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Event Horizons in Static Electrovac Space-Times

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Abstract. The following theorem is established. Among all static, asymptotically flat electrovac fields with closed, simply-connected equipotential surfaces $g_{00} = \text{const.}$, the only ones which have regular event horizons $g_{00} = 0$ are the Reissner-Nordström family of spherisymmetric solutions with $m \geqq G^{1/2}|e|/c$. In the special case where the gravitational coupling of the electromagnetic energy density is neglected ($G = 0$) all solutions are computed explicitly, thus extending an earlier result of GINZBURG for a magnetic dipole in SCHWARZSCHILD'S space-time. Possible implications for gravitational collapse are briefly discussed.

1. Introduction

Of central importance to the theory of gravitational collapse is the question whether event horizons are a fairly normal characteristic of very intense gravitational fields, or whether they are merely quirks of the special highly symmetric solutions which have so far been studied.

If we restrict ourselves to the class of asymptotically flat, static vacuum fields, it is already known [1] that the only regular event horizons are spherical. More precisely: among all fields in this class with closed, simply-connected equipotential surfaces $g_{00} = \text{const.}$, Schwarzschild's solution is the only one with a regular event horizon $g_{00} = 0$. This means that no static asymmetric perturbation of the Schwarzschild field which originates from sources *within* the critical surface $g_{00} = 0$ ($r = 2m$) can preserve a regular event horizon. (On the other hand, perturbations due to exterior sources, such as distant masses, leave the qualitative character of the event horizon unaffected [2].)

Quite generally, in the case of an arbitrary asymptotically flat field, it therefore seems natural to ask whether the regularity of an event horizon is destroyed by any asymmetric perturbation due to interior sources (e.g. mass quadrupole [3], magnetic dipole field [4], outgoing gravitational waves; an exception has to be made here for rotation — the Kerr manifold has a regular event horizon [5]).[1] If this were true, it would force a drastic reappraisal of our current ideas on the nature of gravitational collapse [6].

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[1] For instance, it might be conjectured that every vacuum field which has a regular event horizon and which is asymptotically flat (with an outgoing radiation condition) is algebraically special.

The present paper represents a small step towards a definitive answer to this question. We shall study the class of electrovac spaces — i.e. regions which are the seat of electromagnetic fields but free of charge and mass (all sources are assumed to be immured within the surface $g_{00} = 0$). The main object of the paper is to prove the following result: of all static, asymptotically flat electrovac fields with closed, simply-connected equipotential surfaces $g_{00} = \text{const.}$, the only ones which possess regular event horizons $g_{00} = 0$ are the spheri-symmetric Reissner-Nordström solutions for a charged particle. (A precise formulation is given in Sec. 4.)

In the special case where one neglects the gravitational effect of the electromagnetic energy density [4], it is a straightforward matter to compute the solutions explicitly (Sec. 7).

In the general case, we proceed by reformulating the given conditions in terms of the geometry of the surfaces $g_{00} = \text{const.}$ (Sec. 2—4), showing that the equipotential surfaces of the electric (or magnetic) field necessarily coincide with these surfaces (Sec. 5), and finally proving that they are spheres (Sec. 6). In the interests of mathematical simplicity we shall confine ourselves to the situation where the field is purely electric or purely magnetic. However, there should be no essential difficulty in extending the proof given here to the slightly more general situation of crossed electrostatic and magnetostatic fields.

2. Static Fields

In this section we shall deal with a general static field. Our aim is to reformulate Einstein's static field equations as conditions on the geometry of the equipotential surfaces.

The notation follows reference 1. (Signature of metric $-+++$. Capitalized Latin indices (range 0—3) refer to space-time tensors. Three-dimensional and two-dimensional subtensors are distinguished by Greek indices (range 1—3) and by lower-case Latin indices (values 2, 3). Covariant differentiation with respect to the 4-dimensional, 3-dimensional, and 2-dimensional metrics is denoted by ∇, a stroke and a semicolon respectively.)

A space-time is called "static" if it admits a regular vector field ξ which satisfies Killing's equations:

$$0 = \nabla_A \xi_B + \nabla_B \xi_A = \xi^C \partial_C g_{AB} + g_{AC} \partial_B \xi^C + g_{CB} \partial_A \xi^C \,, \tag{1}$$

is hypersurface-orthogonal:

$$\xi_{[A} \nabla_C \xi_{B]} = 0 \,, \tag{2}$$

and time-like over some domain. With V defined by

$$V = (-\xi_A \xi^A)^{1/2}, \tag{3}$$

the identity

$$\partial_{[A} (V^{-2} \xi_{B]}) = 0 \tag{4}$$

follows from (1) and (2), and shows that a simply-connected domain which has $\xi_A \xi^A < 0$ throughout admits a scalar field $t(x^A)$ such that

$$V^{-2} \xi_A = - \partial_A t \,. \tag{5}$$

In this domain, we can therefore introduce "static co-ordinates". Let the 4-scalars x^α be any three independent solutions of

$$\xi^A \partial_A x^\alpha = 0 \,. \tag{6}$$

In the co-ordinate system $x^0 \equiv t$, x^α we derive $\xi^\alpha = 0$ from (6), then $\xi^0 = 1$, $\xi_\alpha = g_{\alpha 0} = 0$ from (3) and (5), finally $\partial g_{AB}/\partial t = 0$ from (1). Thus the metric is reducible (in the domain where $\xi_A \xi^A < 0$) to the standard form

$$\left.\begin{aligned} ds^2 &= g_{\alpha\beta}(x^1, x^2, x^3)\, dx^\alpha dx^\beta - V^2 dt^2 \,, \\ V &= V(x^1, x^2, x^3) > 0 \,. \end{aligned}\right\} \tag{7}$$

