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On Local One-Particle Approximations and Locally Conserved Currents

A. H. VÖLKEL*

II. Institut für Theoretische Physik — Universität Hamburg

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Abstract. Local one-particle approximations are constructed for matrix elements of two local field operators. If one of the fields is a locally conserved current the approximation is extended in such a way that both locality and current conservation are valid in the approximation.

I. Introduction

The most basic difficulty in any theoretical treatment of elementary particle physics is the fact that we have to deal with an infinite number of intercorrelated functions.

In relativistic quantum field theory such intercorrelations are induced by the infinite set of possible intermediate particle states in matrix elements of field operators. As a first approximation, one can try to take only the discrete one-particle states out of this infinite set and drop all the continuous states as intermediate states. Such an approximation would only be reasonable if all the general properties of the theory are not destroyed by this approximation. In relativistic quantum field theory, it is locality which causes some trouble in this respect.

Because locality is destroyed by the simple one-particle approximation for the commutator matrix element

$$\langle \mathbf{p}|\, [A_0(x), B_0(y)]\, |\mathbf{p}\rangle \approx \langle \mathbf{p}|\, A_0(x)\, |1\rangle \langle 1|\, B_0(y)\, |\mathbf{p}\rangle - \\ - \langle \mathbf{p}|\, B_0(y)\, |1\rangle \langle 1|\, A_0(x)\, |\mathbf{p}\rangle$$

FUBINI and FURLAN [1] got an unwanted $\mathbf{p}$-dependence of the corresponding equal time expression and were forced to take a limit $\mathbf{p} \to \infty$ or $\mathbf{p} \to 0$ to get a consistent result.

In the general frame of relativistic quantum field theory [2], [3] "local one-particle approximations" were first constructed by SYMANZIK [4], [5] for retarded functions and by ZIMMERMANN [6], [7] for time ordered functions. STREATER [8] and STORA [9] have investigated the

* Present address: Physics Department-University of Pittsburgh

same problem for the mixed or generalized retarded functions. Starting from the work of Symanzik we have explicitly constructed in an earlier paper [10] the corresponding approximation for the four point Wightman function.

If there is a locally conserved current $\mathrm{j}_u(x)$ in the theory:

$$\partial^\mu \langle \Psi | \, \mathrm{j}_\mu(x) \, |\Phi\rangle = \langle \Psi | \, \partial^\mu \mathrm{j}_\mu(x) \, |\Phi\rangle = 0 \tag{1}$$

any reasonable approximations must be made in such a way, that this equation is also valid in it. Even if the current is not conserved the first part of equation (1) must hold in an approximation for the matrix elements. For if one wants to study any effects caused by breaking of a current conservation, one must not mix the breaking of current conservation of the theory with similar effects induced by the approximation.

In the present paper, we construct a local one-particle approximation for the matrix elements of two local field operators, in which the left part of equation (1) holds, if one of the fields is a current.

II. Local One-Particle Approximation

As already stated in the introduction, our "axiomatic" frame will be local quantum field theory [2], [3] that is, the objects of investigation will be matrix elements of field operators $\{\mathrm{A}_\nu^\alpha(x); \mathrm{B}_\mu^{\gamma_i}(x)\}$ with the usual properties:

Poincaré covariance
(A) Locality
Spectrum condition
Completeness

Notation: In the following $\mathrm{A}_\nu^\alpha(x)$ denotes always a boson field (scalar or vector) with internal symmetry index α and $\mathrm{B}_\mu^{\gamma_i}(x)$ a local field which is associated with the i-th particle of the theory with mass m_i and a set of internal quantum numbers $\gamma_i = \{\gamma_{i_1}, \ldots, \gamma_{i_r}\}$

$$\langle \gamma_i; \mathbf{p}, m_i | \, \mathrm{B}_\mu^{\gamma_i}(0) \, |0\rangle \neq 0$$
$$\langle \gamma_j, \mathbf{p}, m_j | \, \mathrm{B}_\mu^{\gamma_i}(0) \, |0\rangle = 0 \quad \text{for} \quad i \neq j \,. \tag{2}$$

The corresponding asymptotic fields are denoted by $\mathrm{B}_\mu^{\gamma_i}(x)_{ex}$. Then $\bar{\gamma}_i$ means the anti-particle to γ_i. All interacting fields are assumed to be local relative to each other. $T(x)$ is the translation operator and $|\Psi_p\rangle$ is a state of four momentum p.

$$\mathrm{T}(x) \, |\Psi_p\rangle = e^{ipx} |\Psi_p\rangle \quad px =: p^{(0)} x^{(0)} - \mathbf{p}\,\mathbf{x} \,. \tag{3}$$

Last but not least $V_\pm^m$ denotes the open region $\pm p^{(0)} > 0$, $p^2 > m^2$ in Minkowski space and $\overline{V_\pm^m}$ its closure.

We will proceed in three steps. First, we will treat the simple case of only one scalar particle of mass m_i in the theory. Then it will be easy to generalize the result first to the case of several particles of equal or unequal masses and in the last step to particles of spin $s \neq 0$.

a) The Case of One Single Mass m_i

Introducing a complete set of intermediate states between the two field operators, one gets the following structure of the matrix element $\langle \Psi_p | \mathrm{A}^\alpha_\mu(x)\, \mathrm{A}^\beta_\nu(y)\, |\Phi_q\rangle$ in four-momentum space:

$$\begin{aligned}
(2\pi)^{-3} \tilde{F}^{\alpha\beta}_{\mu\nu}(k_1, p, q, k_4)_i &=: (2\pi)^{-3} \langle \Psi_{k_1} | \tilde{\mathrm{A}}^\alpha_\mu(p)\, \tilde{\mathrm{A}}^\beta_\gamma(q)\, |\Phi_{k_4}\rangle \\
&= \delta(p + k_1)\, \delta(q - k_4)\, \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0)\, |0\rangle \langle 0 | \mathrm{A}^\beta_\gamma(0)\, |\Phi_{k_4}\rangle + \\
&+ \delta(k_1 + p + q + k_4)\, \{\delta_+(p_1^2 - m_i^2)\, \langle \Psi_{k_1} | A^\alpha_\mu(0)\, | m_i, \mathfrak{p}_1\rangle \times \\
&\times \langle \mathfrak{p}_1, m_i | \mathrm{A}^\beta_\nu(0)\, |\Phi_{k_4}\rangle + \\
&+ \int_{2m_i}^{\infty} d\varrho(s)\, \delta_+(p_1^2 - s)\, \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0)\, | s, \mathfrak{p}_1\rangle \langle \mathfrak{p}_1, s | \mathrm{A}^\beta_\gamma(0)\, |\Phi_{k_4}\rangle\}
\end{aligned} \tag{4}$$

$$p_1 =: k_1 + p\,.$$

From this equation it follows that the support of $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)$ is contained in the closed set Γ^0_i defined by:

$$\Gamma^0_i =: \left\{ k : \sum_{r=1}^{4} k_r = 0;\ \sum_{r=1}^{n} k_r \in \overline{\overset{\circ}{V}{}^{2m_i}_{+}} \cup \{0\},\ n = 1, 2, 3 \right\} \tag{5}$$

$$\overline{\overset{\circ}{V}{}^{2m}_{\pm}} =: \overline{V^{2m}_{\pm}} \cup \{k : k^2 - m^2 = 0;\ \pm k^{(0)} > 0\}\,. \tag{6}$$

