

On the Factor Type of Equilibrium States in Quantum Statistical Mechanics

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Abstract. A theorem is derived giving sufficient conditions for a factor to be either finite or purely infinite. These conditions are:

i. In the Hilbert space \mathfrak{H} exists a conjugation operator J transforming the factor \mathfrak{R} into its commutant \mathfrak{R}' .

ii. There exists a one parameter abelian group of automorphisms of \mathfrak{R} implemented by unitary operators U_t weakly continuous in t and commuting with J .

iii. There is a cyclic and separating vector Ω , which is invariant for J and which is the only vector in \mathfrak{H} invariant for U_t .

This theorem is of interest for Statistical Mechanics since representations of thermal equilibrium states satisfy these conditions [1]. One finds that the representations of equilibrium states corresponding to one phase are factors of type III.

1. Introduction and Motivation

In this note we shall discuss and prove a theorem giving sufficient conditions for a factor to be either finite or purely infinite (type III). Since these conditions arise from physical considerations we shall use this introductory section to discuss the connection between these conditions and properties of the equilibrium states in quantum statistical mechanics. Some of the consequences of the theorem will be discussed in section 3.

Our starting point will be a C^* -algebra \mathfrak{A} of quasi-local observables and a one parameter group of automorphisms $A \in \mathfrak{A} \rightarrow A_t \in \mathfrak{A}$ corresponding to time-evolution. We shall here take the point of view that a thermal equilibrium state ω is a positive linear form over the C^* -algebra \mathfrak{A} satisfying the following conditions

1. ω is invariant, i. e., $\omega(A_t) = \omega(A)$.
2. $\omega(A^*A) = 0$ implies $A = 0$.
3. For fixed A and B $\omega(A_t B)$ is a function of t which can be continued analytically in the strip $0 > \text{Im} t > \beta$ and is continuous on the boundaries. Similarly the function $\omega(BA_t)$ is analytical in the strip $0 < \text{Im} t < \beta$, and

$$\omega(A_t B)_{t=t_0-i\beta} = \omega(BA_t)_{t=t_0}. \quad (1)$$

Here $\beta = 1/kT$, where k is the Boltzmann constant and T the temperature. This condition is known as the Kubo-Martin-Schwinger boundary condition. For the connection of these three conditions on the state with the well-known definition of the equilibrium state of finite systems by means of Gibbs-ensembles we refer to [1].

A particular situation arises in the case of infinite temperature ($\beta = 0$). We then have from (1) $\omega(AB) = \omega(BA)$ for all A and $B \in \mathfrak{A}$, so that ω is a finite trace over \mathfrak{A}^+ . If on the other hand ω is a finite trace we shall show that necessarily $\beta = 0$ or $T = \infty$. If ω is a trace and $\beta \neq 0$, (1) implies that $f(t) = \omega(A_t B)$ is analytical for $-\beta < \text{Im}t < \beta$, and $f(t) = f(t + i\beta)$ for all $-\beta \leq \text{Im}t \leq 0$. Hence $f(t)$ can be extended to the whole complex plane and is periodic with period $i\beta$. Furthermore $f(t)$ is bounded since for real t $\omega(A_t B) \leq \omega(e) \|A\| \cdot \|B\|$. Consequently $f(t)$ is a constant. This implies that $\omega\{(A - A_t)B\} = 0$ for all A, B and t . Taking $B = (A - A_t)^*$ we get $\omega\{(A - A_t)(A - A_t)^*\} = 0$ and, using condition 2, we find that $A = A_t$ for all A and t , in contradiction to our assumptions. We conclude that $\beta = 0$.