The form (7) can be decomposed further. We suppose that

$$\varrho^{-1} \equiv (V_{|\alpha} V^{|\alpha})^{1/2} \tag{8}$$

vanishes nowhere in the domain of interest (cf. end of Sec. 4). As intrinsic co-ordinates for the equipotential 2-spaces $V =$ const., $t =$ const. introduce functions θ^1, θ^2 which are constant along the orthogonal trajectories: $g^{\alpha\beta}(\partial_\alpha \theta^a)(\partial_\beta V) = 0$. Then the spatial metric reduces to

$$g_{\alpha\beta} dx^\alpha dx^\beta = g_{ab}(V, \theta)\, d\theta^a d\theta^b + [\varrho(V, \theta)]^2\, dV^2 \,. \tag{9}$$

Let $\mathfrak{n}$ be the unit spatial vector normal to the equipotential surfaces,

$$n_\alpha = \varrho \partial_\alpha V = \varrho^{-1} \partial x^\alpha (V, \theta)/\partial V \,, \tag{10}$$

and $\mathfrak{e}_{(a)}$ the tangential base-vectors associated with θ^a,

$$e_{(a)}{}^\alpha = \partial x^\alpha (V, \theta)/\partial \theta^a \,, \qquad \mathfrak{e}^{(a)} \equiv g^{ab} \mathfrak{e}_{(b)} \,. \tag{11}$$

The triad $\{\mathfrak{e}_{(a)}, \mathfrak{n}\}$ spans the 3-space at each point, and the following decompositions are derivable from (9) (by making the special choice of co-ordinates $x^1 = V$, $x^a = \theta^a$, or otherwise [1]):

$$g^{\alpha\beta} = g^{ab} e_{(a)}{}^\alpha e_{(b)}{}^\beta + n^\alpha n^\beta \,, \tag{12}$$

$$\begin{aligned} V_{|\alpha\beta} = \varrho^{-1} K_{ab} e^a{}_{(\alpha)} e^b{}_{(\beta)} &- \varrho^{-2} (\partial_c \varrho)\,(e^{(c)}{}_\alpha n_\beta + e^{(c)}{}_\beta n_\alpha) \\ &- \varrho^{-3} (\partial \varrho/\partial V)\, n_\alpha n_\beta \,. \end{aligned} \tag{13}$$

Here,

$$K_{ab} \equiv \frac{1}{2} \varrho^{-1} \partial g_{ab}/\partial V \tag{14}$$

is the extrinsic curvature of the 2-space $V =$ const., considered as imbedded in the 3-space $t =$ const. Note the related formulas

$$\left.\begin{aligned} \partial g^{ab}/\partial V = -2 \varrho K^{ab} \,, &\qquad \partial g^{1/2}/\partial V = g^{1/2} \varrho K \\ \partial (g^{1/2} R)/\partial V = -2 g^{1/2} [\varrho (K^{ab} &- K g^{ab})]_{;ab} \end{aligned}\right\} \tag{15}$$

which can be deduced from (14); g is the 2×2 determinant of g_{ab}, $R\equiv g^{ab}R_{ab}$ is the two-dimensional curvature invariant (the Gaussian curvature is $-\frac{1}{2}R$), and $K\equiv g^{ab}K_{ab}$ is (twice) the mean curvature. From (13) and (12), we have

$$V^{|\mu}{}_{|\mu}=\varrho^{-1}K-\varrho^{-3}\partial\varrho/\partial V\,. \tag{16}$$

The imbedding relations for the three-dimensional Ricci and Einstein tensors $R_{\alpha\beta}$, $G_{\alpha\beta}$ are [1]

$$G_{\alpha\beta}n^{\alpha}n^{\beta}=\frac{1}{2}\left(K_{ab}K^{ab}-K^{2}-R\right), \tag{17a}$$

$$R_{\alpha\beta}e_{(a)}{}^{\alpha}n^{\beta}=\partial_{a}K-K^{b}_{a;b}\,, \tag{17b}$$

$$R_{\alpha\beta}e_{(a)}{}^{\alpha}e_{(b)}{}^{\beta}=\frac{1}{2}Rg_{ab}+\varrho^{-1}\varrho_{;ab}+KK_{ab}+\varrho^{-1}g_{ap}\partial K^{p}_{b}/\partial V\,. \tag{18}$$

Let X, Y_a and Z_{ab} denote the right-hand sides of (17a), (17b) and (18) respectively. Then the identities

$$g^{-1}(\partial/\partial V)\,(gX)=\varrho Z_{ab}(K^{ab}-Kg^{ab})-\varrho^{-1}(\varrho^{2}Y^{a})_{;a}\,, \tag{19a}$$

$$g^{-1/2}(\partial/\partial V)\,(g^{1/2}Y_{a})=[\varrho(\delta^{b}_{a}Z^{c}_{c}-Z^{b}_{a})]_{;b}+\varrho^{-1}(\varrho^{2}X)_{;a} \tag{19b}$$

are consequences of (14). The left-hand sides of (17) and (18) of course satisfy corresponding identities, whose content is merely that of the contracted Bianchi identities $G^{\beta}_{\alpha|\beta}=0$. Thus, if (14) and (18) are regarded (in a given 3-space and for given ϱ) as a system of first-order equations determining the evolution of g_{ab} and K_{ab} as functions of V, then (17a, b) are "involutive constraints": if they are satisfied on *one* surface $V=$ const. then, by virtue of (19), they are satisfied identically.

