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The Center-of-Mass in Einsteins Theory of Gravitation

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Abstract. We prove the existence and uniqueness of a center-of-mass line as well as a center-of-motion line, the latter due to G. DIXON, 1964. The validity of the theorems depends on some assumptions listed in § 2, whose most restrictive ones (in the sense of physics) state a certain weakness of the gravitational field. In the concluding paragraph we give some corrolaries and a very simple application to the problem of motion.

1. Motivation

To determine the motion of a finite number of material particles one needs three sets of equations: (I) the field equations, (II) the equations of motion and (III) the supplementary equations. By (III) we mean all equations (if necessary inequalities too) that determine the problem uniquely, e.g. equations of state. By (II) one usually means a set of differential equations giving as solution timelike curves in the space time V^4 (V^4 meant as solution of (I)), such that the curves are uniquely attached to the particles. Up to now it is unknown whether there exist solutions $g_{ab}(x)$ of (I) such that the support of the matter field $T^{ab}(x)$ is a finite set of timelike lines. We therefore take the point of view that $T^{ab}(x)$ means a collection of *extended* sources (we make it precise in § 2). Now two problems arise: (a) To find a timelike curve uniquely defined by the given matter distribution; (b) EINSTEINS classical problem, to derive from (I) the equation of motion (II) for the curves already found in (a). Obviously (b) makes sense only when (a) has been solved. This paper is devoted to the problem (a).

Up to now an answer to (a) was given either by taking over the center-of-mass line of Special Relativity to curved space time, which works for weak fields in connection with a suitable approximation method (FOCK, 1939, et al.); or by taking singular sources and taking as the required timelike curve the support of $T^{ab}(x)$ (EINSTEIN, INFELD, HOFFMANN[1], 1938; TAUB, 1964; INFELD, PLEBANSKI, 1960 et al.). Taking the

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[1] Here the sources are singular in the mathematical treatment.

second point of view, for the reason of uniqueness one has to introduce supplementary conditions. Many of them have been proposed but G. DIXON, 1964, was the first, who found an algebraic equation fixing[2] the timelike line without using the equations of motion in its formulation; he explicitely states the logical independence of (a) from (b).

Because of the independence of the spacetime geometry of the material field the center of mass line may be introduced in Special Relativity using mainly $T^{ab}{}_{\|b} = 0$. If we want to include gravitational phenomena we need the whole set of field equations and not the integrability conditions alone, as will become explicit below. All assumptions restricting the generality of V^4 are listed in § 2. Having in mind the use of the center-of-mass line in approximation methods, we demand it to be an extension of the center-of-mass line in Special Relativity; (regarding classical mechanics as the "low velocity"-limit of Special Relativity it is even an extension of the classical center-of-mass concept). We therefore devote § 3 to the problem in Special Relativity; we show two possibilities to define the center-of-mass — one due to SYNGE, 1935; MØLLER, 1949, the other is an improvement of the idea of LANCZOS, 1929; PAPAPETROU, 1939; the theorem 3.1 states their equivalence. In the rest of the paper we try to take over both definitions to General Relativity: First we construct a timelike unit vectorfield in a sufficiently large region, containing the particles under consideration, by the condition, that it makes the real valued function (4.4) minimum. With the aid of this unique vectorfield (4.14) defines a mapping of complete normed space of continuous time like curves into itself. This mapping is contractive. By the Banach-fixpoint-theorem there exists one and only one line mapped into itself. We call it the "center-of-mass" line of the particles[3].

DIXONS condition also fixes a unique "center-of-motion"-line which will be proved essentially by reduction to the proof sketched above. But now, in curved space time, both lines are different in general. Finally, in § 5 we give some properties of the center-of-mass so defined.

2. Basic Assumptions

In this paragraph we list all assumptions needed in the following. In some cases it would be too cumbersome, to give the exact formulation here; we give it in the context below.

The most important quantity will be the matter distribution described by the symmetric tensorfield $T^{ab}(x)$ with the properties:

[2] DIXON did not prove this but he proposed a procedure that can be extended to a proof as has been shown by W. KUNDT (Sem. Hamburg 1965).

[3] A more detailed outline of the proofs was given at the London Conference 1965 [1].

(2.1) $T^{ab}v_b$ is timelike for all v^a timelike or null.

(2.2) $T^{ab}{}_{\|b} = 0$, where differentiation is meant with respect to the metric in (2.7).

By T we denote the support of $T^{ab}(x)$ in V^4, where V^4 is the Riemannian manifold with the metric (2.7) of signature -2. Then:

(2.3) T is timelike and differentiable; i.e. there exists a congruence of timelike, differentiable curves of infinite length with support T.

(2.4) T is space-compact; i.e. the intersection of T with any spacelike hypersurface is compact in the topology induced by the metric topology of V^4 on the hypersurface.

(2.5) T is space-equibounded; i.e. there exists $D > 0$ such that the geodesic distance between any two points of T spacelike to each other is less or equal to D.

(2.6) $T \subset \mathscr{R}(T)$ and e_x^{-1} is diffeomorphic on $\mathscr{R}(T)$; $\mathscr{R}(T)$ is the Riemann-convex hull of T defined as follows: Take all the timelike vectorfields $v_i^a(x)$, $i \in I$ $x \in T$ and construct the geodesic surfaces $\Gamma_{i,x}$ orthogonal to $v_i^a(x)$; then: $\mathscr{R}(T) = \bigcup_{i \in I} \bigcup_{x \in I} e_x \circ k\{e_x^{-1} \circ (\Gamma_{i,x} \cap T)\}$. e_x is the exponential map of the tangentspace at x into V^4 and k means the convex hull in the usual sense. Obviously $\mathscr{R}(T)$ contains the geodesic-convex hull of T and (2.6) implies that the space-sections $\Gamma_{i,x} \cap T$ are covered by a Riemannian coordinate frame. Assumption (2.6) is very strong and we will weaken it in the concluding paragraph of this paper.

(2.7) The metric used in this paper is meant to be the solution of Einsteins field equations $R^{ab} - \frac{1}{2} R g^{ab} = T^{ab}$, where the right hand side is the matter tensor discussed above.

(2.8) g^{ab} is of class $\mathscr{C}^s$, $s \geqq 3$ in $\mathring{T}$ and therefore $T^{ab}(x)$ is of class $\mathscr{C}^r$, $r \geqq 1$ in $\mathring{T}$.

(2.9) Take any timelike curve k in T and take the assembly of all spacelike geodesic starting at k; then this assembly taken as a point set should cover T.

This condition is a consequence of (2.6)[4] but is considerably weaker and will be sufficient for the basic definitions needed below. It just guarantees that the whole of $T^{ab}(x)$ contributes to the center-of-mass line.

(2.10) V^4 is regular in $\mathscr{R}(T)$ in the following sense: $\sup|\Gamma^a_{bc}(\xi)| \leqq S^a_{bc} < \infty$. The sup is taken over all Riemannian coordinate frames adapted to $v^a(x)$ $(v^a = \delta^a_0)$ — $v^a(x)$ varies over a compact region of the unit mass-hyperboloid — and over all ξ, where ξ is a point in the region of T covered by all these coordinate frames.

[4] It follows also from (2.3), (2.4).

(2.11) Condition (4.12) is valid. We need this condition to get lemma 4.3 but it would be to cumbersome to make it explicit here.

(2.12) $P^a(x)$, $P^a_\lambda(x)$ are the total momentum quantities (see appendix A) respective to the observers $v^a(x)$, $v^a_\lambda(x)$ with $\lim\limits_{\lambda \to 0} v^a_\lambda = v^a$. Then we assume for small λ that:

$$v_a \int\limits_{K(\lambda)} 2\Gamma^{(a}_{bc} T^{b)c} \, dx \leqq \gamma \cdot \gamma^2(\lambda) \, \varphi_x(v) \,, \quad 0 \leqq \gamma < 1$$

where

$$\gamma^2(\lambda) \leqq \frac{v^a_\lambda (v_a - v_{\lambda a})}{v^a v_{\lambda a}} + \frac{2\lambda(1 - v^a v_{\lambda a})}{(\lambda(1 - v^a v_{\lambda a}) + 1)^2}$$

and $K(\lambda)$ is the wedge "between" $\Gamma_{v(x)}$ and $\Gamma_{v_\lambda(x)}$.[5] This condition looks very technical. Physically it assures, that the difference in the total momentum quantity measured in the restframe of the two observers goes to zero faster than the difference in the total mass quantity; this is meant in the limit of λ, i.e. relative velocity, goes to zero.

In the terminology of appendix A the upper limit γ_0 of γ is of the order of α_0; i.e. <1, as can be seen by the estimates of appendix B. $\gamma = 0$ in flat space-time (see § 3).

(2.13) The weak field conditions made precise in the appendix.

Weakness of the fields is meant in the following sense: Measured in the Riemannian coordinate frame adapted to $u^a(x)$ (resp. $p^a(x)$) — timelike vectorfields defined in § 4 (resp. § 5) — and given in e.g. CGS-units, the fields should be numerically small. Because the units are adapted to other physical (e.g. electrical) fields, $|\Gamma^a_{bc}| < 1$ means, that the gravitational field is small compared to the above (electrical) field measured by the same observer. In this sense we use weakness in appendix B.

We gave all assumptions very explicitly and in course of the proofs we will refer to the numbers in this paragraph. It is worth to note, that most of the above assumptions are fulfilled by general physical considerations. Just the assumptions (2.6), (2.11)—(2.12) are somewhat restrictive and of a very technical character. Except of (2.11), which has to be verified in any special problem, they state, that the fields are "not to strong". In appendix B we give some numerical estimates, that show, that in practical cases they are physically not very restrictive.

Throughout this paper the system T is free from nongravitational exterior forces and, for simplicity, $T = \mathscr{R}(T)$ in §§ 3—5.

3. The Center-of-Mass in Special Relativity

In (2.7) we replace the solution of the field equations by the Minkowski-metric; then (2.6)—(2.13) are fulfilled automatically.

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[5] The integral and the quantities φ, Γ are defined in § 4.

