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Infinite Volume Limit of a $\lambda\phi^4$ Field Theory

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We study a boson field with a $\lambda\phi^4$ self interaction. Such a model has already been studied by one author in the case of a theory with an ultra-violet cut-off, quantized in a box with periodic boundary conditions [1—2]. In this note we study the vacuum-state functional of that theory in the limit of infinite volume, keeping a fixed ultra-violet cut-off. We show that the vacuum functional converges to a regular state of the Weyl algebra of the canonical commutation relations. This state is translation invariant. We believe that this infinite volume state is not given by a density matrix in Fock space, although we have no proof at this time.

The theory in the box of volume V is described in terms of canonical fields

$$\phi(V;\mathrm{x}) = \frac{1}{(2V)^{1/2}} \sum_{\mathbf{k}} \{a(\mathbf{k})\, e^{i\mathbf{k}\mathrm{x}} + a^*(\mathbf{k}) e^{-i\mathbf{k}\mathrm{x}}\} \frac{1}{\sqrt{\omega(\mathbf{k})}}\,,$$

and

$$\pi(V;\mathrm{x}) = \frac{-i}{(2V)^{1/2}} \sum_{\mathbf{k}} \{a(\mathbf{k})\, e^{i\mathbf{k}\mathrm{x}} - a^*(\mathbf{k}) e^{-i\mathbf{k}\mathrm{x}}\} \sqrt{\omega(\mathbf{k})}\,,$$

where $\mathbf{k} = 2\pi V^{-1/3}(\nu_1, \nu_2, \nu_3)$ for integers ν_j, $\omega(\mathbf{k}) = (\mathbf{k}^2 + m^2)^{1/2}$, and $[a(\mathbf{k}), a^*(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}$. The corresponding Weyl algebra is generated by $\exp i\phi(V;f)$, $\exp i\pi(V;g)$, where $f, g \in \mathfrak{D}(V)$, the set of infinitely differentiable real functions with support strictly contained inside the cube V centered at the origin and having volume V. The Weyl relation is

$$\exp i\phi(V;f) \exp i\pi(V;g) = \exp - i(f,g) \exp i\pi(V;g) \exp i\phi(V;f)$$

where $(f,g) = \int f(\mathrm{x})\, g(\mathrm{x})\, d\mathrm{x}$.

The Hamiltonian for the theory in the box is

$$H(V) = \int_V H(V;\mathrm{x})\, d\mathrm{x}\,,$$

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and the Hamiltonian density $H(V; \mathrm{x})$ is expressed in terms of the fields by

$$H(V; \mathrm{x}) = \frac{1}{2} : \pi^2(V; \mathrm{x}) + (\nabla\phi(V; \mathrm{x}))^2 + m^2\phi^2(V; \mathrm{x}) : + \lambda : \phi_K^4(V; \mathrm{x}) : .$$

The cut-off field $\phi_K(V; \mathrm{x})$ is defined by

$$\phi_K(V; \mathrm{x}) = \frac{1}{(2V)^{1/2}} \sum_{\mathbf{k}} \{a(\mathbf{k})\, e^{i\mathbf{k}\mathbf{x}} + a^*(\mathbf{k})\, e^{-i\mathbf{k}\mathbf{x}}\} \frac{1}{\sqrt{\omega(\mathbf{k})}} \chi_K(\mathbf{k}),$$

and

$$\chi_K(\mathbf{k}) = \begin{cases} 1, \text{ if } |k_j| \leq K,\ j = 1, 2, 3 \\ 0 \text{ otherwise}. \end{cases}$$