We are now ready to decompose the Einstein field equations

$$G_{AB}=-8\pi\gamma T_{AB} \tag{20}$$

($\gamma=7.3\times 10^{-29}$ cm/gm is Newton's constant of gravitation divided by c^2). Under a change of spatial co-ordinates $x^{\alpha'}=x^{\alpha'}(x)$, the 3×3 submatrix $T_{\alpha\beta}$ of the energy tensor T_{AB} transforms as a 3-tensor, and T^{0}_{0} is invariant. The $3+1$ split of (20) yields [1]

$$\frac{1}{2}g^{\alpha\beta}R_{\alpha\beta}=8\pi\gamma T^{0}_{0}\,, \tag{21a}$$

$$0=8\pi\gamma T_{\alpha 0}\,, \tag{21b}$$

$$G_{\alpha\beta}=-8\pi\gamma T_{\alpha\beta}-V^{-1}(V_{|\alpha\beta}-V^{|\mu}{}_{|\mu}g_{\alpha\beta})\,. \tag{21c}$$

We can immediately deduce a relativistic analogue of Poisson's equation:

$$V^{-1}V^{|\mu}{}_{|\mu}=4\pi\gamma(T^{\alpha}_{\alpha}-T^{0}_{0})\,. \tag{22}$$

From (16) and (22),

$$\varrho^{-2}\partial\varrho/\partial V=K-4\pi\gamma V\varrho(T^{\alpha}_{\alpha}-T^{0}_{0})\,. \tag{23}$$

From (18), (21) and (13),

$$V^{-1} g^{-1/2} \partial (g^{1/2} V K_a^b)/\partial V = - \varrho_{;a}{}^{;b} - \frac{1}{2} \varrho R \delta_a^b - 8\pi\gamma\varrho \left(T_{\alpha\beta} e_{(a)}{}^\alpha e_{(b)}{}^\beta - \frac{1}{2} T_A^A g_{ab} \right) . \tag{24}$$

From (17), (21) and (13),

$$\frac{1}{2} (K_{ab} K^{ab} - K^2 - R) = - 8\pi\gamma T_{\alpha\beta} n^\alpha n^\beta + K/(\varrho V) , \tag{25}$$

$$\partial_a K - K_{a;b}^b = - 8\pi\gamma T_{\alpha\beta} e_{(a)}{}^\alpha n^\beta + (\partial_a \varrho)/(\varrho^2 V) . \tag{26}$$

Eqs. (14), (23) and (24) form a complete system for determining the evolution of g_{ab}, ϱ, K_a^b as functions of V. The tangential stresses $T_{\alpha\beta} e_{(a)}{}^\alpha e_{(b)}{}^\beta$ can be assigned arbitrarily over the 3-space; the conservation identities $\nabla_B T^{AB} = 0$ are satisfied automatically if the normal stresses $T_{\alpha\beta} n^\beta$ are determined by (25) and (26).

Finally, we record the expression [1]

$$\frac{1}{4} R_{ABCD} R^{ABCD} = G_{\mu\nu} G^{\mu\nu} + V^{-2} V_{|\mu\nu} V^{|\mu\nu} \tag{27}$$

for the square of the four-dimensional Rieman tensor. Evaluating the second term with the aid of (13), we find

$$\frac{1}{4} R_{ABCD} R^{ABCD} = G_{\mu\nu} G^{\mu\nu} + \varrho^{-2} V^{-2} K_{ab} K^{ab} + 2\varrho^{-4} V^{-2} \varrho_{;a} \varrho^{;a} + \varrho^{-6} V^{-2} (\partial\varrho/\partial V)^2 . \tag{28}$$

3. Static Electrovac Fields

An electromagnetic field

$$F_{BC} = \partial_B A_C - \partial_C A_B \tag{29}$$

in a static space-time is itself static if the Lie-derivative of the 4-potential **A** vanishes:

$$\xi^C \nabla_C A^B - A^C \nabla_C \xi^A = 0 . \tag{30}$$

The field is purely electric (or, by an obvious re-interpretation, purely magnetic) if F_{BC} is a simple bivector of the form

$$F_{BC} = 2 V^{-1} \xi_{[B} E_{C]} . \tag{31}$$

The "electric vector" E_A, defined by (31), may be taken without loss of generality (as long as $\xi_B \xi^B \neq 0$) to be purely spatial ($\xi^A E_A = 0$), and is then given explicitly by

$$E_A = V^{-1} F_{AB} \xi^B , \tag{32}$$

i.e.

$$E_A = - V^{-1} \partial_A \varphi , \quad \varphi \equiv - A_B \xi^B = - A_0 . \tag{33}$$

It is readily verified that

$$\nabla_B F^{AB} = \xi^A \nabla_B (V^{-1} E^B) = \xi^A V^{-1} E^\beta{}_{|\beta} \tag{34}$$

so that, in the absence of charge and current, the Maxwell equations $\nabla_B F^{AB} = 0$ can be written

$$E^\alpha{}_{|\alpha} = (V^{-1} g^{\alpha\beta} \partial_\beta \varphi)_{|\alpha} = 0 \,. \tag{35}$$

By virtue of (9), this can also be written as

$$V g^{-1/2} (\partial/\partial V)\, (g^{1/2} V^{-1} \psi) = -\, (\varrho\, \varphi^{;a})_{;a} \,, \tag{36}$$

with ψ defined by

$$\partial \varphi / \partial V = \varrho\, \psi \,. \tag{37}$$

For an electrovac field (electromagnetic field without matter) the energy tensor is given by

$$\begin{aligned} 4\pi T^{AB} &= F^{AC} F^B{}_C - \frac{1}{4} g^{AB} F_{DC} F^{DC} \\ &= \frac{1}{2} g^{AB} E^2 - E^A E^B + E^2 V^{-2} \xi^A \xi^B \,. \end{aligned} \tag{38}$$

More explicitly, we have

$$\left.\begin{aligned} 4\pi T^0_0 &= -\frac{1}{2} E^2 \,, \\ 4\pi T^{0\alpha} &= 0 \,, \\ 4\pi T^{\alpha\beta} &= \frac{1}{2} g^{\alpha\beta} E^2 - E^\alpha E^\beta \,, \end{aligned}\right\} \tag{39}$$

with

$$V E_\alpha = -\, \psi n_\alpha - e^{(a)}{}_\alpha \partial_a \varphi \,, \tag{40}$$

$$E^2 \equiv E_\alpha E^\alpha = V^{-2} (\psi^2 + \varphi_{;a} \varphi^{;a}) \,. \tag{41}$$