We shall now consider the cyclic representation of \mathfrak{A} defined by ω and obtained by means of the Gelfand-Naimark-Segal construction. Let $R(A)$ represent A as a bounded operator in the Hilbert-space \mathfrak{H} , let $\Omega \in \mathfrak{H}$ be the cyclic vector such that $\omega(A) = (\Omega, R(A)\Omega)$, and let \mathfrak{R} be the von Neumann algebra generated by $R(\mathfrak{A})$. On the basis of the conditions 1, 2 and 3 one proves [1] that

i. There exists a conjugation operator J (an anti-unitary operator J with $J^2 = 1$) such that $J\mathfrak{R}J = \mathfrak{R}'$ and $J\Omega = \Omega$.

ii. There exists a unitary operator U_t continuous in t which implements the automorphism and has the properties

$$U_t \Omega = \Omega \text{ and } [U_t, J] = 0$$

In section 2 we shall make one more assumption about the state ω .

iii. Ω is the only invariant state. This property is a direct consequence of the fact that an equilibrium state of a system consisting of one phase only is extremal invariant or ergodic [2], [3]. Using this property and the K.M.S.-boundary condition one can show that \mathfrak{R} is a factor. This means that the invariance for time-translation cannot be broken spontaneously.

2. Theorem

Let \mathfrak{R} be a factor and Ω a cyclic vector in the Hilbert-space \mathfrak{H} satisfying the conditions

i. There exists a conjugation operator J such that $J\mathfrak{R}J = \mathfrak{R}'$ and $J\Omega = \Omega$.

ii. There exists a one parameter abelian group of automorphisms of \mathfrak{R} implemented by unitary operators U_t continuous in t with the properties

$$U_t \Omega = \Omega \text{ and } [U_t, J] = 0 .$$

iii. Ω is the only invariant state in \mathfrak{S} . Then there are the following two possibilities:

- a. \mathfrak{R} is of finite type and Ω is a trace vector.
- b. \mathfrak{R} is of type III.

To prove the theorem we assume that \mathfrak{R} is semi-finite and show that possibility a) occurs. Let $\phi(R)$ be a normal semi-finite trace, $\mathfrak{R} < \mathfrak{R}$ the two-sided ideal of all elements $\kappa \in \mathfrak{R}$ such that $\phi(\kappa^* \kappa) < \infty$.

Lemma 1. \mathfrak{R} is invariant for U_t , i.e., $U_t \mathfrak{R} U_t^{-1} = \mathfrak{R}$.

Proof. Consider the form $\phi_t(R) = \phi(R_t)$, where $R_t = U_t R U_t^{-1}$. $\phi_t(R^* R) = \phi(R_t^* R_t) = \phi(R_t R_t^*) = \phi_t(R R^*)$. Hence $\phi_t(R)$ is again a trace. Since R is a factor we find that $\phi_t(R) = \lambda_t \phi(R)$ where λ_t is a positive number. If $\kappa \in \mathfrak{R}$, $\phi(\kappa_t^* \kappa_t) = \phi_t(\kappa^* \kappa) = \lambda_t \phi(\kappa^* \kappa) < \infty$, hence $\kappa_t \in \mathfrak{R}$, which proves the lemma.

We shall now make use of condition i. The set \mathfrak{R} is a linear space. With the scalar product $(\kappa_1, \kappa_2) = \phi(\kappa_1, \kappa_2)$ \mathfrak{R} is a prehilbert space. Let \mathfrak{S}' be its closure. Then \mathfrak{R} is isomorphic with the Von Neumann algebra $U(\mathfrak{R})$ generated by the left representation of \mathfrak{R} in $\mathfrak{B}(\mathfrak{S}')$ [4]. Like \mathfrak{R} , $U(\mathfrak{R})$ is a factor, with the property that there exists a conjugation operator J' such that $J' U(\mathfrak{R}) J' = U(\mathfrak{R})'$. Therefore \mathfrak{R} and $U(\mathfrak{R})$ are spatially isomorphic [5], i.e. there exists an isometric mapping S of \mathfrak{S}' onto \mathfrak{S} such that $S U(R) S^{-1} = R$. The dense set of vectors $\kappa \in \mathfrak{S}'$ is then mapped onto a dense set of vectors $\chi_\kappa \in \mathfrak{S}$. J' is mapped onto J . This leads immediately to the following relations for the vectors χ_κ

$$R \chi_\kappa = \chi_{R \kappa}$$

and

$$J \chi_\kappa = \chi_{\kappa^*}.$$

We next define the (unbounded) operator Q which transforms χ_κ into $\psi_\kappa = \kappa \Omega$:

$$\psi_\kappa = Q \chi_\kappa.$$

Lemma 2. Q is symmetrical.

