_2(\xi) = -\tilde{a}_2(-\xi)$,

(iii) $|\tilde{a}_1(\xi)|^2 + |\tilde{a}_2(\xi)|^2 \leqq 1$.

Proof. This is proved by a straightforward calculation, using (21) and (22).

In an actual problem one can use the pairs of functions $(\tilde{a}_1, \tilde{a}_2)$ satisfying (i)—(iii) as parameters for the set of quasi-free states or simple functions of $\tilde{a}_1$ and $\tilde{a}_2$ (cf. [1] and appendix A).

Now we give another application of the operators A_1 and A_2 by proving the following simple results.

Proposition 3. *For a translation-invariant state* ω_A *the dimension of the null space of* A *is equal to* 0 *or* ∞.

Proof. By (22), the Fourier transform of the equation $A\,\varphi = 0$ is

$$\tilde{a}_1(\xi)\,\tilde{\varphi}(\xi) + \bar{\tilde{a}}_2(-\xi)\,\bar{\tilde{\varphi}}(-\xi) = 0 . \tag{23}$$

We decompose the purely imaginary function $\tilde{a}_1$ in to its symmetric and antisymmetric parts $\tilde{a}_{1s}$ and $\tilde{a}_{1a}$, the antisymmetric function $\tilde{a}_2$ in to its real and imaginary parts $\tilde{a}_{2r}$ and $i\,\tilde{a}_{2i}$, and, with the same notations, the function $\tilde{\varphi}$ as follows

$$\tilde{\varphi} = \tilde{\varphi}_{rs} + \tilde{\varphi}_{ra} + i(\tilde{\varphi}_{is} + \tilde{\varphi}_{ia}) \,.$$

This gives rise to the following system of equations equivalent to (23)

$$\begin{pmatrix} \tilde{a}_{1s} & \tilde{a}_{1a} + \tilde{a}_{2i} & 0 & \tilde{a}_{2i} \\ \tilde{a}_{1a} - \tilde{a}_{2r} & \tilde{a}_{1s} & -\tilde{a}_{2i} & 0 \\ 0 & \tilde{a}_{2i} & \tilde{a}_{1s} & \tilde{a}_{1a} - \tilde{a}_{2i} \\ \tilde{a}_{2i} & 0 & \tilde{a}_{1a} - \tilde{a}_{2r} & \tilde{a}_{1s} \end{pmatrix} \begin{pmatrix} \varphi_{rs} \\ \varphi_{ra} \\ \varphi_{is} \\ \varphi_{ia} \end{pmatrix} = 0 \,. \tag{24}$$

Let $\mathcal{M}$ be the set of points ξ, where the determinant of the coefficient matrix of (24) is equal to 0. We have the following two cases

1. $\mathcal{M}$ has measure 0. Then the solution of (24) and hence of (23) is 0 almost everywhere, and $\dim \mathfrak{N}_A = 0$.

2. $\mathcal{M}$ has positive measure m. Then we can divide $\mathcal{M}$ into a sequence of disjoint sets $\mathcal{M}_i$, $i = 1, 2, \ldots$, of measure $m/2^i$ and for each i construct a non-trivial solution of (24) $(\varphi^{(i)}_{rs}, \varphi^{(i)}_{ra}, \varphi^{(i)}_{is}, \varphi^{(i)}_{ia})$ with support in $\mathcal{M}_i$. Since the determinant is symmetric in ξ, the right symmetry properties can be obtained by use of the above construction in a half-space and reflection. Thus we have obtained a sequence of non-zero orthogonal functions in the null space of A, hence $\dim \mathfrak{N}_A = \infty$.

By means of proposition 3 we can prove

Proposition 4. *For every translation-invariant, quasi-free state* ω_A *there exists a translation-invariant operator* J *satisfying* $J^+ = -J$, $J^2 = -1$, *and such that*

$$[A, J]_- = 0 \,,$$

i. e. such that A *is* J*-gauge-invariant.*

Proof. Since $\dim \mathfrak{N}_A$ is 0 or ∞, we can use the construction of Theorem 2, a, 1°. We need only to add, that J can be chosen translation-invariant on $\mathfrak{N}_A$. On the complement of $\mathfrak{N}(A)$ this is satisfied because A is translation-invariant, and $J = A(A^+ A)^{-1/2}$ on $\mathfrak{N}_A^{\perp}$.

Appendix A

We establish the relation between our formalism and that of ref. [1]

Proposition A 1. *For every euclidean space* (H, s) *and Hilbert structure* J *on* (H, s), *there exists a closed subspace* E *of* H *and an orthogonal linear operator* Λ *on* H, *satisfying* $H = E \oplus J E$ *and* $\Lambda^2 = 1$, $[\Lambda, J]_+ = 0$.

Proof. Let $\{\varepsilon_i, \varphi_i \mid i \in I\}$ be an orthonormal basis of H related to J i. e. $J\,\varepsilon_i = \varphi_i$. The linear operator Λ is defined by $\Lambda\,\varepsilon_i = \varphi_i$ and $\Lambda\,\varphi_i = \varepsilon_i$ for all $i \in I$; Λ is a hermitean orthogonal operator satisfying $[\Lambda, J]_+ = 0$.

The projections $P = \frac{1+A}{2}$ and $Q = \frac{1-A}{2}$ are orthogonal, complementary and satisfy $A P = P A = P$, $A Q = Q A = -Q$ and $J P = Q J$, then $E = P H$ and $H = E \oplus J E$ where $J E = Q H$.