Hence (22) reduces to

$$V^{|\mu}{}_{|\mu} = \gamma V E^2 \,. \tag{42}$$

We substitute (39) into the basic Eqs. (23)—(26) of the previous section. This leads to the following *complete first-order system for determining the march of the variables g_{ab}, φ, ψ, ϱ, K^b_a as functions of V:*

Geometrical equation:

$$\partial g_{ab} / \partial V = 2 \varrho K_{ab} \,. \tag{14}$$

Electrostatic equations:

$$\partial \varphi / \partial V = \varrho\, \psi \,, \tag{37}$$

$$V g^{-1/2} \partial (g^{1/2} V^{-1} \psi) / \partial V = -\, (\varrho\, \varphi^{;a})_{;a} \,. \tag{36}$$

Gravitational equations:

$$\varrho^{-2} \partial \varrho / \partial V = K - \gamma V^{-1} \varrho (\psi^2 + \varphi_{;a} \varphi^{;a}) \,, \tag{43}$$

$$\begin{aligned} V^{-1} g^{-1/2} \partial (g^{1/2} V K^b_a) / \partial V &= -\, \varrho_{;a}{}^{;b} - \frac{1}{2} \varrho R \delta^b_a \\ &\quad + \gamma V^{-2} \varrho [2 \varphi_{;a} \varphi^{;b} - \delta^b_a (\psi^2 + \varphi_{;c} \varphi^{;c})] \,. \end{aligned} \tag{44}$$

Involutive constraints:

$$\frac{1}{2}(K_{ab}K^{ab} - K^2 - R) = \gamma V^{-2}(\psi^2 - \varphi_{;a}\varphi^{;a}) + \varrho^{-1}V^{-1}K\,, \tag{45}$$

$$\partial_a K - K^b_{a;b} = 2\gamma V^{-2}\psi\varphi_{;a} + \varrho^{-2}V^{-1}\varrho_{;a}\,. \tag{46}$$

That the constraints (45), (46) are respected by the equations of evolution can be verified explicitly with the aid of the identities (19).

The following result, which will be needed later, is obtained by contracting (44) and eliminating R by means of (45):

$$V\partial(V^{-1}K)/\partial V = -\varrho_{;a}{}^{;a} - \frac{1}{2}\varrho K^2 - \varrho\Lambda_{ab}\Lambda^{ab} - 2\gamma\varrho V^{-2}\varphi_{;a}\varphi^{;a}\,. \tag{47}$$

Here,

$$\Lambda_{ab} \equiv K_{ab} - \frac{1}{2}K g_{ab} \tag{48}$$

is a measure of the deviation from spherical symmetry.

Combining (28) and (43), we have finally

$$\begin{aligned}\frac{1}{4}R_{ABCD}R^{ABCD} = {} & G_{\mu\nu}G^{\mu\nu} + \varrho^{-2}V^{-2}K_{ab}K^{ab} + 2\varrho^{-4}V^{-2}\varrho_{;a}\varrho^{;a} \\ & + \varrho^{-2}V^{-2}[\gamma V^{-1}\varrho(\psi^2 + \varphi_{;a}\varphi^{;a}) - K]^2\,.\end{aligned} \tag{49}$$

A rather complicated explicit expression for the term $G_{\mu\nu}G^{\mu\nu}$ in terms of the field variables can be obtained from (21c), (39) and (13). For our purposes it will be sufficient merely to note that this term is obviously non-negative.

4. Statement of Theorem

In a static space-time, let Σ be any spatial hypersurface $t = $ const., *maximally extended* consistent with $\xi_A\xi^A < 0$. We consider the class of static fields such that the following conditions are satisfied on Σ:

(i) Σ is an electrovac space (i.e. free of charge and matter).

(ii) Σ is regular, non-compact and "asymptotically Euclidean". The last statement means that there exist co-ordinates x^α in terms of which the metric (7) has the asymptotic form

$$\left.\begin{aligned} g_{\alpha\beta} &= \delta_{\alpha\beta} + 0(r^{-1})\,, & \partial_\gamma g_{\alpha\beta} &= 0(r^{-2})\,, \\ V &= 1 - (m/r) + \eta\,, & m &= \text{const.}, \\ \eta &= 0(r^{-2})\,, \quad \partial_\alpha\eta = 0(r^{-3})\,, & \partial_\alpha\partial_\beta\eta &= 0(r^{-4}) \end{aligned}\right\} (r\to\infty) \tag{50}$$

where $r \equiv (\delta_{\alpha\beta}x^\alpha x^\beta)^{1/2}$.

(iii) The electrovac field is purely electrostatic (or purely magnetostatic). The asymptotic form of the electrostatic (or magnetostatic) scalar potential is

$$\begin{aligned} \varphi &= (e/r) + \zeta\,, & e &= \text{const.}, \\ \zeta &= 0(r^{-2})\,, & \partial_\alpha\zeta &= 0(r^{-3})\,. \end{aligned} \quad (r\to\infty) \tag{51}$$

(iv) The equipotential surfaces $V = \text{const.} > 0$, $t = \text{const.}$ are a regular family of simply-connected closed 2-spaces.

(v) If the greatest lower bound of V on Σ is zero, then the geometry of the equipotential surfaces $V = \varepsilon$ approaches a limit as $\varepsilon \to 0+$, corresponding to a closed regular 2-space of finite area.

(vi) The invariant $R_{ABCD}R^{ABCD}$ is bounded on Σ.

Theorem. *The only static space-time which satisfies conditions (i) — (vi) is the spherically symmetric Reissner-Nordström solution*

$$\left.\begin{aligned} ds^2 &= V^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\Phi^2) - V^2dt^2 \\ V^2 &= 1 - 2m/r + \gamma e^2/r^2\,, \qquad \varphi = e/r\,, \end{aligned}\right\} \tag{52}$$

with $m \geqq \gamma^{1/2}|e|$.

(We here assume $\gamma > 0$. In the exceptional case $\gamma = 0$, discussed in Sec. 7, the Theorem does not hold in quite this form.)