It is well known (e.g. SYNGE, 1956) that the total momentum vector $P^a \equiv \int\limits_{\Sigma} T^{ab}\, d\overset{*}{x}_b$[6] is timelike (see (2.1)) and independent of the special choice of the spacelike surface $\sum$ (because of (2.2), (2.4)). We redefine that quantity:

$$P^a(x, v(x)) \equiv \int\limits_{\Gamma_{v(x)}} T^{ab}\, d\overset{*}{x}_b \tag{3.1}$$

where $v^a(x) \in \mathscr{K}^1_x$ ($\mathscr{K}^1_x$ is the set of all timelike unitvectors at x pointing in the future) and $\Gamma_{v(x)}$ is the hypersurface spanned by the geodesics in x orthogonal to $v^a(x)$. Then the above statement says that $P^a(x, v(x))$ is independent of x, $v^a(x)$; i.e. *a constant vectorfield on* $M^4 \times \mathscr{K}^1_x$.

Next we define the minimal vectorfield $u^a(x)$, $u^a(x) \in \mathscr{K}^1_x$, by the condition: $\min\limits_{v \in \mathscr{K}^1_x} v_a(x)\, P^a(x, v(x)) =: u_a(x)\, P^a(x, u(x))$ for all $x \in M^4$. The constancy of P^a and the hyperbolic character of the metric gives immediately: $u^a(x)$ *is a constant vectorfield on* M^4 *and* $u^{[a} P^{b]} = 0$. This involves two statements: 1. The total mass $M(x) = u_a(x)\, P^a(x, u(x))$ is constant. 2. For all $x' \in \Gamma_{u(x)} \equiv \Gamma_x$ we have $\Gamma_x = \Gamma_{x'}$.

We define a map $\mathscr{S} : M^4 \to M^4$ by:

$$x^a \xrightarrow{\mathscr{S}} x^a_M = (u_r \int\limits_{\Gamma_x} T^{rs}\, d\overset{*}{x}_s)^{-1}\, u_b \int\limits_{\Gamma_x} \xi^a T^{bc}\, d\overset{*}{x}_c + x^a \tag{3.2}$$

where $\xi^a \in \Gamma_x$; i.e. ξ^a is in the tangent space T_x at x fulfilling there $u_a(x)\,\xi^a = 0$. Physically (3.2) means: Calculate the center of mass in the inertial-frame of an observer in x, who measures minimal total rest-mass (or equivalently: who measures $P^\alpha = 0$, $\alpha = 1, 2, 3$) — we call him u^a-observer. To (3.2) we apply a theorem well known in classical mechanics: The center-of-mass of a positive (see (2.1)) measure with compact support (see (2.4)) normed to unity on a locally convex, positive definite vectorspace (here Γ_x) lies in the convex hull of the measures support; i.e. $x_M \in T$.

We may enlarge our map $\mathscr{S}$ in a natural manner to timelike curves $x(s)$ by applying $\mathscr{S}$ pointwise such getting the curve $x_m(s)$, the *center-of-mass*-line of the given matter distribution $T^{ab}(x)$[7]. It lies in the convex hull of T; and because of the statement 2 above $x_M(s)$ is independent of the $x(s)$ we were starting with[8] (the Lebesgue-measure dx_a is translation-invariant!). This, together with (2.3), (3.2) involves: $s \to x_M(s)$ *is differ-*

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[6] $\Sigma(u, v, w)$ be the parameter representation of the 3-surface Σ; then $d\overset{*}{x}_a = \sqrt{-g}\, \delta_{abcd} \frac{\partial x^b}{\partial u} \frac{\partial x^c}{\partial v} \frac{\partial x^d}{\partial w}\, du \wedge dv \wedge dw$; δ_{abcd} is the alternating Kronnecker-tensor.

[7] The definition via the inertial-frames shows: $x_M(s)$ is a preferred line (because of the minimum condition defining $u(x)$!) in the centroid of C. LANCZOS, 1929; A. PAPAPETROU, 1939; see J. SYNGE, 1956.

[8] Consequently it is unique and completely fixed by the given $T^{ab}(x)$.

entiable. We calculate its tangent vector t^a_M: ($dT \equiv u_a T^{ab} d\overset{*}{x}_b$, $\underset{u}{\mathscr{L}}$ is the Lie-derivative in u^a-direction)

$$\frac{dx^a_M}{ds} = \frac{1}{M}\frac{d}{ds}\int_{\Gamma_x} \xi^a\, dT = \frac{1}{M}\left(\int_{\Gamma_x} \underset{u}{\mathscr{L}}\, \xi^a\, dT + A^a\right)^{9} = \frac{1}{M}\int_{\Gamma_x} u^a\, dT = u^a .$$

The vanishing of A^a is most easily seen in the u^a-inertial-frame at $x^a(s)$:

$$A^\gamma = \int_{\Gamma_x} \xi^\gamma T^{00}{}_{|0}\, d^3x = -\int_{\Gamma_x} (\xi^\gamma T^{0\alpha})_{|\alpha} d^3x + \int_{\Gamma_x} T^{0\gamma} d^3x = \int_{\Gamma_x} T^{\gamma b}\, d\overset{*}{x} = P^\gamma = 0$$

where we were aware of (2.2), (2.4), $u^{[a}P^{b]} = 0$. Such we get $t_M{}^{[a}u^{b]} = 0$ and therefore $t_M{}^{[a}P^{b]} = 0$.

We gave the procedure leading to the center-of-mass line in considerable detail out of two reasons: 1. It contains all the physical ideas serving lateron as a background for the generalisation to gravitational theory, 2. Essentially, the much more complicated proofs in § 4 follow the same outline given here.

Various authors (J. Synge, 1935, 1960; C. Møller, 1949; C. Pryce, 1949) proposed a different definition of the center-of-mass. They define the total angular momentum J^{ab} with respect to $x_0 \in \Gamma_{p(x)}$ ($p^a(x) \equiv (P_r P^r)^{-1/2} P^a(x)$, $\Gamma_{p(x)}$ is the 3-surface orthogonal to $p^a(x)$ at x; because of $u^{[a}p^{b]} = 0$ it is identical with Γ_x) by:

$$J^{ab}(x, x_0) \equiv S^{ab}(x) - 2(x_0 - x)^{[a}P^{b]} \tag{3.3}$$

where S^{ab} is the spin quantity

$$S^{ab}(x) \equiv \int_{\Gamma_x} \xi^{[a}T^{b]c}\, d\overset{*}{x}_c . \tag{3.4}$$

Evidently, $J^{ab}(x, x_0)$ is independent of $x \in \Gamma_{x_0}$. On the other hand, there exists a preferred x_0 — we call it the *center-of-motion* x_B — defined by:

$$J^{ab}(x_B) P_b = 0 . \tag{3.5}$$

By this we get a map $\mathscr{S}_B : x \to x_B$. Again we enlarge it to timelike curves $x(s) \to x_B(s)$. The above says, that it is independent of the special choice of $x(s)$. A simple algebraic calculation combining (3.3)—(3.5) shows:

$$x^a_B(s) = \Big(p_r \int_{\Gamma_x} T^{rs}\, d\overset{*}{x}_s\Big)^{-1} p_b \int_{\Gamma_x} \xi^a T^{bc}\, d\overset{*}{x}_c + s p^a . \tag{3.6}$$

i.e. $x_B(s)$ exists uniquely, it is a timelike geodesic with tangentvector t^a_B parallel to p^a (see J. Synge, 1960). Comparing the properties of $x_M(s)$, $x_B(s)$, especially (3.2), (3.6) we proved

Theorem 3.1. *In flat space the center-of-mass line is identical with the center-of-motion line. It is a timelike geodesic lying in the convex hull of T; the vectors P^a, u^a, and t^a_M are parallel to each other.*

Lateron we will show (theorem 6.1, 6.3, 6.4) that in curved spacetime this theorem is no longer valid exept for special cases.

[9] For tie differentiation of an integral see e.g. Schouten, J.A., 1954, p. 111. The slight generalisation used here can be found in Dixon, G., 1964.

4. The Center-of-Mass in Curved Space-Time

1. The minimal vectorfield on T

In the tangent space T_x at $x \in T$ we define the vector-valued 3-form

$$\omega_x^a(\xi) \equiv \Pi \circ T^{ar}(e_x(\xi)) \sqrt{-g(e_x(\xi))}\, d\overset{*}{x}_r\,. \tag{4.1}$$

For $d\overset{*}{x}_r$ see § 3; $\Pi \circ T^{ab}$ means a tensor at x one gets by parallel propagation of $T^{ab}(e_x(\xi))$ from $e_x(\xi)$ to x along the geodesic $g(e_x(\xi), x)$ with initial direction ξ^a and length $|g(e_x(\xi), x)| = |\xi^a|$. Using the product integral of Schlesinger, 1931, we may express it explicitly (see also appendix A): $\Pi \circ t^a(e_x(\xi)) = t^r(e_x(\xi)) \int_{e_x(\xi)}^{x} (\delta^a{}_r + \Gamma^a_{rs}(s)\, ds^s)$[10]. Whenever ω_x^a is a differentiable form, a simple calculation using (2.2) leads to the 4-form:

$$d\omega_x^a(\xi) = -\Pi \circ (\Gamma^r_{st} T^{ts} + \Gamma^s_{st} T^{rt})\,(e_x(\xi))\,d\overset{*}{x}\,. \tag{4.2}$$

The form (4.1) gives the integral (see G. de Rham, 1955) we will use throughout this paper; we introduce the notation

$$\int_\Sigma \omega_x{}^a \equiv \int_\Sigma T^{ab}\, d\overset{*}{x}_b\,. \tag{4.3}$$

It is a vector at x well defined by $\sum$ and the matter distribution $T^{ab}(x)$. With its aid we define

$$\mu_x(v) := v_a(x) \int_{\Gamma_v} T^{ab}\, d\overset{*}{x}_b \tag{4.4}$$

a real valued function on $\mathscr{H}^1_x$. $\mu_x(v) \geqq 0$ because of (2.1) and $\mu_x(v) = 0$ if and only if $T^{ab} = 0$ almost everywhere on $e_x(\Gamma_v)$. As v^a approaches the light cone, $\mu_x(v)$ increases. We are interested in the minimum of $\mu_x(v)$ and therefore we restrict our arguments on a compact[11] domain K suitably chosen in $\mathscr{H}^1_x$.