In the infinite volume case, the canonical variables are chosen to be

$$\phi(\mathrm{x}) = \frac{1}{\sqrt{2}\,(2\pi)^{3/2}} \int d\mathbf{k} \{a(\mathbf{k})\, e^{i\mathbf{k}\mathbf{x}} + a^*(\mathbf{k})\, e^{-i\mathbf{k}\mathbf{x}}\} \frac{1}{\sqrt{\omega(\mathbf{k})}},$$

and

$$\pi(\mathrm{x}) = \frac{1}{\sqrt{2}\,(2\pi)^{3/2}} \int d\mathbf{k} \{a(\mathbf{k})\, e^{i\mathbf{k}\mathbf{x}} - a^*(\mathbf{k})\, e^{-i\mathbf{k}\mathbf{x}}\} \sqrt{\omega(\mathbf{k})},$$

with $[a(\mathbf{k}), a^*(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')$. The corresponding Weyl algebra is generated by the operators satisfying

$$\exp i\phi(f) \exp i\pi(g) = \exp - i(f, g) \exp i\pi(g) \exp i\phi(f).$$

For each volume V, we identify the corresponding Weyl algebra as a sub-algebra of the standard algebra $\mathfrak{A}$ for infinite volume. In particular, for each function $f \in \mathfrak{D}(V)$ we let $\exp i\phi(V; f)$ correspond to $\exp i\phi(f)$ in the infinite volume algebra. We call the image of the Weyl algebra for volume V the sub-algebra $\mathfrak{A}(V)$ of the standard algebra $\mathfrak{A}$.

In [1—2] it was shown that for each finite volume V, the Hamiltonian H_V is a self adjoint operator on Fock space, and it has a unique, translation-invariant ground state vector $\Psi_0(V)$, the physical vacuum. This vector corresponds to the state ω_V on $\mathfrak{A}(V)$. According to the above correspondence, if $f, g \in \mathfrak{D}(V)$, then

$$\omega_V(\exp i\phi(f) \exp i\pi(g)) = (\Psi_0(V), \exp i\phi(V; f) \exp i\phi(V; g)\, \Psi_0(V)).$$

Furthermore, the vacuum expectation values of the fields are defined for each finite volume V.

Theorem. *Let ω_n be the vacuum state for the above theory in volume V_n, where the sequence of cubes V_n increases to cover $\mathbb{R}^3$ as $n \to \infty$. Let P be the polynomial algebra of the Weyl algebra of the canonical commutation relations. Then there is a subsequence $\omega_{k(n)}$ of ω_n such that for all $a \in P$, $\omega_{k(n)}(a)$ converges to $\omega(a)$ as $k(n) \to \infty$. Furthermore ω is a regular translation-invariant state for the canonical commutation relations on $\mathfrak{A}$.*

Remarks. 1. The state ω need not be a density matrix in Fock space.

2. The state ω is regular in the sense that $\omega(\exp i\phi(f) \exp i\pi(g))$ is defined for all $f, g \in \mathfrak{D}(\mathbb{R}^3)$, and it is continuous in f and g in a suitable topology.

Lemma 1. *Whenever $f, g \in \mathfrak{D}(V_n)$,*

$$|\omega_n(\phi(f)\,\phi(g)| \leqq C|f|\,|g|\,,$$

and

$$|\omega_n(\pi(f)\,\pi(g))| \leqq C|f|\,|g|\,,$$

where C is independent of n and

$$|f| = \|f\|_{L^1} + \|(1-\Delta)f\|_{L^2}\,.$$

Proof. This result is essentially contained in [3]. Following that reference, we split the Fock Hilbert space for the theory in volume V_n, into a tensor product $\mathfrak{H}_1(V_n) \otimes \mathfrak{H}_2(V_n)$. The n-particle states of the modes which enter the interaction Hamiltonian have their image in $\mathfrak{H}_1(V)$, while $\mathfrak{H}_2(V)$ is associated with the remaining modes. Note that ω_n corresponds to the expectation value in the vector $\Psi_0(V_n)$, and the ground state of $H(V_n)$ factors in $\mathfrak{H}_1(V_n) \otimes \mathfrak{H}_2(V_n)$. This vector is represented by $\Psi_{1n} \otimes \Psi_{2n}$. Since the Hamiltonian $H(V_n)$ is invariant under $\phi(V_n) \to -\phi(V_n)$ and $\pi(V_n) \to -\pi(V_n)$, we infer that for $f, g \in \mathfrak{D}(V_n)$,

$$\omega_n(\phi(f)\,\phi(g)) = \omega_{1n}(\phi_1(f)\,\phi_1(g)) + \omega_{2n}(\phi_2(f)\,\phi_2(g))\,,$$

and

$$\omega_n(\pi(f)\,\pi(g)) = \omega_{1n}(\pi_1(f)\,\pi_1(g)) + \omega_{2n}(\pi_2(f)\,\pi_2(g))\,.$$