It is well known [2], [3] how to subtract from $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)$ in a local manner the contribution from the intermediate vacuum state by introducing the so-called truncated matrix elements

$$\begin{aligned}
\tilde{F}^{\alpha\beta}_{\mu\nu}(k_1, k_2, k_3, k_4)^T_i =: \tilde{F}^{\alpha\beta}_{\mu\nu}(k_1, k_2, k_3, k_4) &- \\
- \langle \Psi_{k_1} | \tilde{\mathrm{A}}^\alpha_\mu(k_2)\, |0\rangle \langle 0 | \tilde{\mathrm{A}}^\beta_\nu(k_3)\, |\Phi_{k_4}\rangle &- \\
- \langle \Psi_{k_1} | \tilde{\mathrm{A}}^\beta_\nu(k_3)\, |0\rangle \langle 0 | \tilde{\mathrm{A}}^\alpha_\mu(k_2)\, |\Phi_{k_4}\rangle &- \\
- \langle \Psi_{k_1} |\, |\Psi_{k_4}\rangle \langle 0 | \tilde{\mathrm{A}}^\alpha_\mu(k_2)\, \tilde{\mathrm{A}}^\beta_\nu(k_3)\, |0\rangle &.
\end{aligned} \tag{7}$$

The truncated matrix elements are covariant, local and do not contain any vacuum singularity except the δ-function for over-all four-momentum conservation.

The support of $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^T_i$ is given by:

$$\Gamma^T_i = \left\{ k_j : \sum_{r=1}^{4} k_r = 0,\ \sum_{r=1}^{n} k_r \in \overline{\overset{\circ}{V}{}^{2m_i}_{+}},\ n = 1, 2, 3 \right\}. \tag{8}$$

Now we go one step further and subtract from $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)_i$ and at the same time from $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^T$ in a similar way the contribution from the intermediate one-particle state, i.e. the second term in equation (4). In other words, we wish to define a new matrixelement $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i$ by subtracting from $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^T_i$ expressions of the form

$$\int d^4p \, \langle \Psi_{k_1} | \, \tilde{\mathrm{A}}^\alpha_\mu(k_2) \, \tilde{\mathrm{B}}^{\gamma_i}(p) \, |0\rangle \, K(p) \, \langle 0| \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-p) \, \tilde{\mathrm{A}}^\beta_\nu(k_3) \, |\Phi_{k_4}\rangle \tag{9}$$

such that $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i$ is Poincaré covariant, local and has furthermore the following support property:

The support of $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i$ in momentum space is contained in Γ^{II}_i:

$$\Gamma^{II}_i = \left\{ k : \sum_{r=1}^{4} k_r = 0 \, , \;\; k_1 \in \overline{\overset{\circ}{V}{}^{2m_i}_{+}} , k_1 + k_2 \in \overline{V^{2m_i}_{+}} , \;\; k_1 + k_2 + k_3 \in \overline{\overset{\circ}{V}{}^{2m_i}_{+}} \right\} . \tag{10}$$

This condition means that we remove from $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i$ the discrete one-particle singularity in the variable $k_1 + k_2$.

A matrix element $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i$ which has all the properties we demand is given by:

$$\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II}_i =: \tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{T}_i - \tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{I}_i \tag{11}$$

where $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{I}_i$ is defined by:

$$\begin{aligned}
&\tilde{F}^{\alpha\beta}_{\mu\nu}(k_1, p, q, k_4)^I_i =: \int d^4u \; \delta_+(u^2 - m_i^2) \times \\
&\times \{ \langle \Psi_{k_1} | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, | m_i, \mathbf{u}, \gamma_i \rangle \, \langle \gamma_i, \mathbf{u}, m_i | \, \tilde{\mathrm{A}}^\beta_\nu(q) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \tilde{\mathrm{A}}^\beta_u(q) \, | m_i, \mathbf{u}, \gamma_i \rangle \, \langle \gamma_i \, \mathbf{u}, m_i | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-u)_{out} | \Phi_{k_4}\rangle \, \langle 0 | \, \tilde{\mathrm{B}}^{\gamma_i}(u)_{out} \tilde{\mathrm{A}}^\alpha_\mu(p) \, \tilde{\mathrm{A}}^\beta_\nu(q) \, |0\rangle + \\
&+ \langle \Psi_{k_1} | \tilde{\mathrm{B}}^{\gamma_i}(u)_{out} | \Phi_{k_4}\rangle \, \langle 0 | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, \tilde{\mathrm{A}}^\beta_\nu(q) \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-u)_{out} \, |0\rangle \} - \\
&- 2\pi \int d^4u \{ i \langle \Psi_{k_1} | \, \tilde{\Re}(-u^{\bar{\gamma}_i} | q^\beta_\nu) \, |0\rangle \overleftrightarrow{\Delta}(u) \, \langle 0 | \, \tilde{\Re}(u^{\gamma_i} | p^\alpha_\mu) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-u) \, |0\rangle \overleftrightarrow{\Delta}_{ret}(u \, \langle 0 | \, \tilde{\Re}(u^{\gamma_i} | q^\beta_\nu) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-u) \, \tilde{\mathrm{A}}^\beta_\nu(q) \, |0\rangle \overleftrightarrow{\Delta}_{ret} \, (u) \langle 0 | \, \tilde{\Re}(u^{\gamma_i} | p^\alpha_\mu) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \tilde{\Re}(-u^{\bar{\gamma}_i} | q^\beta_\nu) \, |0\rangle \overleftrightarrow{\Delta}_{av}(u) \, \langle 0 | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, \tilde{\mathrm{B}}^{\gamma_i}(u) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \, \Re(-u^{\gamma_i} | p^\alpha_\mu) \, |0\rangle \overleftrightarrow{\Delta}_{av}(u) \, \langle 0 | \, \tilde{\mathrm{B}}^{\gamma_i}(u) \, \tilde{\mathrm{A}}^\beta_\nu(q) \, |\Phi_{k_4}\rangle + \\
&+ \langle \Psi_{k_1} | \, \tilde{\mathrm{B}}^{\bar{\gamma}_i}(-u) \, |\Phi_{k_4}\rangle \overleftrightarrow{\Delta}_{ret}(u) \times \\
&\times [\langle 0 | \, \tilde{\mathrm{A}}^\alpha_\mu(p) \, \tilde{\Re}(u^{\gamma_i} | q^\beta_\nu) \, |0\rangle + \langle 0 | \, \tilde{\Re}(u^{\gamma_i} | p^\alpha_\mu) \, \tilde{\mathrm{A}}^\beta_\nu(q) \, |0\rangle] \} \, .
\end{aligned} \tag{12}$$