Because of (2.2), (2.9), (2.8), (2.4) $\mu_x : \mathscr{H}^1_x \to \mathbb{R}^+$ is continuous (see also Kobayashi, Nomizu, 1963, proposition III, 8.1). Therefore $\mu_x(v)$ takes its minimum on K in say $u^a(x)$. We prove:

Lemma 4.1. *$u^a(x)$ is unique in $\mathscr{H}^1_x$.*

Without[12] loss of generality we choose K such that

$$\hat{K} := \{\alpha v \mid v \in K,\ \alpha \in \mathbb{R}^+\}$$

becomes a convex cone. We define by $\Phi_x(v) := \mu_x\left(\frac{v}{|v|}\right)$ a continuous function: $\Phi_x : \hat{K} \to \mathbb{R}^+$. We show that Φ_x is strictly convex; i.e.

$$\Phi_x(\lambda v + (1-\lambda) w) < \lambda \Phi_x(v) + (1-\lambda)\, \Phi_x(w); \quad (0 < \lambda < 1) \tag{4.5}$$

for all $v, w \in \hat{K}$ not collinear to each other. If so, Φ_x takes its minimum

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[10] R. Brehme, B. S. de Witt, 1960; G. Dixon, 1964, called the product integral the bitensor of parallel propagation; it reduces to δ^a_b in flat space time.

[11] We take the topology induced by Euclidian topology of T_x.

[12] In the proof we omit the coordinate indices, if no confusions may arise.

in exactly one ray, whose intersection with $\mathscr{K}_x^1$ we call $u(x)$. By construction $u \in K$ and

$$\mu_x(u) < \mu_x(v) \quad \text{for all} \quad v \neq u, \; v \in \mathscr{K}_x^1 .$$

It suffices to take v, $w \in K$ in (4.5). Than (4.5) is equivalent to ($v(\lambda) := = \lambda v + (1 - \lambda) w$)

$$\frac{1}{|v(\lambda)|}\left(\lambda v \int\limits_{\Sigma_{v(\lambda)}} T\, d\overset{*}{x} + (1-\lambda)\, w \int\limits_{\Sigma_{v(\lambda)}} \ldots \right) < \lambda v \int\limits_{\Sigma_v} \cdots + (1-\lambda) w \int\limits_{\Sigma_w} \ldots$$

with $|v(\lambda)| = (v^a(\lambda)\, v_a(\lambda))^{1/2}$, $1 < |v(\lambda)| < \infty$. This means:

$$\left[\left(\frac{1}{|v(\lambda)|} - 1\right) \lambda\, \mu(v) + \frac{\lambda}{|v(\lambda)|}\, v \left(\int\limits_{\Sigma_{v(\lambda)}} \ldots - \int\limits_{\Sigma_v} \ldots\right)\right] +$$

$$+ \;[\text{the same replacing } \lambda \to (1-\lambda),\; v \to w] < 0 . \qquad (4.6)$$

This is trivially true if the second summand (s.s.) in each bracket is ≤ 0 (e.g. flat space time); we therefore assume it to be > 0. Using Stokes theorem (see also Schlesinger, 1928) and (4.2) we calculate for (s.s.):

$$(\text{s.s.})_I = -v_a \int\limits_{K(\lambda)} d\xi \{(\Gamma^r_{bc} T^{cb} + \Gamma^b_{bc} T^{rc})\,(e_x(\xi)) \int\limits_{e_x(\xi)}^{x} (\delta^a_r + \Gamma^a_{rs}\, d\tau^s)\} . \quad (4.7)$$

Where $K(\lambda)$ the section of the support of ω^a_x "between $\Sigma_{v(\lambda)}$ and Σ_v". For $(\text{s.s.})_{\text{II}}$ in the second bracket we get an analogous result. Obviously (4.6) is true if

$$\frac{(|v(\lambda)| - 1)\mu_x(v)}{(s \cdot s)_I} > 1 . \qquad (4.8)$$

But (4.8) is fulfilled because of assumption (2.12). The same argument holds for the second bracket.

We remark, that all our valuations are made in a fixed Riemann-normal-coordinate frame adapted to any vector of K chosen once for ever. Then everything is well defined, because of the compactness of K, the compactness of the rotation group SO_3 and our regularity assumption (2.10). Obviously the special choice of our coordinate frame affects not our result, so we proved the lemma.

The above construction applied to all x gives us a uniquely defined timelike unit vectorfield $u^a(x)$ in V^4. We set $\mu_x(u) =: M(x)$ in the following and call it the "mass quantity". The set of all points lying on the geodesics starting from x orthogonal to $u^a(x)$ is called Γ_x; it is a hypersurface in the neighborhood of x, where the exponential map is diffeomorphic (e.g. Eisenhart, 1949; Kobayashi u. Nomizu, 1963).

Lemma 4.2. *$u^a(x)$ is continuous in T.*

Proof. For a suitable neighborhood N_x of the zero vector in $T_x e_x$ is a diffeomorphism (Kobayashi u. Nomizu, 1963). Assumption (2.6) implies: Taken as a mapping of $\bigcup_{x \in T} N_x \to V^4$ the exponential map is a diffeomorphism, and furtheron, that the support of ω_x^a lies in N_x for all $x \in T$. Then, as a consequence of (2.4), (2.8), (2.9), (2.10), $\mu(x, v)$ is continuous for all $x \in T$ taken as a mapping of the principal fibre bundle $\mathscr{B} := \mathscr{R}(T) \times \mathscr{K}_x^1$ into $\mathbb{R}^+$. (Obviously the structure group $\mathscr{L}_+^\uparrow$ (proper orthochronous Lorentzgroup) acts transitively on the fibre). By lemma 4.1 $\sigma(x) : x \to (x, u(x))$ defines a cross section on $\mathscr{B}$. We have to prove, that σ is continuous on T.

Be $x_0 \in T$ and $\{x_k\} \to x_0$ a converging sequence in a suitable neighborhood $\mathscr{U}(x_0)$, and be $\tau(x)$ a continuous cross section over $\mathscr{U}(x_0)$ with $\tau(x_0) = \sigma(x_0)$. For any k there exists a $g_k \in \mathscr{L}_+^\uparrow$ defined by $g_k \tau(x_k) = \sigma(x_k)$, and our remark to lemma 4.1 shows, that the $\{g_k\}$ lay in compact domain of $\mathscr{L}_+^\uparrow$; i.e. $\{g_k\} \to g_0$. We set $f(x) \equiv \mu(x, \tau(x))$, which is a continuous function into $\mathbb{R}^+$ with the property $f(x_k) \geqq \mu(x_k, g_k \tau(x_k)) > 0$. This and the uniqueness of the minimum of μ imply:

$$\mu(x_k, g_k \tau(x_k)) \to \mu(x_0, g_0 \tau(x_0)) = \mu(x_0, \sigma(x_0));$$

therefore $g_0 = e$ and finally $\sigma(x_k) \to \sigma(x_0)$.

As a consequence we get

Corollary. *$M(x)$ is continous on T.*

Remark. In flat space time $u^a(x)$, $M(x)$ are constant and defined all over M^4. In this case $M(x)$ is the total rest mass, which justifies the terminology "mass quantity".

With the aid of lemma 4.2 we are in position to prove

Lemma 4.3. *$u^a(x)$ is differentiable in T.*

Proof. Let $U(x_0, v_0)$ be a neighborhood of $(x_0, v_0) \in U' \times \mathscr{K}_x^1$, U' open in T, fixed once for ever and $v_0 \equiv u(x_0)$. In an arbitrary fixed coordinate frame covering U' let $v^a = (v^0, v^1, v^2, v^3)$. We have to prove, that the functions $u^0(x), \ldots, u^3(x)$ are differentiable in T. Because of the definition of the minimum vectorfield $u(x)$, those functions have to fulfill:

$$\begin{aligned} &\text{(a)}\ \ g_{ab}(x)\, u^a(x)\, u^b(x) = 1 \\ &\text{(b)}\ \ (x, v) \to \mu_x(v) \quad \text{becomes minimal for} \quad v = u(x);\ x \in T\,. \end{aligned} \tag{4.9}$$

Set $\Big(\int\limits_{\Gamma v(x)} T^{ab}\, d\overset{*}{x}_b\Big)\, g_{ai}(x) \equiv \Phi_i(v^0, \ldots, v^3; x)$. Then $(v^0, \ldots, v^3) \to$ $\to \Phi_i^x(v^0, \ldots, v^3)$ is of class $\mathscr{C}^s$ for all $i = 0, \ldots, 3$ because of assumption (2.6) (2.8); further $x \to \Phi_i^{v^0, \ldots, v^3}(x)$ is differentiable. To see this, one has to set $v(x') = \tau(x')$, $x' \in U'$, where $v(x_0) = \tau(x_0)$ and τ is the cross-section related to the parallel propagation. Then $x \to \Gamma_{\tau(x)}$ is of class $\mathscr{C}^s$ in $x \in T$. Because of (2.8) we see $x \to \Phi_i^v(x)$ is of class $\mathscr{C}^r$ $(r \geqq 1!)$ in T.

In the above terminology we may replace (4.9) by

$$\begin{aligned} &\text{(a)}\ \ P^x(v^0, \ldots, v^3) = 1 \\ &\text{(b)}\ \ \mu_x(v^0, \ldots, v^3) = \sum_{i=0}^{3} v^i \Phi_i^x(v^0, \ldots, v^3) \ \text{minimal}\,. \end{aligned} \tag{4.10}$$

We have to look for the minimum with respect to v for fixed $x \in U'$. It is the solution of:

$$\Phi_i^x(v^0, \ldots, v^3) + \sum_{k=0}^{3} v^k \Phi_{ik}^x(v^0, \ldots, v^3) - \lambda Q_i^x \equiv G_i(x, v) = 0 \quad (4.11)$$

where λ is a Lagrangian multiplicator, $Q_i^x := \frac{\partial}{\partial v^i} P^x$ a polynomial in $v^0, \ldots v^3$ and $\Phi_{ik}^x := \frac{\partial}{\partial v^k} \Phi_i^x$. By the preceeding arguments we know: $x \to G_i(x, v)$ is differentiable and so is $v \to G_i(x, v)$ because of (2.8). By lemmata 4.1 and 4.2 we know, there exists a continuous and unique solution $v^a = u^a(x)$ $(a = 0, \ldots, 3)$ of (4.11). An elementary calculation shows

$$\frac{\partial u^i}{\partial x^j} = (D_v)^{-1\,is}(D_x)_{sj}\big|_{(x_0, v_0)} \text{ }^{13}$$

where $(D_v)_{rs}(D_v)^{-1\,rs} = \mathbb{1}$ and $(D_\alpha)_{rs}$ $(\alpha = x, v)$ is the rs-component of the derivation matrix of G^r with respect to α^s. Therefore the four functions $u^a(x)$ are differentiable if and only if $\det(D_v)_{rs} \neq 0$ in T. This means:

$$\det\left(2\Phi_{rs}^x + \sum_{k=0}^{3} u^k \Phi_{rks}^x + \lambda Q_{rs}^x\right) \neq 0 \quad (4.12)$$

where $\Phi_{rks}^x \equiv \frac{\partial}{\partial v^s} \Phi_{rk}^x$ and $Q_{rs}^x \equiv \frac{\partial}{\partial v^s} Q_r^x$. Our assumption (2.11) concludes the proof.