That the Reissner-Nordström manifold with $m \geqq \gamma^{1/2}|e|$ actually does satisfy (i)—(vi) is well-known, and can be easily verified from the explicit formulas

$$\left.\begin{aligned} g_{ab}d\theta^a d\theta^b &= r^2(d\theta^2 + \sin^2\theta\, d\Phi^2)\,, \\ \varrho = V^{-1}dr/dV &= r^3/(mr - \gamma e^2)\,, \quad \psi = -eV/r^2\,, \\ K_{ab} &= Vg_{ab}/r\,, \end{aligned}\right\} \tag{53}$$

obtained by applying the definitions (8), (37) and (14) to the metric (52).

There are two major steps in the proof of the theorem. In Sec. 5 we show, mainly with the aid of the electrostatic Eqs. (36) and (37), that (i)—(vi) can only be satisfied if φ is constant on the equipotential surfaces $V = \text{const.}$: $\varphi = \varphi(V)$. This simplifies the gravitational Eqs. (43) and (44), from which it can then be deduced (Sec. 6) that the equipotential surfaces are spherical.

There is one trivial case for which the proof can be quickly disposed of here. Suppose that V has a positive lower bound. Then the (maximally extended) 3-space Σ is complete. From (35) and the boundary condition (51) we deduce $\varphi \equiv 0$ with the aid of Green's theorem. Eq. (42) now reduces to Laplace's equation $V^{|\mu}{}_{|\mu} = 0$; together with the boundary conditions (50) this yields $V \equiv 1$, showing that space-time is flat and establishing the theorem.

We may assume henceforth that V comes arbitrarily close to zero on Σ. The 2-space $V = 0+$ then forms an inner boundary of Σ and encloses an internal "hole". By (iv), every equipotential surface is homotopic to $V = 0+$. It follows that the gradient of V cannot vanish at any interior point P of Σ. If it did, then by (42) V would have a minimum at P (unless E vanishes identically) and the equipotential surfaces near P could be shrunk to a point. [If $E \equiv 0$, P would be a point of bifurcation of the equipotential surfaces [1], which also contradicts (iv).] Hence ϱ *remains finite, and the metric form* (8) *regular, at all interior points of* Σ. The possibility that $\varrho \to \infty$ as $V \to 0+$ is, however, not excluded.

We conclude this section by recording the exterior and interior boundary conditions in a form convenient for later application. For the asymptotic forms (50) and (51) we find from (8), (15) and (37)

$$\left.\begin{aligned} r \to \infty\,, \quad \varrho/r^2 \to m^{-1}\,, \quad rK \to 2\,, \\ r\varphi \to e\,, \quad r^2\psi \to -e \quad \text{as} \quad V \to 1\,. \end{aligned}\right\} \tag{54}$$

According to (vi) and (49), the regularity of the manifold at the inner boundary $V = 0+$ requires that

$$\left.\begin{aligned} K_{ab} = 0(\varrho V)\,, \quad \varrho_{;a} = 0(\varrho^2 V)\,, \\ \psi = 0(V)\,, \quad \varphi_{;a} = 0(V) \quad \text{as} \quad V \to 0+ \end{aligned}\right\} \tag{55}$$

Thus φ and ϱ^{-1} are constant on the event horizon:

$$\left.\begin{aligned} \varphi(0, \theta^1, \theta^2) = \varphi_0 = \text{const.}, \\ \varrho^{-1}(0, \theta^1, \theta^2) = 1/\varrho_0 = \text{const. (possibly zero)}\,. \end{aligned}\right\} \tag{56}$$

5. The Electrostatic Field

In this section, our interest will center on the electrostatic Eqs. (36) and (37). A number of integral relations will be derived which enable us to show that the electrostatic and gravitational fields must be chained together by the condition $\varphi_{;a} = 0$ if (i)—(vi) are to be satisfied, and we shall determine the function $\varphi = \varphi(V)$ explicitly.

Let $F(V, \varphi)$, $G(V, \varphi)$ be (for the moment, arbitrary) differentiable functions. From (36), (37), (43) and (15) we readily obtain the identity

$$\begin{aligned} & g^{-1/2} \frac{\partial}{\partial V} g^{1/2} [V^{-1} F(V, \varphi)\, \psi + \varrho^{-1} G(V, \varphi)] \\ & = A(V, \varphi)\, \varrho(\psi^2 + \varphi_{;a}\varphi^{;a}) + B(V, \varphi)\, \psi + \varrho^{-1} \partial G/\partial V \\ & \qquad - V^{-1}(F \varrho \varphi^{;a})_{;a}\,, \end{aligned} \tag{57a}$$

where

$$AV = \gamma G + \partial F/\partial\varphi\,, \qquad B = V^{-1}\partial F/\partial V + \partial G/\partial\varphi\,. \tag{57b}$$

To obtain integral conservation laws from (57), let us require that

$$A = B = \partial G/\partial V = 0\,. \tag{58}$$

The general solution of this (over-determined) linear system of differential equations for F, G is a linear combination of the three particular solutions

$$F = 1\,, \qquad G = 0\,, \tag{59a}$$

$$F = \gamma\varphi\,, \qquad G = -1\,, \tag{59b}$$

$$F = \gamma\varphi^2 + V^2\,, \qquad G = -2\varphi\,. \tag{59c}$$

Taking each of these values for F, G in turn, we integrate (57a) over Σ — i.e. we form $\iiint_\Sigma [(57\mathrm{a})]\, g^{1/2} d\theta^1 d\theta^2\, dV$ — noting that the integral of the last term (2-divergence) vanishes when taken over any closed 2-space $V = \text{const}$. The results express the equality of the surface integrals of the expression in square brackets over the two boundary surfaces $V = 1$ and $V = 0+$ (both have to be understood as limits):

$$\int (\psi/V)\, d\mathrm{S}\Big|_{V=0+}^{V=1} = 0\,, \tag{60a}$$

$$\int (\lambda/\varrho)\, d\mathrm{S}\Big|_{V=0+}^{V=1} = 0\,, \tag{60b}$$

$$\int [(V + \gamma\varphi^2/V)\,\psi - 2\varphi/\varrho]\, d\mathrm{S}\Big|_{V=0+}^{V=1} = 0\,. \tag{60c}$$

