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A Note on the Decrease of Truncated Wightman Functions for Large Space-like Separation of the Arguments

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Abstract. The truncated Wightman functions cannot decrease arbitrarily fast for large space-like separation of the arguments. For certain configurations they can fall off at most exponentially.

Upper bounds on the decrease of truncated Wightman functions were established a long time ago [1–5]. For instance, for a relativistic quantum field theory of a self-interacting neutral, scalar field $A(x)$ H. Araki [2] (compare the footnote in [5]) proved the following theorem: Under the assumptions of a) Lorentz invariance, b) temperedness of the Wightman functions, c) the existence of a lowest non-zero mass, the truncated vacuum expectation value (TVEV)

$$\langle A(x_0) \ldots A(x_n)\rangle^T$$

vanishes *at least* exponentially for $x_{i-1} - x_i = \xi_i + \lambda \xi_i'$ $i = 1, \ldots, n$ where $\xi_i + \lambda \xi_i'$ should be a Jost point for sufficiently large λ and $\lambda \to +\infty$, ξ_i, ξ_i' fixed (with at least one $\xi_i' \neq 0$).

Here we want to point out that a *lower* bound on the decrease of the TVEV for similar configurations can be obtained as well. We do not assume locality or the existence of a lowest non-zero mass.

To begin with, let us consider the 2-point function. Lorentz invariance, temperedness and positive definiteness imply the well-known Källén-Lehmann representation

$$\langle A(x_0)\, A(x_1)\rangle^T = \langle A(x_0)\, A(x_1)\rangle = i \int_0^\infty d\varrho(\mu)\, \Delta_\mu^+(x_0 - x_1)\,,$$

$\varrho(\mu)$ a positive tempered measure

$$\sim \frac{\text{const}}{-(x_0 - x_1)^2}$$

$$\left(\text{or} \sim \frac{\sqrt{m}}{2^{5/2}\, \pi^{3/2} \sqrt{-(x_0 - x_1)^2}^{3/2}} \exp\{-m \sqrt{-(x_0 - x_1)^2}\} \text{ in case of the existence of a lowest non-zero mass } m \text{ in the theory}\right).$$

Next, we turn to the 3-point function. It is analytic in the "extended tube" $\mathscr{T}'_{0,1,2}$ the boundaries of which are explicitly known in terms of the invariants [6]. Consider

$$W_2^T(x_0, x_1, x_2) = \langle A(x_0)\, A(x_1)\, A(x_2)\rangle^T$$

for $x_{i-1} - x_i = \xi_i + \lambda\xi'_i$ $i = 1, 2$ with x_0, x_1, x_2 totally space-like in the order $0, 1, 2$ for sufficiently large positive λ, ξ_i, ξ'_i $i = 1, 2$ fixed, not both $\xi'_i = 0$.

Define

$$w_2^T(\lambda; \xi_i, \xi'_i) = W_2^T(x_0, x_1, x_2) .$$

$w_2^T(\lambda; \xi_i, \xi'_i)$ is real-analytic for sufficiently large positive λ and can be analytically continued in λ into a wedge-shaped region with the following angle:

$$\begin{array}{ll}
\pi - \operatorname{arc\,tg}\sqrt{\dfrac{\xi_1'^2\xi_2'^2 - (\xi'_1\cdot\xi'_2)^2}{(\xi'_1\cdot\xi'_2)^2}} & \text{if } \ \xi'_i\cdot\xi'_j < 0 \ \ i,j = 1,2, \ \ \xi_1'^2\xi_2'^2 > (\xi'_1\cdot\xi'_2)^2 \\
\pi & \text{if } \ \xi'_i\cdot\xi'_j < 0 \ \ i,j = 1,2, \ \ \xi_1'^2\xi_2'^2 < (\xi'_1\cdot\xi'_2)^2 \\
\operatorname{arc\,tg}\sqrt{\dfrac{\xi_1'^2\xi_2'^2 - (\xi'_1\cdot\xi'_2)^2}{(\xi'_1\cdot\xi'_2)^2}} & \text{if } \ \xi_1'^2 < 0, \ \xi_2'^2 < 0, \ \xi'_1\cdot\xi'_2 \geqq 0 \\
\pi & \text{if } \ \begin{array}{l} \xi'_1 = 0, \ \xi_2'^2 < 0, \ \xi_1\cdot\xi'_2 < 0 \\ \text{or } \ \xi_1'^2 < 0, \ \xi'_2 = 0, \ \xi'_1\cdot\xi_2 < 0 \end{array} \\
2\operatorname{arc\,tg}\sqrt{\dfrac{\xi_1^2\xi_2'^2 - (\xi_1\cdot\xi'_2)^2}{(\xi_1\cdot\xi'_2)^2}} & \text{if } \ \xi'_1 = 0, \ \xi_2'^2 < 0, \ \xi_1\cdot\xi'_2 > 0 \\
2\operatorname{arc\,tg}\sqrt{\dfrac{\xi_1'^2\xi_2^2 - (\xi'_1\cdot\xi_2)^2}{(\xi'_1\cdot\xi_2)^2}} & \text{if } \ \xi_1'^2 < 0, \ \xi'_2 = 0 \ \ \xi'_1\cdot\xi_2 > 0 .
\end{array}$$

In particular, $w_2^T(\lambda; \xi_i, \xi'_i)$ can be analytically continued in λ into a half-plane for all $(\xi_1, \xi_2, \xi'_1, \xi'_2) \in S_2$:

$$\begin{aligned}
S_2 = \{(\xi_1, \xi_2, \xi'_1, \xi'_2)/\xi'_1, \xi'_2 &\in \text{a plane that contains a time-like vector,} \\
&\xi'_i\cdot\xi'_j < 0 \quad i,j = 1,2\} \\
&\cup \{(\xi_1, \xi_2, \xi'_1, \xi'_2)/\xi'_1 = 0, \quad \xi_2'^2 < 0, \quad \xi_1^2 < 0, \quad \xi_1\xi'_2 < 0\} \\
&\cup \{(\xi_1, \xi_2, \xi'_1, \xi'_2)/\xi_1'^2 < 0, \quad \xi'_2 = 0, \quad \xi_2^2 < 0, \quad \xi'_1\cdot\xi_2 < 0\} .
\end{aligned}$$

This analyticity domain together with the temperedness implies according to theorem 5.1.12 [7] that $w_2^T(\lambda; \xi_i, \xi'_i)$ for $(\xi_1, \xi_2, \xi'_1, \xi'_2) \in S_2$ can decrease at *most* (linearly) exponentially, i.e.

$$\limsup_{\lambda\to+\infty} \frac{\log|w_2^T(\lambda; \xi_i, \xi'_i)|}{\lambda} = -M_2(\xi_i, \xi'_i) > -\infty$$

unless $w_2^T(\lambda; \xi_i, \xi'_i) \equiv 0$.

It is not difficult now to treat the general truncated n-point function by the same method. One considers

$$W_n^T(x_0, \ldots, x_n) = \langle A(x_0) \ldots A(x_n)\rangle^T$$

for $x_{i-1} - x_i = \xi_i + \lambda\xi'_i$ $i = 1, \ldots, n$ with $x_0, \ldots, x_n$ totally space-like in the order $0, 1, \ldots, n-1, n$ for sufficiently large positive λ, ξ_i and ξ'_i fixed, not all $\xi'_i = 0$.