Proof.

$$\begin{aligned} (\chi_{\kappa_1}, Q \chi_{\kappa_2}) &= (\chi_{\kappa_1}, \psi_{\kappa_2}) = (\chi_{\kappa_1}, \kappa_2 \Omega) = (\kappa_2^* \chi_{\kappa_1}, \Omega) = (\chi_{\kappa_2^* \kappa_1}, \Omega) \\ &= (J \Omega, J \chi_{\kappa_2^* \kappa_1}) = (\Omega, \chi_{\kappa_1^* \kappa_2}) = (\Omega, \kappa_1^* \chi_{\kappa_2}) = (\kappa_1 \Omega, \chi_{\kappa_2}) \\ &= (\psi_{\kappa_1}, \chi_{\kappa_2}) = (Q \chi_{\kappa_1}, \chi_{\kappa_2}). \end{aligned}$$

Lemma 3. Q commutes with R

Proof. $R Q \chi_\kappa = R \psi_\kappa = \psi_{R \kappa} = Q \chi_{R \kappa} = Q R \chi_\kappa$.

Lemma 4. Q commutes with U_t .

Proof. It follows from lemma 1 that $\kappa_t \in \mathfrak{R}$ if $\kappa \in \mathfrak{R}$. We define V_t by the equation $V_t \chi_\kappa = \chi_{\kappa_t}$ and we shall prove that $V_t = U_t$. We notice that V_t and V_{-t} are bounded operators. If $R \in \mathfrak{R}$, $V_t R V_t^{-1} \chi_\kappa = V_t R \chi_{\kappa_{-t}} = V_t \chi_{R \kappa_{-t}} = \chi_{R_t \kappa} = R_t \chi_\kappa$; hence $V_t R V_t^{-1} = R_t$. We also have $V_t J \chi_\kappa = V_t \chi_{\kappa^*} = \chi_{\kappa_t^*} = J \chi_{\kappa_t} = J V_t \chi_\kappa$, and hence $[V_t, J] = 0$. As a conse-

quence, the operator $V_t U_t^{-1}$ commutes both with \mathfrak{R} and with \mathfrak{R}' , and is thus a multiple of the identity. Therefore, $V_t = \mu_t U_t$, where μ_t is a complex function of t , satisfying the relation $\mu_{t_1} \cdot \mu_{t_2} = \mu_{t_1 + t_2}$. We want to prove that $\mu_t = 1$.

An immediate consequence of our result so far is that

$$U_t Q = Q U_t \mu_t \tag{2}$$

or, taking matrix elements, and using the symmetry of Q

$$(\chi_{\kappa_1}, U_t \psi_{\kappa_2}) = \mu_t (\psi_{\kappa_1}, U_t \chi_{\kappa_2}). \tag{3}$$

From (3) we see that μ_t is the ratio of two continuous functions of t . Since there is no value of t for which either of these two functions vanishes for all κ_1 and κ_2 , we conclude that μ_t is continuous for all t . Using the multiplication property we find that $\mu_t = \exp(\alpha t)$. Taking $\kappa_1 = \kappa_2$ in equation (3) and taking the complex conjugate at both sides we find that $\mu_t = \mu_t^*$, so that α is real. Suppose $\alpha > 0$. Since both matrix elements in (3) are bounded functions of t , we conclude that $\lim_{t \rightarrow \infty} (\psi_{\kappa_1}, U_t \chi_{\kappa_2}) = 0$ and $\lim_{t \rightarrow -\infty} (\chi_{\kappa_1}, U_t \psi_{\kappa_2}) = 0$. This is clearly in contradiction with the fact that for any $\phi, \psi \in \mathfrak{H}$ the mean value $\lim_{T \rightarrow \infty} \frac{1}{T} \int_T dt (\phi, U(t) \psi)$