We have defined the element of area $dS = g^{1/2}\, d\theta^1 d\theta^2$ and

$$\lambda \equiv \frac{1}{2} V^{-1}\partial(V^2 - \gamma\varphi^2)/\partial V = 1 - \gamma\varrho\psi\varphi/V\,. \tag{61}$$

[Eqs. (60a, b) could also have been inferred somewhat more directly from (35) and (42).] Evaluating the surface integrals for $V = 1$ by means of (54), and taking (56) into account, we find

$$\int_{V=0+} (\psi/V)\, d\mathrm{S} = -4\pi e\,, \tag{61a}$$

$$-\gamma\varphi_0 \int_{V=0+} (\psi/V)\, d\mathrm{S} + \mathrm{S}_0/\varrho_0 = 4\pi m\,, \tag{61b}$$

$$\gamma\varphi_0^2 \int_{V=0+} (\psi/V)\, d\mathrm{S} - 2\varphi_0\, \mathrm{S}_0/\varrho_0 = -4\pi e\,, \tag{61c}$$

where S_0 is the area of the 2-space $V = 0+$ [finite and non-vanishing by (v)]. Solving (61), we find

$$\gamma e\varphi_0 = m - (m^2 - \gamma e^2)^{1/2}\,, \tag{62}$$

$$S_0/\varrho_0 = 4\pi(m^2 - \gamma e^2)^{1/2}\,. \tag{63}$$

(A second solution for φ_0 involving the opposite sign for the square root, is unacceptable because it makes S_0/ϱ_0 negative.) For the existence of a regular event horizon it is thus necessary that $m^2 \geqq \gamma e^2$.

To motivate the next step, we begin by observing from (53) that the manifestly nonnegative expression

$$(\varrho V)^{-1}[\varrho\psi(1 - \gamma e\varphi/m) + eV/m]^2 \tag{64}$$

vanishes for spherical symmetry. Now, the expression (64) resembles the right-hand side of (57a) in form. Accordingly, let us require

$$\begin{gathered} A = V^{-1}(1 - \gamma e\varphi/m)^2\,, \qquad B = 2(e/m)\,(1 - \gamma e\varphi/m)\,. \\ \partial G/\partial V = e^2 V/m^2 \end{gathered} \tag{65}$$

in (57a, b). The resulting linear differential equations for F, G have the particular solution

$$\left.\begin{aligned} m^2 F &= -\frac{1}{2}\gamma e^2 V^2 \varphi + \frac{1}{2}\gamma^2 e^2 \varphi^3 - 2\gamma e m \varphi^2 , \\ m^2 G &= \frac{1}{2} e^2 V^2 - \frac{1}{2}\gamma e^2 \varphi^2 + 2 e m \varphi + \gamma^{-1} m^2 . \end{aligned}\right\} \qquad (66)$$

With F, G given by (66), we have thus the identity

$$g^{-1/2}(\partial/\partial V)\, g^{1/2}(V^{-1}F\psi + \varrho^{-1}G) = (\varrho V)^{-1}[\varrho\psi(1-\gamma e\varphi/m) + eV/m]^2 + \varrho V^{-1}(1-\gamma e\varphi/m)^2 \varphi_{;a}\varphi^{;a} - V^{-1}(F\varrho\varphi^{;a})_{;a} . \qquad (67)$$

Integrating over Σ, we deduce the inequality

$$\int_{S_1} (V^{-1}F\psi + \varrho^{-1}G)\, dS \geqq \int_{S_0} (V^{-1}F\psi + \varrho^{-1}G)\, dS , \qquad (68)$$

where S_0, S_1 are the inner and outer boundary surfaces, $V = 0+$ and $V = 1$ (both understood as limits). Equality in (68) holds if and only if

$$\left.\begin{aligned} &\varphi_{;a} = 0 , \quad \text{i.e.} \quad \varphi = \varphi(V) , \\ &\varrho\psi(1-\gamma e\varphi/m) + eV/m = 0 \end{aligned}\right\} \qquad (69)$$

everywhere on Σ. Now, the surface integrals in (68) can actually by evaluated with the aid of (54), (61a), (62) and (63). A straightforward calculation yields the value

$$4\pi m(\gamma^{-1} + \frac{1}{2} e^2/m^2)$$

for *both* sides of (68). We conclude that (69) must be true. (This argument clearly breaks down for the special case of zero coupling: $\gamma = 0$. This case is dealt with separately in Sec. 7.)

From (69) and (37),

$$\varrho\psi = d\varphi/dV = -eV/(m - \gamma e\varphi) . \qquad (70)$$

Solving the differential Eq. (70) subject to $\varphi = \varphi_0$ [see (62)] when $V = 0$, we find

$$\gamma e\varphi(V) = m - \alpha(V) , \qquad (71)$$

where

$$\alpha(V) \equiv [m^2 - \gamma e^2(1 - V^2)]^{1/2} . \qquad (72)$$

(The other boundary condition, $\varphi \to 0$ when $V \to 1$, now shows that the parameter m introduced in (50) cannot be negative.) From (61), (69) and (71) we have the explicit formulae

$$\lambda(V) = m/\alpha(V) , \qquad (73)$$

$$\varrho\psi = -eV/\alpha(V) . \qquad (74)$$

6. The Gravitational Field

The explicit formulae just obtained for the electrostatic field, when substituted into the gravitational Eqs. (43) and (44), result in a great simplification. We are now in a position to derive two integral relations from (43) and (44) which will enable us to infer that only spherical surfaces $V = \text{const.}$ are compatible with conditions (i)—(vi) of Sec. 4. This will complete the proof of our theorem.