$w_n^T(\lambda; \xi_i, \xi'_i) = W_n^T(x_0, \ldots, x_n)$ is real-analytic for sufficiently large positive λ and can be analytically continued in λ into a half-plane for all

$$(\xi_{i_{i=1,\ldots,n}}, \xi'_{i_{i=1,\ldots,n}}) \in S_n$$

$S_n = S_n^{(1)} \cup S_n^{(2)} \cup S_n^{(3)}$

$S_n^{(1)} = \{(\xi_{i_{i=1,\ldots,n}}, \xi'_{i_{i=1,\ldots,n}}) / \xi'_i \in$ a plane that contains a time-like vector, $\xi'_i \cdot \xi'_j < 0$ for all $i, j \in (1, \ldots, n)\}$

$S_n^{(2)} = \{(\xi_{i_{i=1,\ldots,n}}, \xi'_{i_{i=1,\ldots,n}}) / \xi'_{i_1} = \cdots = \xi'_{i_t} = 0 \;\; t < n-1$, $\xi'_{i_{i \notin (i_1, \ldots, i_t)}} \in$ a plane that contains a time-like vector, not all of the $\xi'_{i_{i \notin (i_1, \ldots, i_t)}}$ on a space-like line, $\xi'_i \cdot \xi'_j < 0$ for all $i, j \notin (i_1, \ldots, i_t)$, $\xi_i \cdot \xi'_j < 0$ for all $i \in (i_1, \ldots, i_t)$, $j \notin (i_1, \ldots, i_t)$, $(\xi'_j (\xi_i, \xi'_l) - \xi'_l (\xi_i, \xi'_j))^2 > 0$ for all $i \in (i_1, \ldots, i_t)$ and for at least two linearly independent ξ'_j, ξ'_l $j, l \notin (i_1, \ldots, i_t)$, $\xi_i^2 < 0$ for all $i \in (i_1, \ldots, i_t)\}$,

$S_n^{(3)} = \{(\xi_{i_{i=1,\ldots,n}}, \xi'_{i_{i=1,\ldots,n}}) / \xi'_{i_1} = \cdots = \xi'_{i_t} = 0 \;\; t < n$, with $\xi'_{i_{i \notin (i_1, \ldots, i_t)}}$ on a space-like line, $\xi'_i \cdot \xi'_j < 0$ for all $i, j \notin (i_1, \ldots, i_t)$, $\xi_i \cdot \xi'_j < 0$ for all $i \in (i_1, \ldots, i_t)$, $j \notin (i_1, \ldots, i_t)$, there exists a time-like vector $\eta, \eta^0 > 0$ such that $(\eta (\xi_i, \xi'_j) - \xi'_j (\xi_i, \eta))^2 + \xi_i^2 (\eta, \xi'_j)^2 > 0$ for all $i \in (i_1, \ldots, i_t), j \notin (i_1, \ldots, i_t)\}$.

Again we may invoke the theorem 5.1.12 of [7] and conclude that for all $(\xi_{i_{i=1,\ldots,n}}, \xi'_{i_{i=1,\ldots,n}}) \in S_n$ $w_n^T(\lambda; \xi_i, \xi'_i)$ can decrease *at most* (linearly) exponentially

$$\limsup_{\lambda \to +\infty} \frac{\log |w_n^T(\lambda; \xi_i, \xi'_i)|}{\lambda} = - M_n(\xi_i, \xi'_i) > -\infty$$

unless $w_n^T(\lambda; \xi_i, \xi'_i) \equiv 0$.

More detailed information about the decrease of $w_n^T(\lambda; \xi_i, \xi'_i)$ for $(\xi_i, \xi'_i) \in S_n$ is given by the following theorem 10.4.1 of [7]: For each positive ε and δ there is a sequence $\{\lambda_n\}$, $\lambda_n \to \infty$ and a positive η such that the subset of $(\lambda_n, \lambda_n + \delta \lambda_n)$ in which

$$\lambda^{-1} \log |w_n^T(\lambda; \xi_i, \xi'_i)| \geqq M_n(\xi_i, \xi'_i) - \varepsilon$$

has measure $\geqq \eta \cdot \lambda_n$.

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