In equation (12) we have introduced the following notations:

$$\overleftrightarrow{\Delta}(u) =: \overleftarrow{(m_i^2 - u^2)}\, \Delta(u)\, \overrightarrow{(m_i^2 - u^2)};\ \Delta(u) = 2\pi i\, \varepsilon(-u^{(0)})\, \delta(u^2 - m_i^2)$$

$$\overleftrightarrow{\Delta}_{av}^{ret}(u) =: \overleftarrow{(m_i^2 - u^2)}\, \Delta_{av}^{ret}(u)\, \overrightarrow{(m_i^2 - u^2)};\ \Delta_{av}^{ret}(u) = \frac{1}{m_i^2 - (u \pm i\varepsilon)^2}\quad \varepsilon > 0$$

$$\tilde{\mathfrak{R}}(u^{\gamma_i} | k_\mu^\alpha) =: -\frac{i}{(2\pi)^5} \int d^4x\, d^4y\, e^{i(ux+ky)} \times \qquad (13)$$

$$\times\, \theta(x^0 - y^0)\, [\mathrm{B}^{\gamma_i}(x); \mathrm{A}_\mu^\alpha(y)]$$

$$\tilde{\mathrm{B}}^{\gamma_i}(\mathrm{k})_{ex} = \theta(k^{(0)})\, \mathrm{b}_{\gamma_i}(\mathrm{k})_{ex} + \theta(-k^{(0)})\, \mathrm{b}_{\bar{\gamma}_i}(-\mathrm{k})_{ex}^* \qquad (14)$$

$$\mathrm{b}_{\gamma\alpha}(\mathrm{k})_{ex}^* |0\rangle = |m_i, \mathrm{k}, \gamma_i\rangle;\ \tilde{\mathrm{B}}^{\bar{\gamma}_i}(k) = \tilde{\mathrm{B}}^{\gamma_i}(-k)^*\,.$$

Before we show that $\tilde{F}_{\mu\nu}^{\alpha\beta}(k)_i^{II}$ has all the properties we demand, let us make a remark on how one can derive formula (12) in some logical manner. Starting from the work of SYMANZIK [4], [5] on retarded functions, we had constructed in [10] an expression similar to (12) for the Wightman-function $W(x_1, x_2, x_3, x_4)^I$. If we drop in this expression all terms which are local in the variables $x_1 - x_2$ and $x_3 - x_4$ we come immediately to equation (12).

The proof of the demanded properties runs in the same way as that for the Wightman-function [10]. Therefore we can restrict ourselves to some remarks.

Because every term in (12) has the support Γ_i^T we have $p \notin \overline{V_+^{m_i}}$ and $q \notin \overline{V_-^{m_i}}$ for $0 < (k_1 + p)^2 < 4m_i^2$. In the allowed region all terms in equation (12) vanish except the first, which is cancelled by the corresponding term from $\tilde{F}_{\mu\nu}^{\alpha\beta}(k)_i^K$ in equation (11).

The third, fourth and tenth term in equation (12) are separately local. If we break up the commutators which occur in the expression for $F_{\mu\nu}^{\alpha\beta}(k_1, x, y, k_4)_i^I - F_{\nu\mu}^{\beta\alpha}(k_1, y, x, k_4)_i^I$ into retarded and advanced parts and use the well known relation $\Delta_{av}(p) - \Delta_{ret}(p) = \Delta(p)$, then the first, second and fifth term in equation (12) and the commuted ones are all cancelled. We are left with a sum of retarded and advanced functions which vanish for space-like separations of x and y.

In the following the expression (12) will be called "*local one-particle approximation*" for the truncated matrix element $\tilde{F}_{\mu\nu}^{\alpha\beta}(k)_i^T$.

As we have seen before, the third, fourth and tenth term in equation (12) are separately local. If we had dropped them, we would not have destroyed the locality and the support properties of $\tilde{F}_{\mu\nu}^{\alpha\beta}(k)_i^{II}$. On the other hand in certain cases these terms correspond to the peripheral model, if one calculates from (12) the amplitude for the process $k_4 + q \rightarrow$ $\rightarrow k_1 + p$. Therefore we will not drop them.

b) The Case of Several Scalar Particles

If there are several particles in the theory with masses $0 < m_1 \leqq \leqq m_2 \leqq \cdots \leqq m_N$ and a set of internal quantum numbers

$$\gamma_i = \{\gamma_{i_1}, \ldots, \gamma_{i_r}\} \quad (i = 1, \ldots, N)$$

then equation (4) reads:

$$\begin{aligned}
&(2\pi)^{-3} \tilde{F}^{\alpha\beta}_{\mu\nu}(k_1, p, q, k_2) =: (2\pi)^{-3} \langle \Psi_{k_1} | \tilde{\mathrm{A}}^\alpha_\mu(p)\, \tilde{\mathrm{A}}^\beta_\nu(q)\, |\Phi_{k_2}\rangle \\
&= \delta(k_1 + p)\, \delta(q - k_2) \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0) |0\rangle \langle 0| \mathrm{A}^\beta_\nu(0) |\Phi_{k_2}\rangle + \\
&\quad + \delta(k_1 + p + q - k_2) \Bigg\{ \sum_{i=1}^{N} \delta_+((p + k_1)^2 - m_i^2) \times \\
&\quad \times \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0) | m_i, \mathbf{p}_1, \gamma_i\rangle \langle \gamma_i, \mathbf{p}_1, m_i | \mathrm{A}^\beta_\nu(0) |\Phi_{k_2}\rangle + \\
&\quad + \sum_{i=1}^{N} \int d\varrho(s)\, \theta(s - M_{\gamma_i})\, \delta_+(p_1^2 - s) \times \qquad (15)\\
&\quad \times \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0) | s, \mathbf{p}_1, \gamma_i\rangle \langle \gamma_i, \mathbf{p}_1, s | \mathrm{A}^\beta_\nu(0) |\Phi_{k_2}\rangle + \\
&\quad + \sum_{\substack{\alpha' \neq \gamma_i \\ (i = 1, \ldots, N)}} \int d\varrho(s)\, \theta(s - M_\alpha)\, \delta_+(p_1^2 - s) \times \\
&\quad \times \langle \Psi_{k_1} | \mathrm{A}^\alpha_\mu(0) | s, \mathbf{p}_1, \alpha'\rangle \langle \alpha', \mathbf{p}_1, s | \mathrm{A}^\beta_\nu(0) |\Phi_{k_2}\rangle \Bigg\} \\
&\qquad\qquad p_1 =: p + k_1 .
\end{aligned}$$

Here M_α denotes the threshold mass of the intermediate continuous states with internal quantum numbers $\alpha = \{\alpha_1, \ldots, \alpha_r\}$. If in each channel with the internal quantum numbers γ_i there is a gap between the one-particle mass m_i and the threshold M_{γ_i} then one can construct in the same manner as in case a) a one-particle approximation for the truncated matrix element by:

$$\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^I =: \sum_{i=1}^{N} \tilde{F}^{\alpha\beta}_{\mu\nu}(k)^I_i \qquad (16)$$

where $\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^I_i$ is given by equation (12).