Remembering that φ is a function of V only, we have from (47), (43), (36) and (37)

$$\begin{aligned}(\partial/\partial V)\,[(g\lambda/\varrho)^{1/2}\,V^{-1}K] = &-[g\lambda(V)]^{1/2}\,V^{-1}\,[2\,(\varrho^{1/2})^{;a}{}_{;a} \\ &+ \frac{1}{2}\,\varrho^{-3/2}\varrho_{;a}\varrho^{;a} + \varrho^{1/2}\Lambda_{ab}\Lambda^{ab}]\,.\end{aligned} \tag{75}$$

The integration $\iiint\limits_{\Sigma}(75)\,d\theta^1 d\theta^2 dV$ yields

$$\int\limits_{S_1}(\lambda/\varrho)^{1/2}(K/V)\,dS \leq \int\limits_{S_0}(\lambda/\varrho)^{1/2}(K/V)\,dS\,, \tag{76}$$

with equality if and only if

$$\Lambda_{ab} = 0\,, \quad \varrho_{;a} = 0 \tag{77}$$

everywhere on Σ. According to (54), the left-hand side of (76) has the value $8\pi m^{1/2}$. For the right-hand side we have from (55), (45) and (74)

$$\begin{aligned}\lim_{V\to 0+}(K/V) &= -\frac{1}{2}\,\varrho_0 R(0,\theta^1,\theta^2) - \gamma\varrho_0 \lim_{V\to 0+}(\psi/V)^2 \\ &= -\frac{1}{2}\,\varrho_0 R(0,\theta^1,\theta^2) - \gamma e^2/(\varrho_0\alpha_0^2)\,,\end{aligned} \tag{78}$$

where

$$\alpha_0 \equiv \alpha(0) = (m^2 - \gamma e^2)^{1/2}\,. \tag{79}$$

Noting that

$$\int R(c,\theta^1,\theta^2)\,dS = -8\pi \tag{80}$$

for any closed, simply-connected 2-space $V = c$ (Gauss-Bonnet theorem), we obtain the value

$$4\pi(m/\alpha_0)^{1/2}\,[\varrho_0{}^{1/2} - \gamma e^2/(\alpha_0\varrho_0{}^{1/2})] \tag{81}$$

for the right-hand side of (76). We thus arrive at the inequality

$$\alpha_0\varrho_0 \geqq (m + \alpha_0)^2\,. \tag{82}$$

(If $\varrho_0^{-1} = 0$, (82) is properly interpreted as an inequality for $\lim\limits_{V\to 0+}(\varrho\alpha)$.)

We shall next derive a second inequality giving an *upper* bound for $\alpha_0\varrho_0$. If $f(V)$, $h(V)$ are arbitrary differentiable functions, we have

$$\begin{aligned}(\partial/\partial V)\,(fK + \varrho^{-1}h) = (f' + V^{-1}f - h)\,K - \frac{1}{2}\,f\varrho K^2 + \gamma h\varrho\,V^{-1}\psi^2 \\ + \varrho^{-1}h' - f\varrho^{;a}{}_{;a} - f\varrho\Lambda_{ab}\Lambda^{ab}\,.\end{aligned} \tag{83}$$

In the term involving K^2 on the right-hand side, we substitute the value

$$\frac{1}{2}\,K^2 = \Lambda_{ab}\Lambda^{ab} - R - 2\,(\varrho V)^{-1}K - 2\gamma\,V^{-2}\psi^2 \tag{84}$$

obtained from (45), and we substitute for ψ^2 from (74). We also note the result (easily derived from the formulae of Sec. 3)

$$\partial (g^{1/2} \varrho^{-1} \lambda)/\partial V = 0 \, . \tag{85}$$

We thus obtain the identity

$$\begin{aligned}
&(\partial/\partial V)\, [g^{1/2} \varrho^{-1} \lambda (f K + \varrho^{-1} h)] = \lambda(V)\, f(V)\, g^{1/2} R \\
&+ g^{1/2} \varrho^{-1} \lambda\, [(f' + 3 V^{-1} f - h)\, K + \varrho^{-1} \{h' + \gamma V e^2 \alpha^{-2} (h + 2 V^{-1} f)\}] \\
&- \lambda(V)\, f(V)\, g^{1/2}\, [(\ln \varrho)^{;a}{}_{;a} + \varrho^{-2} \varrho_{;a} \varrho^{;a} + \Lambda_{ab} \Lambda^{ab}] \, ,
\end{aligned} \tag{86}$$

valid for arbitrary $f(V)$, $h(V)$. Let us now choose f, h so as to make the second line of (86) vanish, i.e.

$$\begin{aligned}
&f' + 3 V^{-1} f - h = 0 \, , \\
&h' + \gamma V e^2 \alpha^{-2} (h + 2 V^{-1} f) = 0 \, .
\end{aligned} \tag{87}$$

A particular solution is

$$\begin{aligned}
f(V) &= V [\alpha_0 + \alpha(V)]^{-2} \, , \\
h(V) &= 2 [\alpha_0 + \alpha(V)]^{-1} [\alpha(V)]^{-1} \, .
\end{aligned} \tag{88}$$

With these values for f, h (which, it should be noted, are positive on Σ) we form $\iiint_\Sigma$ (86) $d\theta^1 d\theta^2 d V$. We find, recalling (80),

$$\begin{aligned}
&\int_{S_1} \varrho^{-1} \lambda (f K + \varrho^{-1} h)\, dS \\
&\qquad \leqq \int_{S_0} \varrho^{-1} \lambda (f K + \varrho^{-1} h)\, dS - 8\pi \int_0^1 f(V)\, \lambda(V)\, d V \, ,
\end{aligned} \tag{89}$$

with equality if and only if (77) holds everywhere on Σ. Evaluating the surface integrals with the aid of the boundary conditions (54), (55) and (63) yields

$$\int_{S_1} \cdots = 0 \, , \qquad \int_{S_0} \cdots = \lambda(0)\, S_0 (\varrho_0 \alpha_0)^{-2} = 4\pi m \varrho_0^{-1} \alpha_0^{-2} \, . \tag{90}$$