We remark that (4.12) is true in flat space-time and in the Schwarzschild-solution filled with a perfect fluid, on the central line of spherical symmetry[14].

Corollary. *$M(x)$ is differentiable for $x \in T$.*

2. The Space $Z(T)$

We call $Z'(T)$ the set of all timelike, differentiable curves $k (\equiv x(s))$ in T[15]. With the aid of our u^a-field we define a distance function on $Z'(T)$:

$$\langle k', k'' \rangle \equiv \sup_{x(s) \in Z'(T)} \sup_{s \in \mathbb{R}} |g(\Gamma_{x(s)} \cap k', \Gamma_{x(s)} \cap k'')| \,. \quad (4.13)$$

Obviously $\langle k', k'' \rangle = \langle k'', k' \rangle$, $0 \leqq \langle k', k'' \rangle < \infty$, because of (2.5) and lemma 4.2 and $\langle k', k'' \rangle = 0 \Leftrightarrow k' = k''$. (Be $k' \neq k''$ then there exists $x'(s_0)$ with $|g(\Gamma_{x(s_0)} \cap k', \Gamma_{x(s_0)} \cap k'')| > 0$, which implies $\langle k', k'' \rangle > 0$!) For the geodesic distance (g spacelike!) we have the inequality $|g(x, y)| + |g(y, z)| \geqq |g(x, z)|$ which leads immediately to $\langle k', k \rangle + \langle k, k'' \rangle \geqq$

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[13] In flat-spacetime $(D_x) = 0$, $(D_v)^{-1} \sim \eta^{ab}$ and consequently $u^a(x)$ is constant affirming our result in § 3.

[14] Whenever all the other of our assumptions are fulfilled, which is true in the Schwarzschild-field of not to large rest mass, then (4.12) is a consequence of differentiability of $u(x)$; but the latter is true, $u(x)$ being the tangentvector to the (geodesic, timelike) central line defined covariantly by the spherical symmetry.

[15] $\Gamma_x \equiv e_x(\Gamma_{u(x)})$, where $\Gamma_{u(x)} \in T_x$ as described in § 3.

$\geqq \langle k', k'' \rangle$. All the properties stated above show, that $\langle\ ,\ \rangle$ *is a metric on* $Z'(T)$. By $Z(T)$ we mean the completation, with respect to this metric, of $Z'(T)$. Then $Z(T)$ is a complete, normed space. Because of (2.4) all its elements are continuous curves in T.

3. The Center-of-Mass Line

For any point $x \in T$ we define the map

$$x \xrightarrow{\mathscr{S}} x_M \equiv e_x\{M^{-1}(x)\, u_r(x) \int_{\Gamma_{u(x)}} \xi^a \omega_x^r(\xi)\}\,. \tag{4.14a}$$

In the following we will use the symbolic notation:

$$x \xrightarrow{\mathscr{S}} x_M \equiv M^{-1}(x)\, u_r(x) \int_{\Gamma_x} \xi^a T^{rs} d\overset{*}{x}_s\,. \tag{4.14b}$$

The real valued 3-form $M^{-1}(x)\, u_r(x)\, \omega_x^r(\xi)$ defines a positive, normed measure with compact support ((2.1), (2.4)) on $\Gamma_{u(x)}$. We may identify $\Gamma_{u(x)}$ with $\mathbb{R}^3$ and then it is well known, that this measure has a center-of-mass and that the latter lays in the convex hull of its support. This means in our case:

Lemma 4.4. *x_M in T, whenever $x \in T$.*

We extend (4.14) to $x(s) \in Z(T)$ defining $x(s) \to x_M(s)$ pointwise (i.e. for each $s \in \mathbb{R}$) by (4.14). This extended map also will be called $\mathscr{S}$ and we prove

Lemma 4.5. *$\mathscr{S}$ is a mapping of $Z'(T)$ into $Z'(T)$.*

Proof. Be $x(s) \in Z'(T)$; the composition $s \to x(s) \to x_M(s)$ together with lemma 4.3 shows, that $x_M(s)$ is a differentiable curve; lemma 4.4 shows that $x_M(s) \in T$ for all $s \in \mathbb{R}$. It remains to prove, that $t_M^a := \frac{d}{ds} x_M^a$ is timelike. To do this we need some preliminary steps:

a) *The k-map.* (2.6), (2.8) assures the existance of a neighborhood $N(s)$ covering $T \cap B(\sigma)$, where $B(\sigma)$ is the sandwich "between" Γ_s and $\Gamma_{s+\sigma}$, such that $e_{x(s)}^{-1}: N \to T_{x(s)}$ is diffeomorphic. In $T_{x(s)}$ we introduce an orthonormal tetrad $e_i(s)$, $\langle i = 0, 1, 2, 3\rangle$, with $e_0(s) = u(s)$ and propagate it by a generalized Fermi-transport along $x(s)$[16], namely:

$$\frac{d}{ds} e_i^a + 2u \frac{{}^{[a}du^{b]}}{ds} g_{bc} + u^b N_{bc}^a)\, e_i^c \overset{*}{=} 0 \tag{4.15}$$

where $N_{bc}^a \equiv \left(\frac{d}{ds} u^a\right) u_b u_c - 2u^a \left(\frac{d}{ds} u_{(b}\right) u_{c)}$ (i.e. u^a is transported into u^a and so is its orthogonal space). We relate those points of Γ_s and $\Gamma_{s+\sigma}$, whose corresponding vectors $\xi_{(s)}^a$, $\xi_{(s+\sigma)}^a$ have the same components in the (4.15)-related tetrads; all points related to $\xi(s)$ lie on a curve $k_\xi(s)$. Because of lemma 4.3 $k_\xi(s)$ is differentiable. If σ is small enough the curves $k_\xi(s)$, $\xi \in \Gamma_{x(s)} \cap T$ do not intersect and therefore they constitute

[16] G. DIXON [4] already used this propagation with a somewhat different meaning of u^a.

a one-to-one map of $\Gamma_s \to \Gamma_{s+\sigma}$ $(\sigma > 0)$ called the k-map[17]. We find always a positive σ — (2.13) —, such that the above is true; so our map has meaning in a finite slice containing Γ_s for all s. The tangent vectors to the k-map give a continuous vectorfield in the above mentioned slice. Using (4.15) and remembering, that ξ^a depends on $x(s)$ as well as on $\xi \in \Gamma_{x(s)}$ we get:

$$k^a(\xi) = X^a{}_r\Big(u^r \frac{du^b}{ds} \xi_b - \xi^g{}_{\|s} t^s\Big) \tag{4.16}$$

where $t^a = \dfrac{dx^a}{ds}$ and $X^a{}_r$ is a tensor at $x(s)$ constructed as follows: Propagate ξ^a parallely along $g(x(s), \xi)$ and differentiate the vector so obtained with respect to $x^r(\xi)$. This gives a tensor, whose inverse parallely propagated along $g(\xi, (x^r))$ to $x(s)$ is $X^a{}_r$. We observe that $k^a(0) = t^a$, $k_0(s) = x(s)$, which shows — together with lemma 4.3 — that k^a is a timelike vectorfield. In flat space-time (4.16) immediately gives $k^a = t^a$, as it should be, according to the prescribed meaning of the k-map. As parameter on the k-lines we use the induced-one by the Γ_s-layers, i.e. induced by s in $x(s)$.

b) Now we are in position to prove our statement. To do this we calculate t^a_M explicitely ($\underset{k}{\mathscr{L}}$ means Lie-derivation in k-direction):

$$\begin{aligned} t^a_M = \frac{d}{ds} x^a_M(s) &= \left[\frac{dM}{ds}\cdot\frac{1}{M} x^a_M(s) + \frac{1}{M}\frac{du_r}{ds}\int_{\Gamma_s} \xi^a T^{rs}\, d\overset{*}{x}_s + \right. \\ &\left. + \frac{1}{M}\int_{\Gamma_s} \xi^a \underset{k}{\mathscr{L}}\, T^{rs}\, d\overset{*}{x}_s\right] + \frac{1}{M} u_r \int_{\Gamma_s} \underset{k}{\mathscr{L}} \xi^a T^{rs}\, d\overset{*}{x}_s \equiv [t^a_r] + t^a_z . \end{aligned} \tag{4.17}$$

Obviously $t^a_r u_a = 0$, i.e. t^a_r is spacelike and $= 0$ in flat space-time. The following we calculate in the $\mathscr{R}(s)$-system used in appendix A. The estimates given there show that $|t^a_r| \leqq 2D(\alpha_0 + \alpha_0') \dfrac{|P|}{M} \equiv A_0 \dfrac{|P|}{M}$; it means $|t^a_r| \leqq A \dfrac{|P|}{M}$ with $0 \leqq A \ll 1$ and A_0 is the upper limit of A. The numerical value we have to expect is roughly estimated in appendix B. It remains to discuss t^a_z. By definition of the k-map we see immediately

$t^a_z = \int_{\Gamma_s} k^a(\xi)\, dT$, where $dT = M^{-1} u_b \int_{x(s)}^{\xi} (\delta^a_b + \Gamma^a_{bc}\, ds^c)\, T^{bs}\, d\overset{*}{x}_s$ for abbreviation. Using (4.16), (A.3) we calculate in the $\mathscr{R}(s)$-system[18]:

$$\begin{aligned} k^a(\xi) = \xi^a + &\left(1 - \int_0^{|\xi|} \Gamma\, ds + \frac{1}{2}\int_0^{|\xi|}\int_0^{|\xi|} \Gamma(s)\,\Gamma(s')\, ds\, ds' + \cdots\right)^a_b \\ H^{-1}{}^b_s(\xi) \times &\left(1 + \int_0^{|\xi|} \Gamma\, ds + \frac{1}{2}\int_0^{|\xi|}\int_0^{|\xi|} \Gamma(s)\,\Gamma(s')\, ds\, ds' + \cdots\right)^s_r \\ &((\dot{u}\,\xi)\, u^r + t^r) \end{aligned}$$

[17] It has the important property that $g(x(s), \xi(s)) \to g(x(s+\sigma), \xi(s+\sigma))$ where $\xi(s+\sigma)$ is the k-picture of $\xi(s)$.