exists and equals $(\phi, E_0 \psi)$ where E_0 is the projector on the space spanned by the invariant states [6]. We conclude that $\alpha \leq 0$. In the same way one proves that $\alpha \geq 0$, and hence $\alpha = 0$ and $U_t = V_t$. Now $U_t \chi_\kappa = \chi_{\kappa_t}$ and $U_t \psi_\kappa = \psi_{\kappa_t}$, so that $U_t Q \chi_\kappa = U_t \psi_\kappa = \psi_{\kappa_t} = Q \chi_{\kappa_t} = Q U_t \chi_\kappa$, which completes the proof of lemma 4.

Since Q is symmetrical it has a closure \bar{Q} .

Lemma 5. \bar{Q} is a multiple of the identity.

Proof. We know that $[R, Q] = 0$ and $[U_t, Q] = 0$ for all $R \in \mathfrak{R}$ and all t . Since U_t and \mathfrak{R} are bounded operators, these equalities can be extended to the closure \bar{Q} , in other words $[R, \bar{Q}] = 0$ and $[U_t, \bar{Q}] = 0$. We shall now use condition iii of the theorem. An immediate consequence of this condition is that the von Neumann algebra generated by all $R \in \mathfrak{R}$ and U_t equals $\mathfrak{B}(\mathfrak{H})$. Therefore, since \bar{Q} commutes with a weakly dense set in $\mathfrak{B}(\mathfrak{H})$ one can conclude that \bar{Q} commutes with $\mathfrak{B}(\mathfrak{H})$ and is therefore a multiple of the identity.

The proof of the theorem is now almost completed. We have $\psi_\kappa = q \chi_\kappa$, where q is a complex number. Hence $\phi(\kappa^* \kappa) = |q|^{-2} (\Omega, \kappa^* \kappa \Omega)$ for all $\kappa \in \mathfrak{R}$, or $\phi(\varrho) = |q|^{-2} (\Omega, \varrho \Omega)$ for all $\varrho \in \mathfrak{R}$ with finite trace. The form at the righthand side is finite for all $R \in \mathfrak{R}$. We prove that it is a finite trace. For each $R \in \mathfrak{R}$ there exists a sequence ϱ_n of relative trace-class operators such that $R = \text{weak limit } \varrho_n$. Let $U \in \mathfrak{R}$ be unitary, then

$$\begin{aligned} (\Omega, U R U^{-1} \Omega) &= \lim (\Omega, U \varrho_n U^{-1} \Omega) = \lim \phi(U \varrho_n U^{-1}) = \lim \phi(\varrho_n) \\ &= \lim (\Omega, \varrho_n \Omega) = (\Omega, R \Omega). \end{aligned}$$

We conclude that ϕ is a finite trace and that Ω is a trace-vector, which proves the theorem.

3. Discussion

In the case of interest for physics \mathfrak{R} is the von Neumann-algebra generated by the representation of \mathfrak{Q} . Since \mathfrak{Q} contains more than a finite number of independent elements, type I_n is excluded. We shall see that for finite temperature T also type II_1 with Ω as trace-vector, cannot occur. Indeed, this situation implies that the state ω itself is a trace over \mathfrak{Q} . As discussed in section 1 this in turn implies that $T = \infty$.

An example of a factor satisfying all conditions of the theorem is provided by the equilibrium state of the free Bose-gas as discussed by ARAKI and WOODS [7]. Since in that example it is evident that the state ω of the algebra of the canonical commutation relations is not a trace, it follows that one has a type III representation. A different proof that the representation of the free Bose-gas at a temperature larger than the Bose-Einstein-transition temperature is a factor of type III was given some years ago by ARAKI [8].

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