The matrix element

$$\tilde{F}^{\alpha\beta}_{\mu\nu}(k)^{II} =: \tilde{F}^{\alpha\beta}_{\mu\nu}(k)^T - \tilde{F}^{\alpha\beta}_{\mu\nu}(k)^I \qquad (17)$$

does not contain contributions from the N one-particle intermediate states, i.e. the equation (10) is valid for it for every m_i $(i = 1, 2, \ldots, N)$.

c) The Case of Particles with Spin s

We use the $(2s + 1)$-component spinor formulation for the fields in the notation of THEIS [11]. To generalize equation (12) to this case, we have only to make the following insertions:

$$\begin{aligned}
&\tilde{\mathrm{B}}^{\gamma}(p) \to \tilde{\mathrm{B}}^{\gamma}_{A}(p) \qquad\qquad A = -s, -s+l, \ldots, s\\
&\tilde{\mathrm{B}}^{\bar\gamma}(p) \to \tilde{\mathrm{B}}^{\bar\gamma}_{\dot A}(p) =: \tilde{\mathrm{B}}^{\gamma}_{A}(-p)^{*}\\
&\overleftrightarrow{\Delta}_{\binom{ret}{av}}(p) \to \pi_s^{\dot A C}\left(\frac{p_\mu}{m}\right)\overleftrightarrow{\Delta}_{\binom{ret}{av}}(p)\\
&\int \frac{d^3p}{2w_p}\,|m, \mathbf{p}, \gamma\rangle\langle\gamma, \mathbf{p}, m| \to \sum_{s_3=-s}^{+s}\int\frac{d^3p}{2w_p}\,|m, \mathbf{p}, s_3, \gamma\rangle\langle\gamma, s_3, \mathbf{p}, m|\,.
\end{aligned} \tag{18}$$

The spinor fields $\mathrm{B}^{\gamma}_{A}(x)$ transform under the proper homogeneous Lorentz transformation Λ according to

$$\mathrm{B}^{\gamma}_{A}(x) \xrightarrow[(\Lambda)]{} \mathrm{D}^{(s,0)}_{AA'}(\Lambda^{-1})\,\mathrm{B}^{\gamma}_{A'}(\Lambda x) \tag{19}$$

where $\mathrm{D}^{(s,0)}$ is the $(2s+1)$-dimensional, irreducible representation of the proper homogeneous Lorentz group.

$$\begin{aligned}
&\pi_s^{\dot A C}\left(\frac{p_\mu}{m}\right) =: \left(\frac{\sqrt{p^2}}{m_i}\right)^{2s}\mathrm{D}^{(s,0)}_{\dot A C}\left(\frac{1}{\sqrt{p^2}}\{p^0 - \mathbf{p}\boldsymbol{\sigma}\}\right)\\
&\mathrm{L}_p^{-1}\cdot p = (m, \mathbf{0})\,.
\end{aligned} \tag{20}$$

The free fields are given by:

$$\begin{aligned}
\tilde{\mathrm{B}}^{\gamma}_{A}(p)_{\mathrm{ex}} = \sum_{s_3=-s}^{+s}\{&\theta(p_0)\,\mathrm{D}^{(s,0)}_{A,s_3}(\mathrm{L}_p)\,\mathrm{b}_\gamma(\mathbf{p}, s_3)_{\mathrm{ex}} +\\
&+\theta(-p_0)\,(-1)^{s-A}\,\mathrm{D}^{(s,0)}_{s_3,-A}(\mathrm{L}_p^{-1})\,\mathrm{b}_{\bar\gamma}(-\mathbf{p}, s_3)^{*}_{\mathrm{ex}}\}
\end{aligned} \tag{21}$$

where the creation and annihilation operators $\mathrm{b}_\gamma(\mathbf{p}, s_3)^{*}_{\mathrm{ex}}$, $\mathrm{b}_\gamma(\mathbf{p}, s_3)_{\mathrm{ex}}$ obey the usual commutation and anticommutation relations.

III. Current Conservation

As already stated in the introduction, for a reasonable one-particle approximation, one has to demand the following property, if one of the fields is a conserved or even a non-conserved current $\mathrm{j}_\mu(x)$:

$$\begin{aligned}
\partial^\mu_x\langle\Psi_{p_1}|\,\mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i &=: \partial^\mu_x F^{\alpha\beta}_\mu(p_1, x, y, p_2)^I_i\\
&= \langle\Psi_{p_1}|\,\partial^\mu \mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i
\end{aligned} \tag{22}$$

where both $\langle\Psi_{p_1}|\,\mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i$ and $\langle\Psi_{p_1}|\,\partial^\mu\mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i$ are defined by equation (12). Because in this equation there occur retarded matrix elements, we get for the difference of the right and left hand side of (22):

$$\begin{aligned}
&\partial^\mu_x\langle\Psi_{p_1}|\,\mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i - \langle\Psi_{p_1}|\,\partial^\mu\mathrm{j}^\alpha_\mu(x)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle^I_i\\
&= -\int d^4z\{\langle\Psi_{p_1}|\,\mathrm{B}^{\bar\gamma_i}(z)\,\mathrm{A}^\beta(y)\,|0\rangle\,(\overleftarrow{\Box_z + m_i^2})\,\times\\
&\times[\partial^0_x\langle 0|\,\Re(z^{\gamma_i}/x^\alpha_0)\,|\Phi_{p_2}\rangle - \langle 0|\,\Re(z^{\gamma_i}/\partial^0 x^\alpha_0)\,|\Phi_{p_2}\rangle] +\\
&+[\partial^0_x\langle\Psi_{p_1}|\,\Re(z^{\bar\gamma_i}/x^\alpha_0)\,|0\rangle - \langle\Psi_{p_1}|\,\Re(z^{\bar\gamma_i}/\partial^0 x^\alpha_0)\,|0\rangle]\,\times\\
&\times(\overrightarrow{\Box_z + m_i^2})\,\langle 0|\,\mathrm{B}^{\gamma_i}(z)\,\mathrm{A}^\beta(y)\,|\Phi_{p_2}\rangle +\\
&+\langle\Psi_{p_1}|\,\mathrm{B}^{\bar\gamma_i}(z)\,|\Phi_{p_2}\rangle\,(\Box_z + m_i^2)\,\times\\
&\times[\partial^0_x\langle 0|\,\Re(z^{\gamma_i}/x^\alpha_0)\,\mathrm{A}^\beta(y)\,|0\rangle - \langle 0|\,\Re(z^{\gamma_i}/\partial^0 x^\alpha_0)\,\mathrm{A}^\beta(y)\,|0\rangle]\}\,.
\end{aligned} \tag{23}$$