[18] We use the matrix notation of appendix A and set $(g_{ab}(\xi)\, \xi^a \xi^b)^{\frac{1}{2}} \equiv |\xi|$.

where

$$H_s^b(\xi) = \left(\int_{x(s)}^{\xi} (1 + \Gamma ds)_c^b \xi^c)_{\| s} \right)$$

which gives explicitely:

$$H_s^b(\xi) = \left(1 + \int_0^{|\xi|} \Gamma ds + \cdots \right)_c^b \delta_s^c + \Gamma_{st}^b(\xi) \left(1 + \int_0^{|\xi|} \Gamma ds + \cdots \right)_c^b \xi^c +$$

$$+ \xi^c \left\{ \frac{\partial}{\partial \xi^s} \int_0^{|\xi|} \Gamma ds + \frac{\partial}{\partial \xi^s} \int_0^{|\xi|} \int_0^{|\xi|} \Gamma(s) \Gamma(s') ds\, ds' + \cdots \right\}_c^b .$$

By a simple calculation we find

$$\frac{\partial}{\partial \xi^s} |\xi| = - \frac{1}{|\xi|} (g_{r(c} \Gamma_{b)s}^r \xi^c + g_{sb}) \xi^b \equiv a_s(\xi)$$

and therefore

$$\left(\frac{\partial}{\partial \xi^s} \int_0^{|\xi|} \Gamma ds \right) = \Gamma \left(a_s(\xi) - \int_0^{a(\xi)} \Gamma(\tau) d\tau \right) .$$

With notation of appendix A we get

$$|a_s(\xi)| \leqq \sup_{\Gamma_s} |a_s(\xi)| \leqq \|\hat{g}\| (1 + \|\Gamma\| D)$$

leading immediately to the inequality: $(D \|\Gamma\| \equiv c)$

$$|H_b^a(\xi) - \delta_b^a| \leqq (e^c - 1)(1 + c) + c(1 + \|\hat{g}\| (1 + c) e^c) = B_0\text{[19]} .$$

Therefore we are allowed to write:

$$k^a(\xi) = \xi^a + h^a{}_b(u) t^b + (ut) u^a + z'^a \tag{4.18}$$

where $h^a{}_b(u)$ is the projection on the u-restspace in $x(s)$. The vector z'^a depends on Γ_{bc}^a and vanishes in flat space-time; to give its explicit form would be cumbersome and fortunately we do not need it.

Going back to (4.17) we get

$$t_M{}^a = u^a + \frac{1}{M} \int_{\Gamma_s} z'^a(\xi) + t_r^a \equiv u^a + z^a . \tag{4.19}$$

We have normalized the parameter s such that $u^b t_b = 1$ and have used the fact that $h^a{}_b(u) t^b$ is a fixed vector in Γ_s.

Using the result we have got for t_M^a, the second statement of appendix A and the inequality for $|H(\xi) - \mathbf{1}|$ we get[20]

$$|z^a| \leqq \frac{|P|}{M} (A_0 + B_0 + \alpha_0' D) = C_0 .$$

[19] For the examples estimated in appendix B we get $B_0 \cong a$; a is tabulated there.

[20] Looking at the estimates in appendix B we see, that C_0 is of the same order of magnitude as A_0.

Then it is easy to see that $t^a_M t_{M\,a} > 0$, i.e. t^a_M is timelike, such proving the lemma.

At this stage of our investigations we are in position to say what we mean by the center-of-mass line of a given matter distribution $T^{ab}(x)$. *We define it, being the line* $x(s) \in Z(T)$ *with* $x_M(s) = x(s)$; in other words: it is characterized by the fact, that in its $\mathscr{R}(s)$-system $x^a_M(x) = 0$ for all s.

To prove the existence of such lines, we construct by (4.14) the continuous vectorfield $x^a_M(x)$, $x \in T$. Each of those vectors is spacelike and therefore vanishes if and only if its projection on a spacelike hypersurface vanishes. As such a surface we take Γ_{x_0} (x_0 fixed in T) and project $x^a_M(x')$, $x' \in \Gamma_{x_0} \cap T$ onto Γ_{x_0}. So we get a continuous vectorfield on $\Gamma_{x_0} \cap T$. Assuming T being geodesic convex and calculating $x^a_M(x'')$, $x'' \in \langle$surface of $\Gamma_{x_0} \cap T\rangle$ we see that $x^a_M(x'')$ points to the interior of T and therefore the projection points to the interior of $\Gamma_{x_0} \cap T$. So, finally we have a closed 3-domain and on this domain a continuous vectorfield pointing into the interior everywhere on the surface; then, by BROUWERS fix-point-theorem, we get: It exists at least one point in $\Gamma_{x_0} \cap T$ where our vectorfield vanishes. This means by our argument above that $x^a_M = 0$ at those points, such proving our statement, when we replace x_0 by a differentiable curve $x_0(s) \in Z'(T)$.

This proof (given by J. MADORE[21]) obviously shows a bit more, namely:

Lemma 4.6. *The center-of-mass line exists and lies in the geodesic hull of the support of* $T^{ab}(x)$. *It is a continuous timelike curve.*

The last statement is almost obvious remembering that $Z(T)$ consists of timelike, continuous curves.

Up to now nothing is said about uniqueness and it still might happen that we have several center-of-mass lines, the number of which is completely undetermined. We want to get rid of this ambiguity and prove:

Lemma 4.7. *The map* $\mathscr{S}: Z'(T) \to Z'(T)$ *is contractive with respect to the norm given in* § 4.2.

Take $k, k' \in Z'(T)$ and take the parameter s induced by k for all curves of interest in the following. We have to estimate

$$|x_M(s) - x'_M(s)| = \Big| \int_{\Gamma_{x(s)}} \xi^a \, d\,T_{x(s)} - e^{-1}_{x(s)} \circ e_{x'(s+\delta)} \circ \int_{\Gamma_{x'(s+\sigma)}} \xi'^{\,a} \, d\,T'_{x'(s+\sigma)} \Big|$$

where $e_{x(s)}$ is the exponential map $T_{x(s)} \to V^4$ and $x'(s+\sigma)$ is the point on k' defined by: $\mathscr{S} x'(s+\sigma) = x'_M(s)$. Without restriction to generality we may assume that $|g(x(s), x'(s))| \equiv |\varDelta x|$ is small (but > 0!). For

[21] Oral communication via W. KUNDT 1966.

abbreviation we set $\sigma' \equiv |g(x(s), x'(s+\sigma))|$ and $x'(s+\sigma) \equiv x + \sigma'$; then

$$|x_M - x'_M| =^{22} u_r(x) \left\{ \frac{M(x+\sigma') - M(x)}{M(x+\sigma')} \int\limits_{\Gamma_s} \xi^a \, d\, T^r_{x(s)} + \right.$$
$$\left. + \frac{1}{M(x+\sigma')} \int\limits_{\Gamma_s} \xi^a (d\, T^r_x - k \circ d\, T^r_{x+\sigma'}) \right\}$$

where the generalized Fermi-propagation

$$k \circ v^a = v^a + \sigma' \frac{d v^a}{d s} + \frac{\sigma'^2}{2} \frac{d^2 v^a}{d s^2} + \ldots$$

and

$$\frac{d v^a}{d s} \equiv -u^a v_b u^b{}_{\|c} t^c + u^a{}_{\|c} t^c u^b v_b$$

with t^a tangent to $g(x(s), x+\sigma')$. Using the estimates of appendix A and collecting the powers of σ' we get:

$$|x'_M - x| \leqq \frac{|P|}{M} D \sigma' \{\alpha_0^{||} + 2\|\Gamma\| \, e^{D\|\Gamma\|} + \alpha'_0 |1 - \|\hat{g}\|| \, e^{\frac{|P|}{M} D \alpha'_0 |1 - \|\hat{g}\||}\}$$

$\Gamma_{x(s)}$, $\Gamma_{x(\sigma)}$, are spacelike, u^a is differentiable; then because of (2.5) we get for $|\varDelta x|$ small enough:

$$|\sigma'| \leqq (1 + \alpha'_0 D) \, |\varDelta x| \,.$$

In the exponential we replace $\sigma' < D$ by D which leads to the inequality:

$$\begin{aligned} |x_M - x'_M| \leqq \frac{|P|}{M} D (1 + \alpha'_0 D) \{\alpha_0{}'' + 2 \, \|\Gamma\| \, e^{D\|\Gamma\|} + \\ + \alpha'_0 |1 - \|\hat{g}\|| \, e^{\sigma' \frac{|P|}{M} D \alpha_0 |-1 \|\hat{g}\|}\} |\varDelta x| \equiv \tilde{\gamma}_0 |\varDelta x| \end{aligned} \tag{4.20}$$

i.e. $|x_M(s) - x'_M(s)| \leqq \tilde{\gamma} |\varDelta x|$, where $0 \leqq \tilde{\gamma} < 1$ and $\tilde{\gamma}_0$ is the upper limit of $\tilde{\gamma}$. We made this estimate independently of s and therefore we get finally: $\langle \mathscr{S} k, \mathscr{S} k' \rangle \leqq \tilde{\gamma} \langle k, k' \rangle$ with the above $\tilde{\gamma}$, such proving our lemma.

Estimates of appendix B applied to $\tilde{\gamma}_0$ show that $\tilde{\gamma}_0 \cong A_0$, i.e. $\ll 1$ in practical cases (with $|P| \cong M$).

Applying the Banach fixpoint-theorem to $\mathscr{S} : Z'(T) \to Z'(T)$ we get the main result of this paragraph:

Theorem 4.1. *A space-compact, extended timelike matter-distribution $T^{ab}(x)$ in a Riemannian manifold V^4 obeying Einsteins field equations $G^{ab} = T^{ab}$, possesses one and only one center-of-mass line. It is a continuous, timelike curve lying in the geodesic-convex hull of the support of $T^{ab}(x)$.*

We hint at the fact, that the center-of-mass line has not to be differentiable in the general case.

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[22] We may almost identify $|x_M - x'_M|$ in $T_{x(s)}$ with $|g(x_M, x''_M)|$ because of (2.13).