Using the expression (73) for $\lambda(V)$, we obtain further

$$\int_0^1 f \lambda \, d V = \frac{m}{\gamma e^2} \left(\frac{1}{2\alpha_0} - \frac{1}{m + \alpha_0} \right) . \tag{91}$$

The inequality (89) can now be reduced to

$$\alpha_0 \varrho_0 \leqq (m + \alpha_0)^2 \, . \tag{92}$$

Comparing the inequalities (82) and (92), we infer that (77) holds everywhere on Σ, i.e.

$$\varrho = \varrho(V) \, , \qquad K_{ab} = \frac{1}{2} g_{ab} K(V) \, . \tag{93}$$

This implies that the equipotential surfaces are spherical. Indeed, if we introduce a function $r(V)$ defined by $\varphi = e/r$, we can readily deduce the formulae (53) which characterize the Reissner-Nordström field. We recall [remarks following (63) and (72)] that the parameters had to be restricted by $m \geqq \gamma^{1/2}|e|$. Our proof is thus complete.

7. Zero Coupling

We now take up the exceptional case of zero coupling between the gravitational and electromagnetic fields [$\gamma = 0$ in the Einstein field Eq. (20)]. This had to be excluded from the previous considerations [see the remarks after Eq. (69)]. In this case, the simplest procedure is to exhibit the explicit solutions, which are in any case of interest in their own right.

Our problem is to obtain solutions of the vacuum equations $G_{AB} = 0$ and the electrostatic Eq. (35) which satisfy conditions (i)—(vi) of Sec. 4. It is already known [1] that the only vacuum space-times compatible with (i)—(vi) are the Schwarzschild solutions with $m \geqq 0$. The problem thus reduces to finding well-behaved electrostatic fields defined on the Schwarzschild background:

$$\begin{aligned} g_{\alpha\beta} dx^\alpha dx^\beta &= (1 - 2m/r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\Phi^2) \,, \\ V &= (1 - 2m/r)^{1/2} \,. \end{aligned} \tag{94}$$

The electrostatic Eq. (35), which is linear in φ, reduces to

$$\left(1 - \frac{2m}{r}\right) \frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \varphi}{\partial \Phi^2} = 0 \,. \tag{95}$$

Separable solutions which are regular on the axis have the form

$$\varphi = R(r) \, P_n^M(\cos\theta) \, e^{iM\varphi} \,, \tag{96}$$

where R satisfies

$$x(1 + x) \, d^2R/dx^2 + 2x dR/dx - n(n + 1) \, R = 0 \,, \tag{97}$$

$$x \equiv (r/2m) - 1 \,. \tag{98}$$

For $n = 0$, we have the spherically symmetric solution

$$\varphi + \text{const.} = e/r \tag{99}$$

which is regular for $2m \leqq r < \infty$ and satisfies (i)—(vi).

For $n = 1$, $R = x$ is an obvious solution of (97), and the second solution can be found by variation of parameters. We thus obtain the axially symmetric solutions ($M = 0$)

$$\varphi(r, \theta) = c_1(r - 2m)\cos\theta\,, \tag{100a}$$

$$\varphi(r, \theta) = c_2[-1 + m/r - (r/2m - 1)\ln(1 - 2m/r)]\cos\theta\,. \tag{100b}$$

The first represents a uniform electrostatic or magnetostatic field. The second is the static field of an electric or magnetic dipole; its asymptotic form is

$$\varphi \approx c_2 \cos\theta / r^2 \; (r \to \infty)\,. \tag{101}$$

Results equivalent to (100b) have been given previously by GINZBURG [4].

For general n, (97) is reducible to Legendre's equation, and has the linearly independent solutions

$$R = c_1[x/(1+x)]^{1/2}\mathfrak{P}_n^1(1+2x), \quad R = c_2[x/(1+x)]^{1/2}\mathfrak{Q}_n^1(1+2x) \tag{102}$$

where $\mathfrak{P}_n^1(z)$, $\mathfrak{Q}_n^1(z)$ are the associated Legendre functions, normalized to be real for real $z > 1$. For the electrostatic field we thus have the two families of solutions

$$\varphi = c_1\left(1 - \frac{2m}{r}\right)^{1/2}\mathfrak{P}_n^1\left(\frac{r}{m} - 1\right)P_n^M(\cos\theta)\,e^{iM\Phi}\,, \tag{103a}$$

$$\varphi = c_2\left(1 - \frac{2m}{r}\right)^{1/2}\mathfrak{Q}_n^1\left(\frac{r}{m} - 1\right)P_n^M(\cos\theta)\,e^{iM\Phi}\,, \tag{103b}$$

Since

$$\mathfrak{P}_n^1(z) \sim z^n, \quad \mathfrak{Q}_n^1(z) \sim z^{-(n+1)} \; (z \to \infty\,, \quad n \geqq 1)\,, \tag{104}$$

only the second solution has the correct behaviour at infinity. To examine the behaviour of (103b) near $r = 2m$, we observe from (41) that the field strength E is given by

$$E^2 = \left(1 - \frac{2m}{r}\right)^{-1}\left[\psi^2 + \frac{1}{r^2}\left(\frac{\partial\varphi}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial\varphi}{\partial\Phi}\right)^2\right]. \tag{105}$$

Now, $\mathfrak{Q}_n^1(1+2x) \sim x^{-1/2}$ as $x \to 0$, so that the radial factor (102) approaches a constant non-zero limit. Hence, as $r \to 2m$, $\partial\varphi/\partial\theta$ and (for $M \neq 0$) $\partial\varphi/\partial\Phi$ remain of order unity for $n \geqq 1$, and E is of order $(1 - 2m/r)^{-1/2}$. We conclude that the only electrostatic field on a Schwarzschild background which is well-behaved for $2m \leqq r < \infty$ is spherically symmetric. Nevertheless, *all* the solutions (103b) are compatible with (i)—(vi), since the geometry is regular at $r = 2m$ despite the singularity of the energy density, on account of the zero coupling.

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