Now we must try to re-define the one-particle approximation by adding to equation (12) a local solution of the differential equation:

$$\partial_x^\mu f_\mu^{\alpha\beta\gamma_i}(p_1, x, y, p_2)^I = \langle \Psi_{p_1}| \, \partial^\mu \mathrm{j}_\mu^\alpha(x) \, \mathrm{A}^\beta(y) \, |\Phi_{p_2}\rangle_i^I - \partial_x^\mu \langle \Psi_{p_1}| \, \mathrm{j}_\mu^\alpha(x) \, \mathrm{A}^\beta(y) \, |\Phi_{p_2}\rangle \,. \tag{24}$$

Furthermore, this solution must not destroy the four-momentum space properties (9), (10) of the matrix element $\tilde{H}_\mu^{\alpha\beta\gamma_i}(k_1, k_2, k_3, k_4)_i^{II}$ which is defined in analogy to (11) by:

$$\tilde{H}_\mu^{\alpha\beta\gamma_i}(k)^{II} =: \tilde{F}_\mu^{\alpha\beta}(k)_i^T - F_\mu^{\alpha\beta}(k)_i^I - f_\mu^{\alpha\beta\gamma_i}(k)^I$$

or

$$\tilde{H}_\mu^{\alpha\beta\gamma_i}(k)^{II} = F_\mu^{\alpha\beta}(k)_i^{II} - f_\mu^{\alpha\beta\gamma_i}(k)^I \,. \tag{25}$$

In other words $f_\mu^{\alpha\beta\gamma_i}$ must itself satisfy the spectrum conditions (9), (10) and locality. From equation (23) it is easy to see that the right hand side of the differential equation (24) has all these properties. Furthermore, because of locality, the support of the three brackets in (23) is concentrated in the point $z - x = 0$. Therefore, they are given by a finite sum of δ-functions and their derivatives [12], [13].

$$\langle 0| \, \partial_x^{(0)} \mathfrak{R}(z^{\gamma_i}/x_0^\alpha) \, |\Phi_{p_2}\rangle - \langle 0| \, \mathfrak{R}(z^{\gamma_i}/\partial^{(0)} x_0^\alpha) \, |\Phi_{p_2}\rangle = e^{-ip_2x} \sum_r \Box_z^{(r)} \delta(z-x) \, g_r^{\alpha\gamma_i}(p_2)_2 \tag{26}$$

$$\langle \Psi_{p_1}| \, \partial_x^{(0)} \mathfrak{R}(z^{\bar{\gamma}_i}/x_0^\alpha) \, |0\rangle - \langle \Psi_{p_1}| \, \mathfrak{R}(z^{\bar{\gamma}_i}/\partial^{(0)} x_0^\alpha) \, |0\rangle = e^{ip_1x} \sum_r \Box_z^{(r)} \, \delta(z-x) \, g_r^{\alpha\bar{\gamma}_i}(p_1)_1 \tag{27}$$

$$\langle 0| \, \partial_x^{(0)} \mathfrak{R}(z^{\gamma_i}/x_0^\alpha) \, \mathrm{A}^\beta(y) \, |0\rangle - \langle 0| \, \mathfrak{R}(z^{\gamma_i}/\partial^{(0)} x_0^\alpha) \, \mathrm{A}^\beta(y) \, |0\rangle = \sum_r \Box_z^{(r)} \, \delta(z-x) \, g_r^{\alpha\beta\gamma_i}(y-x)_+ \tag{28}$$

$$\Box_z^{(r)} =: (\partial^{(0)\,2} - \nabla^2)^r \,.$$

For later use we define $g_r^{\alpha\beta\gamma_i}(x)_-$ by the expression which we get if we commute in (28) the field operator with the retarded commutator.

The coefficients g_r in these equations depend on special dynamical assumptions. From our general frame we only know:

1. $g_r(x)_+ - g_r(x)_- = 0$ for $x^2 < 0$
2. $\mathrm{supp.}\,\tilde{g}_r(q)_\mp \subseteqq \overline{V_\pm^{M_3}}$

where M_3 is the lowest mass with

$$\langle M_3| \, \mathrm{A}^\beta(0) \, |0\rangle \neq 0 \,. \tag{29}$$

Inserting equations (26)—(28) into (23) and (24) respectively, performing the z-integrations, using translation invariance and decomposing $f_\mu^{\alpha\beta\gamma}$

according to (23) into a sum of three parts:

$$f^{\alpha\beta\gamma_i}_{\mu+}(p_1, x, y, p_2)^I = e^{i\frac{p_1-p_2}{2}(x+y)} \times \\ \times \sum_{r=1}^{3} e^{i\varepsilon(r)\frac{p_r}{2}(x-y)} f^{\alpha\beta\gamma_i}_{\mu+}(p_1, x-y, p_2)_r \tag{30}$$

$$\varepsilon(1) = 1, \quad \varepsilon(2) = -1, \quad \varepsilon(3) = 0$$

we finally arrive at the three differential equations:

$$(\partial^\mu_x + i a^\mu_r) f^{\alpha\beta\gamma_i}_{\mu+}(p_1, x-y, p_2)_r = h^{\alpha\beta\gamma_i}_{+}(p_1, x-y, p_2)_r \tag{31}$$

$$a_1 = p_1 - \frac{p_2}{2} \quad a_2 = \frac{p_1}{2} - p_2 \quad a_3 = \frac{1}{2}(p_1 - p_2) \,.$$

The inhomogeneous parts of these equations are given by:

$$h^{\alpha\beta\gamma_i}_{1+}(p_1\, x, p_2) =: \left\langle 0 \middle| \mathrm{J}^{\gamma_i}_1\left(\frac{x}{2}\right) \mathrm{A}^\beta\left(-\frac{x}{2}\right) \middle| \Phi_{p_2} \right\rangle \\ h^{\alpha\beta\gamma_i}_{2+}(p_1, x, p_2) = \left\langle \Psi_{p_1} \middle| \mathrm{J}^{\bar\gamma_i}_2\left(\frac{x}{2}\right) \mathrm{A}^\beta\left(-\frac{x}{2}\right) \middle| 0 \right\rangle \tag{32}$$

$$h^{\alpha\beta\gamma_i}_{3+}(p_1, x, p_2) =: (m_i^2 - (p_1 - p_2)^2) \langle \Psi_{p_1} | \mathrm{B}^{\bar\gamma_i}(0) | \Phi_{p_2} \rangle \times \\ \times \sum_r (p_2 - p_1)^{2r} g^{\alpha\beta\gamma_i}_r(-x)_+ \tag{33}$$

where we have introduced the currents $\mathrm{J}^\gamma_n(x)$ by the definition:

$$\mathrm{J}^{\gamma_i}_n(x) =: \sum_r g^{\alpha\bar\gamma_i}_r(p_n)_n \square_x^{(r)} (\square_x^{(1)} + m_i^2) \mathrm{B}^{\gamma_i}(x) \,. \tag{34}$$