5. The Center-of-Motion Line in Curved Spacetime

G. DIXON, 1964, proposed to take over condition (3.5) into curved space-time using it as definition of the center-of-motion line $x_B(s)$ of T. In fact this can be done and will prove it by reducing the problem to § 4.

Any timelike unit vector v^a at $x \in T$ gives raise to the total momentum quantity $P^a(x, v) = \int\limits_{\Gamma_{v(x)}} T^{ab}\, d\overset{*}{x}_b$, the latter being a timelike vector in x.

Lemma 5.1. *There exists one and only one timelike unit vector* $p^a(x)$ *fulfilling* $p^{[a}(x)\, P^{b]}(x,p) =$ *for any* $x \in T$.

To see this we construct the sequence $\{P_k\}_{k=1,2,\ldots}$ at $x \in T$ by the following procedure (it is an improvement of an idea of DIXON proposed by W. KUNDT): Choose any timelike v_0^a and construct $P^a(v_0) \equiv P_1$; then define $v_1^a \equiv P_1^a/|P_1|$ which gives raise to $P^a(v_1) \equiv P_2^a$ etc., such leading to the sequences $\{P_k\}$ and $\{\Gamma_{v_k} \equiv \Gamma_k\}$. Estimating

$$|P_k^a - P_{k+1}^a| = |\int\limits_{\Gamma_{k-1}} T^{ab}\, d\overset{*}{x}_b - \int\limits_{\Gamma_k} T^{ab}\, d\overset{*}{x}_b| = |2 \int\limits_{K(k,k-1)} \Gamma^{(a}_{bc} T^{b)c}\, d\overset{*}{x}| = \beta^a{}_b P_k^b$$

as in appendix A; we get easily $v_{k+1}^a v_{ka} \leqq \varepsilon \cdot v_k^a v_{k-1\,a}$ with $0 \leqq \varepsilon < 1$. The upper limit ε_0 of ε is of the order of magnitude of α_0 and $\varepsilon = 0$ in flat-spacetime. Taking as a complete, metric space the unit-mass hyperboloid in x we get by the Banach fixpoint-theorem our lemma.

Because of this lemma we get a timelike unitvectorfield $p^a(x)$ on T replacing the u-field in § 4. In consequence, we replace $\Gamma_{u(x)}$ by $\Gamma_{p(x)}$. The same procedure as in § 4 shows the continuity of the p-field in T. Starting with a differentiable vectorfield $v_0(x)$ in a suitable neighborhood $U(x_0)$ we see as in § 4 that each $P_k(x)$ of the above sequence depends differentiable on $x \in U$. The sequence is equiconvergent on T and therefore $p^a(x)$ is differentiable in T.

We define the spin quantity (see (3.4)) by:

$$S^{ab}(x) \equiv 2 \int\limits_{\Gamma_x} \xi^{[a} T^{b]c}\, d\overset{*}{x}_c$$

and the total angular momentum quantity J^{ab} with respect to $x_B \in \Gamma_{p(x)}$ (see (3.3)):

$$J^{ab}(x, x_B) \equiv S^{ab}(x) - x_B^{[a} P^{b]}(x)\,.$$

Then by a purely algebraic calculation we get:

Lemma 5.2. *The condition* $J^{ab}(x, x_B)\, p_b(x) = 0$ *is equivalent to*

$$x_B^a = (p_r(x) \int\limits_{\Gamma_x} T^{rs}\, d\overset{*}{x}_s)^{-1}\, p_b(x) \int\limits_{\Gamma_x} \xi^a T^{bc}\, d\overset{*}{x}_c \tag{5.1}$$

such defining a map $\mathscr{S}_B : x \to x_B$.

Replacing (4.14) by (5.1) we follow step by step the arguments given in § 4. So we get:

Theorem 5.1. *Theorem 4.1 is also true, if we replace the word "center-of-mass" by "center-of-motion".*

Here the "center-of-motion" line is defined as the fixpoint of the map $\mathscr{S}_B$ extended to a map $Z'(T) \to Z'(T)$. The terminology is chosen because this line originates from the momentum quantity and usually one associates momentum with motion in physics.

6. Various Consequences. Concluding Remarks

The statements in this paragraph are formulated as theorems, although they have more the character of corollaries to §§ 4, 5.

Because of the uniqueness property we are able to speak of the total spin $S^{ab}_M(s)$, the total momentum $P^a_M(s)$ and the total mass $M_M(s)$ with respect to $x_M(s)$ of the matter distribution $T^{ab}(x)$ at (eigen)time s; namely:

$$S^{ab}_M(x) \equiv \int_{\Gamma_{x_M(s)}} \xi^{[a} T^{b]c} \, d\overset{*}{x}_c \tag{6.1}$$

$$P^a_M(s) \equiv \int_{\Gamma_{x_M(s)}} T^{ab} \, d\overset{*}{x}_b \tag{6.2}$$

$$M(s) \equiv u_a(x_M(s)) \, P^a_M(s) \,. \tag{6.3}$$

They constitute continuous tensorfields on $x_M(s)$. In strict analogy we get $S^{ab}_B(s)$, $P^a_B(s)$ and $M_B(s)$ replacing $x_M(s)$ by $x_B(s)$ and u^a by p^a.

Theorem 6.1. *We have in general* $u^{[a} P^{b]}_M(s) \neq 0$.

This is seen by a variation of $u_a P^a_M$ with respect to u^a resulting in

$$\delta_u(u_a P^a_M) = u_a \int_{K(\delta u)} 2\Gamma^{(a}_{bc} T^{b)c} \, dx \neq 0$$

where $K(\delta u)$ is the wedge "between" Γ_u, $\Gamma_{u+\delta u}$. But we see in flat spacetime and in fields of high symmetry u^a and P^a_M are parallel to each other. Rather trivial is the following

Theorem 6.2. $u_a S^{ab}_M = 0$.

An obvious, but very important consequence of the preceeding theorems is

Theorem 6.3. *In general* $x_M(s)$ *and* $x_B(s)$ *do not coincide.*

This is different to flat spacetime. Physically it means that an observer moving parallel to the total 4-momentum (P_M or P_B) does *not* measure minimal total mass. To see, that the same is true for an observer sitting on the particle, we assume for the rest of this paragraph that $x_M(s)$ is differentiable with tangent vector t^a_M; (similar for $x_B(s)$, t^a_B). Then (4.19) shows

Theorem 6.4. $u^{[a} t^{b]}_M(s) \neq 0$ *in general.*

Theorem 6.5. *In a matter distribution of spherical symmetry (e.g. Schwarzschild fluid-ball) the center-of-mass line coincides with the center-of-motion line and* $u^{[a} t_M^{b]}(s) = 0$. $x_M(s)$ *is identical to the central line defined by the symmetry of the problem; as a consequence* $x_M(s)$ *is a timelike geodesic.*

Tacitely we assumed that no exterior sources are present; the proof is straight forward.

We use this result to get an information on the mass concept (6.3). For simplicity we restrict the following to the static case. Using the metric $ds^2 = -e^{v(r)}\,dt^2 - e^{u(r)}\,dr^2 - r^2\,d\Omega^2$ and setting $t \equiv x^0$ we get:

$$M = e^{v(0)} \int\limits_{t=\mathrm{const}} T^0{}_0 \exp\left\langle \frac{1}{2}(u+v)(r)\right\rangle r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr\,. \tag{6.4}$$

In case of timedepending g_{ab} the expression becomes more complicated; but even in the simple case (6.4) we see, that the mass introduced in (6.3) is different from the mass used by S. A. Eddington, 1924, $(= \int T^0{}_0\, e^{u/2}\, r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr)$ and different from the mass used by H. Bondi, 1964 $(= \int T^0_0\, 4\pi r^2\, d\vartheta\, d\varphi\, dr)$[23].

A detailed inspection of the proofs given in §§ 4, 5 and appendix A shows, that we can weaken assumption (2.7). As long as the "weak field" assumptions remain valid we may interpret the metric g_{ab} as solution of $G^{ab} = T^{ab} + \tau^{ab}$, where τ^{ab} describes any exterior sources τ ($\equiv$ support of τ^{ab}) fulfilling $T \cap \tau = \Theta$ or $\tau^{ab}{}_{\|b} = 0$. Just in theorem (6.5) we have still to exclude exterior sources except they have very high symmetry; otherwise they would split x_M and x_B.

Using the wider interpretation of the field $g_{ab}(x)$ we define a test-particle $T^{ab}(x)$ by the condition that $|\Gamma^a_{bc}(\xi)| \cong 0$, $\xi \in T$, where Γ^a_{bc} is calculated in the Riemannian coordinate frame adapted to $u^a(s)$ (resp. $p^a(s)$) at $x_M(s)$ (resp. $x_B(s)$). Then we get:

Theorem 6.6. *A test particle moves along a geodesic line in the total field generated by* $T^{ab} + \tau^{ab}$; $x_M(s)$ *and* $x_B(s)$ *coincide.*

$t_M^{[a} u^{b]} = 0$ follows from (4.19); the rest is a mere consequence of (A.9)—(A.11) and the proof to theorem (6.1).

Obviously this results of an approximation method assuming that the field is almost constant all over the particle. We have not to split eigenfield and backgroundfield as it would be necessary if speaking of theorem 6.5. as an approximative solution to the motion of bodies of spherical symmetry. The advantage in our test-particle-approach is, that it is absolutely consistent (and covariantly defined) in Einsteins theory; especially it is free of the logical inconsistency discussed in § 1. First it solves *exactly* (a) and then — if the test particle condition is

[23] They have a somewhat different physical meaning: total rest-mass, total baryon-number, total effective mass.

satisfied to high accuracy — answers (b). Nothing is assumed on the shape or the inner structure of the particle; such we get geodesic motion as "leading term" in the motion of particles built by human technique, which affirms the heuristic ansatz in EINSTEINS, 1916-paper in the framework of the final theory.