Furthermore, ω_{2n} corresponds to the expectation value in the Fock no-particle state $\Phi_{02}(V_n)$ in the Hilbert space $\mathfrak{H}_2(V_n)$, and the corresponding free-field vacuum expectation values trivially satisfy the lemma. Hence it is only necessary to prove the result for ω_{1n}, ϕ_1 and π_1. For $f, g \in \mathfrak{D}(V_n)$ we have

$$\begin{aligned}|\omega_{1n}(\phi_1(f)\,\phi_1(g))| &\leqq d\|f\|_{L^1}\|g\|_{L^1}(\Psi_{1n}, \{a + b\,V_n^{-1}H(V_n)\}\Psi_{1n}) \\ &\leqq d\|f\|_{L^1}\|g\|_{L^1}(\Phi_{01}(V_n), \{a + b\,V_n^{-1}H(V_n)\}\,\Phi_{01}(V_n)) \\ &\leqq c\|f\|_{L^1}\|g\|_{L^1}\,,\end{aligned}$$

where the constants a, b, c, d are shown by computation to be independent of n. The analogous result holds for the π's, which proves the lemma.

Lemma 2. *Let $\{\omega_n\}$ be the sequence of vacuum functionals of Lemma 1. Then there is a subsequence $\{\omega_{k(n)}\}$ such that for all $f, g \in \mathfrak{D}$, the numerical sequence $\omega_{k(n)}(\exp i\phi(f)\exp i\pi(g))$ is convergent.*

Proof. Remark that for $k(n)$ sufficiently large, the numerical sequence $\omega_{k(n)}(\exp i\phi(f)\exp i\pi(g))$ makes sense. From the Weyl form of the commutation relations and from Lemma 1 (the boundedness of the two point functions) we infer that the functionals $\omega_n(\exp i\phi(f)\exp i\pi(g))$ are continuous uniformly in n. In fact, one can easily show that

$$|\omega_n(\exp i\phi(f)\exp i\pi(g)) - \omega_n(\exp i\phi(f')\exp i\pi(g'))| \leqq$$
$$\leqq C\{|f-f'| + |g-g'|\}\,.$$

It is important that the constant C does not depend on n. Thus the functionals $\omega_n(\exp i\phi(f)\exp i\pi(g))$ are equicontinuous.

Let $\{f_n\}$ be a countable dense set in $\mathfrak{H}$. (That is, for each $f \in \mathfrak{H}$, a sequence of the f_n's approximates f in the norm $|\cdot|$ defined above.) By the well-known diagonal procedure, one can construct a subsequence $\omega_{k(n)}$ such that the numerical sequences $\omega_{k(n)}(\exp i\phi(f_r) \exp i\pi(f_s))$ tend to a limit as $k(n) \to \infty$ for all r, s. Since the functionals $\omega_{k(n)}(\exp i\phi(f) \exp i\pi(g))$ are equicontinuous and since the $\{f_n\}$ are dense in $\mathfrak{H}$, the lemma follows.

Proof of the Theorem. The sequence $\omega_{k(n)}$ of Lemma 2 converges to a state ω on the polynomial algebra P of the Weyl operators. This limit state is regular, since ω is the limit of equicontinuous functionals. The translation invariance of the state ω follows from the translation invariance of the vacuum functionals $\omega_{k(n)}$ in the box.

References

1. Jaffe, A. M.: Existence theorems for a cut-off $\lambda\phi^4$ field theory. In: Mathematical theory of elementary particles (R. Goodman and I. E. Segal, Ed.). p. 45—58. MIT Press 1966.
2. — Dynamics of a cut-off $\lambda\phi^4$ field theory. Princeton Thesis 1965 and American Mathematical Society Memoir, to appear.
3. — Preliminary results on infinite volume limits in a cut-off $\lambda\phi^4$ field theory. MIT Conference on Constructive Quantum Field Theory, April 1966 (I. E. Segal, Ed.).

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