If we define the distributions $h^{\alpha\beta\gamma}_{n-}$ by the right hand sides of the equations (32), (33) with J^γ_n and A^β commuted and $g^{\alpha\beta\gamma}_{r-}$ inserted for $g^{\alpha\beta\gamma}_{r+}$ respectively, then according to our previous statements in this chapter we have the following properties:

1. *Locality*

$$h^{\alpha\beta\gamma}_n(p_1, x, p_2) =: h^{\alpha\beta\gamma}_{n+}(p_1, x, p_2) - h^{\alpha\beta\gamma}_{n-}(p_1, x, p_2) \\ = 0 \quad \text{for} \quad x^2 < 0 \,. \tag{35}$$

2. *Spectrum condition*

$$\tilde h^{\alpha\beta\gamma}_{n\pm}(p_1, x, p_2) = 0$$

$$\text{for} \quad \{(1/2 p_{3-n} \pm q) \notin \overline{V^{M_{n\pm}}_+} \text{ or } p_j \notin V_+\} \; p_0 =: 0 \tag{36}$$

where $M_{n\pm}$ are the threshold masses between the field operators in equation (32).

$$M_{3+} = M_{3-} = M_{2+} = M_{1-} \,. \tag{37}$$

Now we define the distribution $H^{\alpha\beta\gamma}_\mu(k)^I$ introduced by

$$\tilde H^{\alpha\beta\gamma}_\mu(k)^I =: \tilde F^{\alpha\beta}_\mu(k)^I_i + f^{\alpha\beta\gamma_i}_\mu(k)^I \tag{38}$$

where $\tilde{F}^{\alpha\beta}_{\mu}(k)^I_i$ and $f^{\alpha\beta\gamma_1}_{\mu}(k)^I$ are given by equation (12) and the solution of the boundary value problem (30), (31), (35)—(36) respectively, as the "*one-particle approximation*" for the truncated matrix elements of a current and a field.

All we have to do is to solve the differential equation (31) for $f^{\alpha\beta\gamma}_{\mu+}(k)^I$ with the boundary conditions (35)—(36). The difficulty in doing this arises from the fact that these boundary conditions are defined partially in x-space and partially in q-space. If one solves the equations (31), (35) in x-space then it is very hard to see what the condition (36) mean for this solution and vice versa.

One can avoid this difficulty by means of integral representations, which automatically contain the boundary conditions (35) and (36). For one can insert these representations into the differential equation (31) and solve the corresponding equation for the kernels of these representations.

Such representations are well known [14], [15], [16] for commutators $K^{\alpha\beta\gamma}_{\mu}(k) =: f^{\alpha\beta\gamma}_{\mu+}(k) - f^{\alpha\beta\gamma}_{\mu-}(k)$. In the next chapter, we will first solve our boundary value problem for the commutator by means of the unique Jost-Lehmann-representation [14]—[16] for the case of a symmetric spectrum ($M_{n+} = M_{n-}$). In principle one can solve the nonsymmetric case in the same manner, but the calculations are very involved. Because for many applications it is sufficient to work with the nonunique Dyson-representation [15] [16] (several kernels belong to the same matrix element), we will restrict ourselves to this representation for the case of a nonsymmetric spectrum.

Having in mind this non-uniqueness, we may now go back to the matrix elements $f^{\alpha\beta\gamma}_{\mu\pm}$. The Dyson representation for the commutator reads:

$$K^{\alpha\beta\gamma}_{\mu}(p_1, q, p_2)_n = \int d^4u\, ds\, \varepsilon(q^{(0)} - u^{(0)}) \times \times \delta((q-u)^2 - s)\, \psi^{\alpha\beta\gamma}_{\mu}(p_1, u, s, p_2)_n \tag{39}$$

where the support of ψ is contained in:

$$\Gamma^n_D =: \left\{(u, s) : \left(\frac{1}{2} p_{3-n} \pm u\right) \in V_+, \sqrt{s} \geqq \max\left[0, M_{n+} - \sqrt{\left(\frac{1}{2} p_{3-n} + u\right)^2}, M_{n-} - \sqrt{\left(\frac{1}{2} p_{3-n} - u\right)^2}\right]\right\} \tag{40}$$

By virtue of this support property, we can decompose (39) into its positive and negative frequency parts:

$$f^{\alpha\beta\gamma}_{\mu+}(p_1, q, p_2)_n = \int d^4u\, ds\, \theta(q^{(0)} - u^{(0)}) \times \times \delta((q-u)^2 - s)\, \psi^{\alpha\beta\gamma}_{\mu}(p_1, u, s, p_2)_n \tag{41}$$

$$f^{\alpha\beta\gamma}_{\mu-}(p_1, q, p_2)_n = \int d^4u\, ds\, \theta(u^{(0)} - q^{(0)}) \times \times \delta((q-u)^2 - s)\, \psi^{\alpha\beta\gamma}_{\mu}(p_1, u, s, p_2)_n\,. \tag{42}$$

IV. Solution of the Boundary Value Problem

We will only treat the case of scalar particles, because the generalization to particles with spin is trivial.

a) The Case of a Symmetric Spectrum ($M_{n+} = M_{n-} = M_n$)

The differential equation (31) reads for the commutator

$$\mathrm{K}_{\mu}^{\alpha\beta\gamma} = f_{\mu+}^{\alpha\beta\gamma} - f_{\mu-}^{\alpha\beta\gamma}$$

in momentum space:

$$(q^{\mu} - a_n^{\mu})\, \mathrm{K}_{\mu}^{\alpha\beta\gamma}(p_1, q, p_2)_n = i\, h_n^{\alpha\beta\gamma}(p_1, q, p_2)\,. \tag{43}$$

To take into account locality and the support property (36) in four-momentum space, we use for both sides of (43) a Jost-Lehmann-representation [14]—[16] in the rest frame of $1/2\, p_{3-n}$ ($p_0 = 0$)

$$\begin{aligned} \mathrm{K}_{\mu}^{\alpha\beta\gamma}(p_1, q, p_2)_n = {} & \varepsilon(q^{(0)}) \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s) \times \\ & \times [\Phi_{\mu}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n + q^{(0)} \psi_{\mu}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n] \end{aligned} \tag{44}$$

$$\begin{aligned} h_n^{\alpha\beta\gamma}(p_1, q, p_2) = {} & \varepsilon(q^{(0)}) \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s) \times \\ & \times [\varphi_1^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2) + q^{(0)} \varphi_2^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)]\,. \end{aligned} \tag{45}$$

According to (36) the support of the O_3^+-covariant spectral-functions is contained in:

$$\begin{aligned} \Gamma_{JL}^n = \Big\{(\mathbf{u}, s) : |\mathbf{u}| < \frac{1}{2} p_{3-n}^{(0)}, \sqrt{s} \geqq \max\Big[0, \\ M_n - \sqrt{\left(\frac{1}{2} p_{3-n}^{(0)}\right)^2 - \mathbf{u}^2}\Big]\Big\} \end{aligned} \tag{46}$$

Because the variables $p_j, \alpha, \beta, \gamma, n$ are superfluous in the following calculations, we will drop them.