It is almost obvious how to fit the A. PAPAPETROU-approximation method (1951) into the center-of-mass concept; (here it seems more appropriate to use $x_B(s)$!). It has been elaborated for the quadrupol-particle elsewhere (W. BEIGLBÖCK, Dissertation Hamburg 1965). Just to enlighten somewhat more the meaning of our total mass concept (6.3), we cite the result[24]:

$$\begin{aligned}\frac{dM}{ds} = &\left(\frac{3}{2}\dot{u}_r \dot{u}_s + \frac{1}{6} R_{rs} + u^k u^l R_{rksl}\right) \frac{dQ^{rs}}{ds} + \\ &+ \left(\frac{1}{3} R_{kr} u^k \dot{u}_s - 2\dot{u}_r \ddot{u}_s\right) Q^{rs} + \frac{1}{3} u^k R_{kbcd} \left(Q^{d(b}\dot{u}^{c)} + \frac{7}{2} Q^{bc}\dot{u}^d\right)\end{aligned} \tag{6.5}$$

where the "quadrupol moment" $Q_M^{ab} \equiv u_c \int\limits_{\Gamma_{x_M(s)}} \xi^a \xi^b T^{cd} \, d\overset{*}{x}_d$ and $\dot{u}^a \equiv u^a{}_{\|b} u^b$.

It states mass conservation in flat spacetime and shows, that the spin (6.1) does not contribute to the emmission of gravitons. The result differs from this one given by A. H. TAUB, 1964, in Florence, where he used a center-of-mass concept (and therefore a mass) bearing the difficulties discussed in § 1; his formula shows change of total restmass even in the special relativistic limit.

By theorem 4.1 (resp. 5.1) $x_M(s)$ (resp. $x_B(s)$) lie in the geodesic-convex hull of T ($\equiv h(T)$); our methods demand, that $\Gamma_{v(x)} \cap h(T)$ is covered by the Riemannian normal coordinate system adapted to $v'(x)$, where v, v' are timelike unitvectors in the neighborhood of $u(x)$ for all $x \in h(T)$. Using this as assumption we already weaken (2.6) considerably. But often we can do more. It might happen, that the demanded coordinate condition is not fulfilled for points "near the surface of $h(T)$", but works in the tube $T_1 \subset h(T)$[25]. Then our definitions make sense for all $x \in h(T)$ but $u(x)$ need not to be continuous outside T_1. But if $\mathcal{S}^n(h(T)) \to T_1$ (resp. $\mathcal{S}_B$) ($n < \infty$ may depend on x!), then we can apply our method restricting $Z'(T)$ to $Z'(T_1)$. This makes our method applicable even to rather strong fields, if — for instance — the matter distribution is "almost a ball" in $\Gamma_{u(x)}$.

Using parallel propagation in the definition of the integrals seems at the first sight somewhat superfluous. But it has the advantage of being absolutely covariant; so we can calculate all quantities in any coordinate frame and avoid to introduce Riemannian coordinate frames explicitely, which is a laborous task in most practical cases.

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[24] Reported by the author at the Physikertagung 1965, Frankfurt-Main.

[25] $T_1 = h(T_1)$.

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Appendix A

In the following we are dealing with a purely gravitationally interacting system T^{ab}.

1st statement: $\left|\frac{d P^a}{d s}\right| \leqq \alpha |P^a|$, *where* $0 \leqq \alpha \ll 1$.

Here $P^a(s)$ is the "total momentum quantity" with respect to $x(s)$ defined by

$$P^a(s) \equiv \int_{\Gamma_{x(s)}} T^{ar}\, d\overset{*}{x}_r\,. \tag{A.1}$$

It is a four-vector at $x(s)$ and because of Lemma 4.3 it is differentiable with respect to s whenever $s \to x(s)$ is a differentiable curve. We start with a sandwich $B(\sigma)$ defined as the section "between" $\Gamma_{x(s)}$ and $\Gamma_{x(s+\sigma)}$; because of (2.6) it is guaranteed that $B(\sigma) \cap T$ is covered by the Riemannian normal-coordinate-system in $x(s)$ ($\equiv \mathscr{R}(s)$-system).

For any $\eta \in B(\sigma)$ we assume the geodesic triangle[26]

$$(x(s), \eta, x(s+\sigma)) \equiv \Delta_\eta$$

to be triangulated by small local lassos at $\xi \in \Delta_\eta$. The usual definition of $R^a{}_{bcd}$ by parallel transport leads for the finite Δ_η by summation over all lassos to the formula (see (4.1))

$$\begin{aligned}\omega^a_{x(s+\sigma)}(\eta) = \widehat{\int_{x(s)}^{x(s+\sigma)}} (\delta^a{}_b + \Gamma^a_{bc}(s)\, ds^c) \times \\ \times \left\{\omega^b_{x(s)}(\eta) + \omega^d_{x(s)}(\eta) \int_{\Delta_\eta} R^b{}_{dst}(\xi)\, d\overset{*}{x}{}^{st}\right\}\end{aligned} \tag{A.2}$$

where the product integral [17] $\widehat{\int_{x(s)}^{x(s+\sigma)}} (\delta^a{}_b + \Gamma^a_{bc}(s)\, ds^c)$ is the operator of parallel propagation along $g(x(s), x(s+\sigma))$. The integral at the right hand side means: transport $R^b{}_{dst}(\xi)\, d\overset{*}{x}{}^{st}$ to $x(s)$ along $g(\xi, x(s))$ and multiply the tensor so obtained by $\omega^d_{x(s)}(\eta)$, finally sum up over all $\xi \in \Delta_\eta$. For abbreviation we call this integral $(R\,\omega)^b_{x(s)}(\eta)$.

We proceed with our arguments in the $\mathscr{R}(s)$-system adapted to $u(x(s))$, i.e. $u^a(x(s)) = \delta^a{}_0$. The definition of the product integral leads

[26] It makes no difference in the following to replace $g(x(s), x(s+\sigma))$ by the section of $x(s)$ between the two points in consideration.

immediately to the formula

$$\int\limits_x^{\widehat{y}} (\delta^a_b + \Gamma^a_{bc}\, ds^c) = \delta^a{}_b + \int\limits_x^y \Gamma^a_{bc}(s)\, ds^c + \\ + \frac{1}{2}\int\limits_x^y\int\limits_x^y \Gamma^a_{sc}(s)\, \Gamma^s_{bd}(s')\, ds^c\, ds'^d + \cdots . \tag{A.3}$$[27]

Using the matrix notation: $\Gamma^a_{bc}(s)\, ds^c \equiv \Gamma(s)\, ds$ and substituting $\Gamma^a_{bc}(s) = s'\, d\Gamma^a_{bc}(s) + \frac{s'^2}{2}\, dd\Gamma^a_{bc}(s) + \cdots$, where $d\Gamma^a_{bc}(s) \equiv \Gamma^a_{bc|d} t^d$, t^a beeing the tangent vector to $x(s)$, we get:

$$\int\limits_{x(s)}^{\widehat{x(s+\sigma)}} (\mathbf{1} + \Gamma\, ds) = \mathbf{1} + d\Gamma(s)\frac{\sigma^2}{2} + dd\Gamma(s)\frac{\sigma^3}{3!} + o(\sigma^4) . \tag{A.4}$$

With the aid of (A.2), (4.2) and theorem of Gauß we get easily:

$$P^a_{(s+\sigma)} - P^a(s) = \int\limits_{B(\sigma)} d\,\omega^a_{x(s)}(\eta) + \int\limits_{B(\sigma)} (R\,\omega)^a_{x(s)}(\eta)\, d\eta + \\ + \frac{\sigma^2}{2}\, d\Gamma^a_b(s) \Big\{ \int\limits_{\Gamma_{x(s+\sigma)}} \omega^b_{x(s)}(\eta) + \int\limits_{B(\sigma)} (R\,\omega)^b_{x(s)}(\eta)\, d\eta \Big\} + \cdots \tag{A.5}$$

To proof the inequality of our 1^{st} statement we start with an estimate of:

$$(R\,\omega)^a_{x(s)}(\eta) = \omega^b_{x(s)}(\eta) \int\limits_{\Delta_\eta} \Big(\int\limits_{\xi}^{\widehat{x(s)}} \delta^a_e + \Gamma^a_{ec}\, ds^c \Big) R^e{}_{fst}(\xi)\, d\overset{*}{x}{}^{st} \Big(\int\limits_{\xi}^{\widehat{x(s)}} (\delta^f_b - \Gamma^f_{bc}\, ds^c) \Big).$$

We set $\hat\Gamma(\xi) \equiv \sup\limits_{g(\xi, x(s))} |\Gamma(s)|$ and use (A.3) to get:

$$|(R\,\omega)^a_{x(s)}(\eta)| \leqq |\omega^b_{x(s)}(\eta) \int\limits_{\Delta_\eta} (e^{|g(\xi, x(s))|\,\hat\Gamma(\xi)})^a_e\, R^e{}_{fst}(\xi)\, d\overset{*}{x}{}^{st}\, (e^{\cdots})^f_b .$$

We use $R^a{}_{bcd}\, d\overset{*}{x}{}^{cd} \equiv R(\xi)\, d^2\xi$ and introduce $|g|_\eta \equiv \sup\limits_{\xi \in \Delta_\eta} |g(\xi, x(s))|$,

$$|\hat\Gamma|_\eta \equiv \sup_{\xi\in\Delta_\eta} \hat\Gamma(\xi),\quad |R|_\eta \equiv \sup_{\xi\in\Delta_\eta} |R(\xi)|$$

$$|(R\,\omega)^a_{x(s)}(\eta)| \leqq |\omega^a_{x(s)}(\eta)|\, |R|_\eta\, e^{2|g|_\eta |\hat\Gamma|_\eta} \int\limits_{\Delta_\eta} d^2\xi . \tag{A.6}$$

Assumptions (2.5), (2.10) guarantee that the right hand side is finite, so (A.2) was reasonable.

(4.2) gives:

$$\int\limits_{B(\sigma)} d\,\omega^a_{x(s)}(\eta) = -2 \int\limits_{B(\sigma)} d\overset{*}{x} \int\limits_{\eta}^{\widehat{x(s)}} (\delta^a{}_b + \Gamma^a_{bc}\, ds^c)\, \Gamma^{(b}_{st} T^{t)s}(\eta) .$$

[27] This is valid for any path $x \to y$ in V^4.