Let J be a complex structure on (H, s). We define the operators R and S (see ref. [1]) on the Hilbert space $(H; (.\,,.) = s(.\,,.) + i\, s(J\,.\,,.))$ associated with a quasi-free state ω_A by setting

$$\omega_A(B^-(\psi)\, B^+(\varphi)) = (\psi, R\,\varphi)$$

$$\omega_A(B^+(\psi)\, B^+(\varphi)) = (A\,\psi, S\,\varphi) \quad \text{(in ref. [1] } A\,\psi \text{ is denoted as } \bar{\psi})$$

where $B^{\pm} = \frac{1}{2}(B \mp i\, B \circ J)$.

By identification we obtain

$$A = J(2R - 1 - 2A\,S)$$

and

$$R = \frac{1}{2} - \frac{1}{4}[J, A]_+ ; \quad S = \frac{1}{4} A [J, A]_- .$$

Remark that S is linear and that $-A^2 = 1$ is equivalent with $R - R^2 - S^+ S = 0$.

The connection between (R, S) and (A_1, A_2) is given by

$$R = \frac{1}{2}(1 - J\,A_1) ; \quad S = -\frac{1}{2} J\,A_2 .$$

The usefulness of the operators R and S in physical applications is due to the fact that R and S are simple linear functions of A_1 and A_2 commuting with J.

In this section we also formulate the result of theorem 1 in terms which are commonly used among most physicists. If we have two pure quasi-free states say ω_{J_1} and ω_{J_2} on $\overline{\mathfrak{A}(H, s)}$ then there exists always a Bogoliubov transformation relating the respective creation and annihilation operators.

For the convenience of the reader we write down explicitly the Bogoliubov transformation. The creation and annihilation operators corresponding to the states ω_{J_1} and ω_{J_2} are respectively

$$B^{\pm}_{J_1}(\psi) = \frac{1}{2}\{B(\psi) \mp i\, B(J_1\,\psi)\}$$

$$B^{\pm}_{J_2}(\psi) = \frac{1}{2}\{B(\psi) \mp i\, B(J_2\,\psi)\} ,$$

and the Bogoliubov transformation reads

$$B^{+}_{J_1}(\psi) = B^{+}_{J_2}(U\,\psi) + B^{-}_{J_2}(V\,\psi)$$

$$B^{-}_{J_1}(\psi) = B^{+}_{J_2}(V\,\psi) + B^{-}_{J_2}(U\,\psi)$$

where

$$U = \frac{1}{2}\,[1 - J_2\, T^+ J_2\, T]$$

$$V = \frac{1}{2}\,[1 + J_2\, T^+ J_2\, T]\,.$$

Here T is the operator defined in theorem 1.

Remark that U and V satisfy the well-known consistency equations

$$U^+ U + V^+ V = 1 \quad \text{and} \quad U^+ V + V^+ U = 0\,.$$

Appendix B

In this appendix we construct the Clifford algebra $\overline{\mathfrak{A}(H, s)}$ as an infinite tensor product of finite dimensional C^*-algebras for the case that (H, s) is separable. Another construction can be found in ref. [5]

Proposition B 1. *For every space (H, s) of even dimension, there exists an element $\beta \in \overline{\mathfrak{A}(H, s)}$ anticommuting with all $B(\psi)$, $\psi \in H$, such that $\beta^2 = 1$.*

Proof. Suppose that the dimension of (H, s) is $2n$ and let $\{\psi_1, \psi_2, \ldots, \psi_{2n}\}$ be an orthonormal basis of (H, s) then β can be defined by

$$\beta = i^n\, B(\psi_1)\, B(\psi_2) \ldots B(\psi_{2n})\,.$$

Proposition B 2. *For every space (H, s) where $H = H_1 \oplus H_2$ and H_1 is of even dimension, the C^*-algebra $\overline{\mathfrak{A}(H, s)}$ is isomorphic with $\overline{\mathfrak{A}(H_1, s)} \otimes \overline{\mathfrak{A}(H_2, s)}$ (the tensor product of C^*-algebras [6]).*

Proof. The isomorphism ξ between $\overline{\mathfrak{A}(H, s)}$ and $\overline{\mathfrak{A}(H_1, s)} \otimes \overline{\mathfrak{A}(H_2, s)}$ is defined by the following relations.

$$\xi(B(\psi)) = B(\psi) \otimes 1\!\!1 \quad \text{if} \quad \psi \in H_1$$

$$\xi(B(\psi)) = \beta \otimes B(\psi) \quad \text{if} \quad \psi \in H_2$$

where $\beta \in \overline{\mathfrak{A}(H_1, s)}$ is defined in proposition B 1.

Proposition B 3. *If the space (H, s) is separable, then the C^*-algebra $\overline{\mathfrak{A}(H, s)}$ is isomorphic with $\bigotimes_{i=1}^{\infty} \mathscr{B}_i$ where $\mathscr{B}_i$ is the C^*-algebra of the 2×2 matrices $\left(\bigotimes_{i=1}^{\infty} \mathscr{B}_i\right.$ is the infinite product of C^*-algebras [7]$\left.\right)$.*

Proof. Let $\{\psi_i, \varphi_i \mid i \in N\}$ be an orthonormal basis of (H, s) and let (H_i, s) be the subspace of (H, s) generated by $\{\psi_i, \varphi_i\}$ for each $i \in N$. The C^*-algebras $\mathfrak{A}(H_i, s)$ are isomorphic with the C^*-algebras $\mathscr{B}_i$ for each $i \in N$. The isomorphism η between $\overline{\mathfrak{A}(H, s)}$ and $\bigotimes_{i=1} \mathscr{B}_i$ is defined by

$$\eta(B(\psi)) = \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_{j-1} \otimes B(\psi) \otimes 1 \otimes 1 \otimes \cdots$$

if $\psi \in H_j$. For every $k \in N$ the elements β_k belong to $\overline{\mathfrak{A}(H_k, s)}$ and are defined as in proposition B 1.

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