If we insert the equations (44), (45) into (43), we are led to the calculations of the following two expressions:

$$\begin{aligned} \mathbf{F}(q) &= \mathbf{q} \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s)\, \psi(\mathbf{u}, s) \\ F^0(q) &= (q^{(0)})^2 \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s)\, \psi(\mathbf{u}, s)\,. \end{aligned} \tag{47}$$

Taking into account the δ-functions and the identity

$$\begin{aligned} \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s) \frac{\partial}{\partial s} \int_{-\infty}^{s} ds'\, \psi(\mathbf{u}, s') \\ = \int d^3\mathbf{u}\, ds\, \delta^{(1)}((q^{(0)})^2 - (\mathbf{q} - \mathbf{u})^2 - s)\, \psi(\mathbf{u}, s) \end{aligned} \tag{48}$$

we obtain by partial integrations:

$$\mathbf{F}(q) = \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q}-\mathbf{u})^2 - s) \times \left[\mathbf{u}\,\psi(\mathbf{u},s) - \frac{1}{2}\int\limits_{-\infty}^{s} ds'\, \boldsymbol{\nabla}_u\, \psi(\mathbf{u},s')\right] \tag{49}$$

$$F^0(q) = \int d^3\mathbf{u}\, ds\, \delta((q^{(0)})^2 - (\mathbf{q}-\mathbf{u})^2 - s) \times \left[s\,\psi(\mathbf{u},s) + \frac{1}{4}\int\limits_{-\infty}^{s} ds'\, ((s-s')\,\Delta_u + 6)\,\psi(\mathbf{u},s')\right]. \tag{50}$$

Insertion of (44), (45) into (43), decomposition of the resulting equation into symmetric and antisymmetric parts with respect to inversion of $q^{(0)}$, use of the equations (49), (50) and differentiation with respect to s finally gives the following set of coupled differential equations between the spectral functions:

$$\begin{aligned}\frac{\partial^2}{\partial s^2}[(s-(a^{(0)})^2)\,\psi_0(\mathbf{u},s)] + \frac{3}{2}\frac{\partial}{\partial s}\psi_0(\mathbf{u},s) + \frac{1}{4}\Delta_u\psi_0(\mathbf{u},s) - \\ - \left\{(\mathbf{u}-\mathbf{a})\frac{\partial^2}{\partial s^2} - \frac{1}{2}\boldsymbol{\nabla}_u\frac{\partial}{\partial s}\right\}[\boldsymbol{\Phi}(\mathbf{u},s) + a^{(0)}\boldsymbol{\psi}(\mathbf{u},s)] \\ = i\frac{\partial^2}{\partial s^2}[\varphi_1(\mathbf{u},s) + a^{(0)}\varphi_2(\mathbf{u},s)]\end{aligned} \tag{51}$$

$$\begin{aligned}\Phi_0(\mathbf{u},s) = a^{(0)}\psi_0(\mathbf{u},s) + i\,\varphi_2(\mathbf{u},s) + \\ + (\mathbf{u}-\mathbf{a})\,\boldsymbol{\psi}(\mathbf{u},s) - \frac{1}{2}\int\limits_{-\infty}^{s} ds'\,\boldsymbol{\nabla}_u\boldsymbol{\psi}(\mathbf{u},s').\end{aligned} \tag{52}$$

From these two equations, we need only solve the first one. If we make the ansatz

$$\boldsymbol{\Phi}(\mathbf{u},s) + a^{(0)}\boldsymbol{\psi}(\mathbf{u},s) = -\frac{1}{2}\int\limits_{-\infty}^{s} ds'\,\boldsymbol{\nabla}_u\,\psi_0(\mathbf{u},s) + \boldsymbol{\Pi}(\mathbf{u},s) \tag{53}$$

where $\boldsymbol{\Pi}$ is a new O_3^+-covariant spectral function, we obtain the four dimensional divergence-equation:

$$\begin{aligned}\frac{\partial}{\partial s}\Big[(s-(a^{(0)})^2)\,\psi_0(\mathbf{u},s) - i(\varphi_1(\mathbf{u},s) + a^{(0)}\varphi_2(\mathbf{u},s)) - \\ - (\mathbf{u}-\mathbf{a})\boldsymbol{\Pi}(\mathbf{u},s)\Big] + \frac{1}{2}\boldsymbol{\nabla}_u[\boldsymbol{\Pi}(\mathbf{u},s) + (\mathbf{u}-\mathbf{a})\,\psi_0(\mathbf{u},s)] = 0\end{aligned} \tag{54}$$

with the classical solution

$$\begin{aligned}(s-(a^{(0)})^2)\,\psi_0(\mathbf{u},s) - (\mathbf{u}-\mathbf{a})\,\boldsymbol{\Pi}(\mathbf{u},s) - \\ - i[\varphi_1(\mathbf{u},s) + a^{(0)}\varphi_2(\mathbf{u},s)] = \boldsymbol{\nabla}_u\mathfrak{Z}(\mathbf{u},s) \\ \boldsymbol{\Pi}(\mathbf{u},s) + (\mathbf{u}-\mathbf{a})\,\psi_0(\mathbf{u},s) = -2\left[\frac{\partial}{\partial s}\mathfrak{Z}(\mathbf{u},s) + \mathbf{Z}(u,s)\right] \\ \boldsymbol{\nabla}_u\mathbf{Z}(\mathbf{u},s) = 0\,.\end{aligned} \tag{55}$$

$\mathfrak{Z}$ and $\mathbf{Z}$ are arbitrary tempered distributions with support Γ_{JL}, which transform under rotation like vectors (so-called "Hertz" vectors).