Analogous to the procedure scetched above, we get:

$$\Big| \int_{B(\sigma)} d\,\omega^a_{x(s)}(\eta)\Big| \leqq 2\,\|\Gamma\|_\sigma\, e^{2\|g\|_\sigma\|\Gamma\|_\sigma} \Big| \int_{B(\sigma)} T^{st}\, d\overset{*}{x}\Big|$$

where $\|g\|_\sigma \equiv \sup\limits_{\eta\in B(\sigma)} |g(\eta, x(s))|$, $\|\Gamma\|_\sigma \equiv \sup\limits_{\eta\in B(\sigma)} |\Gamma|_\eta$. Using the k-map defined in § 4.3 and (2.8) we replace $\int\limits_{B(\sigma)} T^{st}\, d\overset{*}{x}$ by the integral $\int\limits_{\Gamma_s} d^3\eta \int\limits_0^{\sigma(\eta)} d\,s\, T^{st}(s, \eta)$ $= \int\limits_{\Gamma_s} d^3\eta\, \sigma(\eta)\, \overset{m}{T}{}^{st}(\eta)$, where the $\int\limits_0^{\sigma(\eta)} d\,s$ is meant along the k-lines and $\overset{m}{T}{}^{st}(\eta)$ stems from the application of the mean value theorem on the path-integral. Obviously $\lim\limits_{\sigma\to 0} \frac{\sigma(\eta)}{\sigma} = 1$ and $\Big|\int\limits_{\Gamma_s} T^{st}\, d^3\eta\Big| < 4\,|P^a|$ because of (2.1). Finally our considerations result in

$$\lim_{o\to 0} \frac{1}{\sigma} \int_{B(\sigma)} d\,\omega^a_{x(s)}(\eta) = \beta^a{}_b\, P^b \tag{A.7}$$

where $|\beta^a{}_b| \leqq 8\,\|\Gamma\|\, e^{\|g\|\|\Gamma\|}$ ($\|g\| \leqq D$ because of (2.5)!). (A.5), (A.6), (A.7) result in

$$\left|\frac{d\,P^a}{d\,s}\right| = \left|\lim_{\sigma\to 0} \frac{P^a(s+\sigma) - P^a(s)}{\sigma}\right| < (8\,\|\Gamma\| + D\,\|R\|)\, e^{2D\|\Gamma\|}\, |P^a| \equiv \alpha_0\, |P^a| \tag{A.8}$$

where $\|R\| \equiv \sup\limits_{\eta\in\Gamma_s} |R|_\eta$ and because of $\Big|\lim\limits_{\sigma\to 0} \frac{1}{\sigma}\int\limits_{\Delta_\eta} d^2\xi\Big| \leqq D$ as is easily seen.

So we proved our first statement with an upper bound α_0 for α. The formula (A.8) shows that $\alpha = 0$ in flat space-time in agreement with the result in § 3.

In appendix B we give the order of magnitude of α_0 by numerical calculation.

2nd statement: $\left|\frac{d\,u^a}{d\,s}\right| \leqq \alpha'$, *where* $0 \leqq \alpha' \ll 1$.

In the $\mathscr{R}(s)$-system we have $\Gamma^a_{bc}(x(s)) = 0$ and therefore we get $\frac{d\,u^a}{d\,s} = \frac{\partial u^a}{\partial x^b}\, t^b$ where t^a is the tangent vector to $x(s)$. We use the formula derived in § 4.1 for the partial derivatives $\frac{\partial u^a}{\partial x^b}$ and we get:

$$\frac{d\,u^a}{\partial x^b}\, t^b = \sum_b \left(2\,g_{a\,r}\, \frac{\partial}{\partial v^b}\, P^r + u^k\, \frac{\partial^2}{\partial v^k\, \partial v^b}\, P^r\, g_{a\,r} + 2\,\lambda\, g_{a\,b}\right)^{-1}$$

$$\left(\frac{\partial}{\partial s}\, g_{b\,r}\, P^r + \frac{\partial}{\partial s}\, P^r\, g_{b\,r} + u^k\, \frac{\partial}{\partial v^k}\, P^r\, \frac{\partial}{\partial s}\, g_{b\,r} + \right. \tag{A.9}$$

$$\left. + u^k\, \frac{\partial}{\partial v^k}\, \frac{\partial}{\partial s}\, P^r \cdot g_{b\,r} + 2\,\lambda\, u^s\, \frac{\partial}{\partial s}\, g_{b\,s}\right)$$

Simultaneously for all $a = 1, 2, 3, 0$ we estimate ($v^a \equiv v$):

$$\left|\frac{\partial P^r}{\partial v}\right| = \left|\lim_{\sigma \to 0} \frac{2}{\sigma} \int_{K(\sigma)} d^4\eta \int_{\eta}^{\widehat{x(s)}} (\delta^r_s + \Gamma^r_{st}\, ds^t)\, \Gamma^{(s}_{tu} T^{t)u}(\eta)\right|$$

where $K(\sigma)$ is the wedge "between" $\Gamma_{u(s)}$ and $\Gamma_{u(s) + \Delta v(\sigma)}$ Considerations analogous to the above ones lead to (see (2.12)):

$$\begin{aligned} \left|\frac{\partial P^r}{\partial v}\right| &\leqq 4\,\|\Gamma\|\, e^{\|D\Gamma\|}\, |P^a| = \tilde{\gamma}_0\, |P^a|^{28} \\ \left|\frac{\partial^2}{\partial v^a\, \partial v^b} P^r\right| &\leqq (4\,\|\Gamma\|)^2\, e^{2D\|\Gamma\|}\, |P^a|\,. \end{aligned} \tag{A.10}$$

It is very simple to see that:

$$\left|\frac{\partial}{\partial s} g_{ab}\right| \leqq |2 g_{d(a} \Gamma^d_{b)c}\, \|t^c| \leqq 2\, \|\hat{g}\|\, \|\Gamma\|$$

where $\|\hat{g}\| \equiv \sup\limits_{\xi \in \Gamma_s} |g_{ab}(\xi)|$.

Using $2\lambda\, g_{as} v^s = g_{as}\left(P^s + u^k \frac{\partial}{\partial v^k} P^s\right)$ for fixed a, we find by (A.10) $|P^a| \leqq 2\lambda \leqq (1 + \tilde{\gamma})\, |P^a|$ as an estimate for the Lagrangian multiplier introduced in (4.11). Observing that the leading term in $(D_v)_{ab}$ is g_{ab} we find easily see ((A.8)):

$$\left|\frac{du^a}{ds}\right| < 2(1 + \tilde{\gamma})^3\, (4\,\|\Gamma\| + \alpha) \equiv \alpha_0' \tag{A.11}$$

where α_0' is an upper bound for α'. Again $\alpha_0' = 0$ in flat space-time as it should be. Numerical estimate of α_0' is given in appendix B.

It will turn out that for practical purpose $4\,\|\Gamma\|$ is the leading term in the second braket of α_0. It stems from the estimate of $\frac{\partial}{\partial s} g_{ab}$ and if necessary we can diminish it by assuming, that the field (resp. T^{ab}) does not vary very much in time i.e. one may replace $4\,\|\Gamma\| \to 4\varrho\, \|\Gamma\|$ where $0 \leqq \varrho'' \ll 1$.

3rd statement: $\frac{dM}{ds} \leqq \alpha''\, |P^a|$, *where* $0 \leqq \alpha'' \ll 1$.

This is an immediate consequence of the 1st and 2nd statement. An upper bound for α'' is $\alpha_0'' = \alpha_0 + \alpha_0'$.

Finally we remark that the inequality $\ll 1$ in all of our three statements is valid under one of the two assumptions:

$$1^\circ \quad (8\,\|\Gamma\| + D\,\|R\|)\, e^{2D\|\Gamma\|} < 1\,. \tag{A.12}$$

2° a) The field varies slowly with time i.e. $|\beta^a_b| \leqq 8\varrho'\, \|\Gamma\|\, e^{D\|\Gamma\|}$

$$\text{and } \left|\frac{\partial}{\partial s} g_{ab}\right| \leqq 2\varrho\, \|\hat{g}\|\, \|\Gamma\| \text{ where } 0 \leqq \varrho, \varrho' \ll 1 \tag{A.13}$$

b) $(8\varrho'\, \|\Gamma\| + D\,\|R\|)\, e^{2D\|\Gamma\|} < 1$.

[28] i.e.: $\tilde{\gamma}_0 = \frac{1}{2}\, |\beta^a{}_b|$.

In the appendix B we give a numerical estimate that makes clear that sometimes it will be wise to use assumption 2°. This gives the precise formulation of (2.13).

Appendix B

To get an impression of the order of magnitudes of the quantities we are dealing with, we give some numerical data in a simplified model. We assume the Schwarzschild-perfect-fluid and assume, that the field has the surface-value all over the ball of radius R, where $R = \frac{\text{radius [km]}}{3 \cdot 10^5 \text{ [km]}}$. For the field quantities we take: $\|R\| \sim \frac{8\pi m}{R^3}$, $\|\Gamma\| \sim 8\pi \frac{m}{R^2}$[29]. The crucial quantities will be $a \equiv 2R\|\Gamma\|$, $b \equiv 2R\|R\|$; with their aid we calculate the following table:

	$\frac{m}{R}$	R	a	b	α_0	α_0'	A_0
Artificial satellite	10^{-24}	10^{-9}	10^{-23}	10^{-14}	$5\cdot10^{-14}$	$1.4\cdot10^{-13}$	$2\cdot10^{-22}$
Earth	$6.7\cdot10^{-6}$	$2\cdot10^{-2}$	$1.7\cdot10^{-8}$	$9\cdot10^{-7}$	$4.5\cdot10^{-5}$	$1.3\cdot10^{-5}$	$2.4\cdot10^{-6}$
Sun	$2.1\cdot10^{-6}$	2.33	$5.5\cdot10^{-5}$	$2.3\cdot10^{-5}$	$1.2\cdot10^{-4}$	$2.7\cdot10^{-4}$	$1.6\cdot10^{-2}$
Dwarf (Sirius B)	10^{-4}	$5\cdot10^{-2}$	$2.5\cdot10^{-3}$	$5\cdot10^{-2}$	$2.5\cdot10^{-1}$	$7\cdot10^{-1}$	$9\cdot10^{-2}$
Giant	10^{-7}	10^{3}	$2.5\cdot10^{-6}$	$2.5\cdot10^{-9}$	$1.3\cdot10^{-8}$	$3.5\cdot10^{-8}$	$9.4\cdot10^{-5}$

We hint at the fact that the leading terms in α_0 resp. α_0' are $5b$ resp. $14b$ both stemming from calculations concerning the time variations of field quantities. Taking into account that our Schwarzschild-field is static, then we get α_0, α_0' of the order of magnitude of $\sim b^2$, which means that $\alpha_0'' \ll 1$ as demanded. But we see from the above table that the introduction of the additional assumption "the field should vary in time slowly" is superfluous in most practical cases.

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[29] m is the Schwarzschildradius: $m \equiv 7.42 \cdot 10^{-29} \times M[g]$.

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