Solving these equations for $\boldsymbol{\Pi}$ and ψ_0 we obtain:

$$\boldsymbol{\Phi}(\mathbf{u}, s) = -a^{(0)}\,\boldsymbol{\psi}(\mathbf{u}, s) - (\mathbf{u}-\mathbf{a})\,\psi_0(\mathbf{u}, s) - \\ -\frac{1}{2}\int_{-\infty}^{s} ds'\,\boldsymbol{V}_u\psi_0(\mathbf{u}, s') - 2\left[\frac{\partial}{\partial s}\mathfrak{Z}(\mathbf{u}, s) + \mathbf{Z}(\mathbf{u}, s)\right] \tag{56}$$

$$\psi_0(\mathbf{u}, s) = [s + (\mathbf{u}-\mathbf{a})^2 - (a^{(0)})^2]^{-1}\Big\{i\,[\varphi_1(\mathbf{u}, s) + a^{(0)}\,\varphi_2(\mathbf{u}, s)] + \\ + \left[\boldsymbol{V}_u - 2(\mathbf{u}-\mathbf{a})\frac{\partial}{\partial s}\right]\mathfrak{Z}(\mathbf{u}, s) - 2(\mathbf{u}-\mathbf{a})\,\mathbf{Z}(\mathbf{u}, s)\Big\} + \delta(s + (\mathbf{u}-\mathbf{a})^2 - \\ -(a^0)^2)\,E(\mathbf{u})\,. \tag{57}$$

$E(\mathbf{u})$ is an arbitrary O_3^+-invariant, tempered distribution on the regular surface $s + (\mathbf{u}-\mathbf{a})^2 - (a^{(0)})^2 = 0$, which vanishes on this surface outside the set Γ_{JL}[1]

If we insert the equations (52) and (56), (57) into (44) and remove the s-integrations by means of equation (49), we find as the solution of the boundary value problem (35), (36) and (43) in the rest frame of p_{3-n}:

$$\widetilde{\mathrm{K}}_0^{\alpha\beta\gamma}(p_1, q, p_2)_n = \varepsilon(q^{(0)}) \int d^3\mathbf{u}\, ds\, \delta(q^{(0)^2} - (\mathbf{q}-\mathbf{u})^2 - s) \times \\ \times \{i\,\varphi_2^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n + (q^{(0)} + a_n^{(0)}) \times \\ \times \psi_0^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n + (\mathbf{q}-\mathbf{a}_n)\,\boldsymbol{\psi}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n\} \tag{58}$$

$$\widetilde{\mathbf{K}}^{\alpha\beta\gamma}(p_1, q, p_2)_n = \varepsilon(q^{(0)}) \int d^3\mathbf{u}\, ds\, \delta(q^{(0)^2} - (\mathbf{q}-\mathbf{u})^2 - s) \times \\ \times \Big\{(\mathbf{q} + \mathbf{a}_n - 2\mathbf{u})\,\psi_0^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n + (q^{(0)} - a^{(0)})\,\boldsymbol{\psi}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n - \\ -2\left[\frac{\partial}{\partial s}\mathfrak{Z}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n + \mathbf{Z}^{\alpha\beta\gamma}(p_1, \mathbf{u}, s, p_2)_n\right]\Big\} \tag{59}$$

where ψ_0 is given by equation (57) and $\boldsymbol{\psi}$ is arbitary like $\mathfrak{Z}$ and $\mathbf{Z}$.

Now we have the result: *For any O_3^+-covariant tempered distributions $\mathfrak{Z}$, $\mathbf{Z}$, $\boldsymbol{\psi}$ and E with support contained in Γ_{JL} a solution of our boundary value problem is given by the equations* (57)—(59).

b) The Case of a Nonsymmetric Spectrum

This case can be treated in exactly the same manner as case a). Therefore we can drop the details and restrict ourselves to some remarks.

To take into account the boundary conditions (35) and (36), we use the Dyson representation [15], [16] and drop again the superfluous variables:

$$\mathrm{K}_\mu(q)_n = \int d^4u\, ds\, \varepsilon(q^0 - u^0)\,\delta((q-u)^2 - s)\,\psi_\mu(u, s)_n \\ h(q)_n = \int d^4u\, ds\, \varepsilon(q^0 - u^0)\,\delta((q-u)^2 - s)\,\varphi(u, s)_n\,. \tag{60}$$

[1] Strictly it is the extension in the sense of Schwartz [12] of a distribution on this surface.

The support Γ_D of the spectral functions is given by equation (40).

If we insert (60) into (43) and use the relation:

$$q_\mu \int d^4u\, ds\, \varepsilon(q^{(0)} - u^{(0)})\, \delta((q-u)^2 - s)\, \psi_\mu(u, s)$$
$$= \int d^4u\, ds\, \varepsilon(q^{(0)} - u^{(0)})\, \delta((q-u)^2 - s) \times \tag{61}$$
$$\times \left\{u_\mu \frac{\partial}{\partial s} + \frac{1}{2} \partial_u^\mu\right\} \int_{-\infty}^{s} ds'\, \psi_\mu(u, s')$$
$$\partial_u^\mu \psi_\mu =: \sum_{\mu=0}^{3} \frac{\partial}{\partial u_\mu} \psi_\mu$$

then we come to a five-dimensional divergence equation, from which we obtain the final solution:

$$\mathrm{K}_\mu^{\alpha\beta\gamma}(p_1, q, p_2)_n = \int d^4u\, ds\, \varepsilon(q^{(0)} - u^{(0)})\, \delta((q-u)^2 - s) \times$$
$$\times \Big\{(q_\mu + (a_n)_\mu)\, \Phi^{\alpha\beta\gamma}(p_1, u, s, p_2)_n - 2u_\mu \Phi^{\alpha\beta\gamma}(p_1, u, s, p_2)_n - \tag{62}$$
$$- 2\left[\frac{\partial}{\partial s} \mathfrak{Z}_\mu^{\alpha\beta\gamma}(p_1, u, s, p_2)_n + \mathbf{Z}_\mu^{\alpha\beta\gamma}(p_1, u, s, p_2)_n\right]\Big\}$$

with Φ defined by:

$$\Phi(u, s)_n = (s - (u-a)^2)^{-1} \Big\{i\, \varphi(u, s)_n +$$
$$+ 2\left[(u - a_n)^\nu \frac{\partial}{\partial s} + \frac{1}{2} \partial_u^\nu\right] \mathfrak{Z}_\nu(u, s)_n + \tag{63}$$
$$+ 2(u - a_n)^\nu Z_\nu(u, s)_n\Big\} + \delta(s - (u - a_n)^2)\, E(u)_n$$
$$\partial_u^\nu Z_\nu(u, s) = 0\,. \tag{64}$$

$\mathfrak{Z}_\nu$ and Z_ν are arbitrary, covariant, tempered distributions with support Γ_D (Z_ν divergence-free). $E(u)$ is again an arbitrary invariant tempered distribution[2] on the surface $s - (u-a)^2 = 0$, which vanishes on this surface outside the set Γ_D. As in the case a) we have the result: *For any Lorentz-covariant tempered distributions $\mathfrak{Z}_\nu$, Z_ν and E with support contained in Γ_D a solution of our boundary value problem is given by the equations* (62)—(64).

V. Final Remarks

The local one-particle approximation $F_{\nu\mu}(k_1, k_2, k_3, k_4)^I$ we have constructed in section II contains besides the mass-shell singularities in $k_1 + k_2$ some of the mass-shell singularities in the variables $k_1 + k_3$ and $k_2 + k_3$. For instance the poles in these variables are given by the seventh, ninth and tenth term in equation (12). By the following arguments we

[2] See Footnote 1.

can conclude that these poles are completely removed from the difference $F^{II} =: F^T - F^I$. In [10] we had constructed a local one-particle approximation $W^I(x_1, x_2, x_3, x_4)$ for the four-point Wightman-function such that the corresponding difference does not contain any singularities in these variables (discussion after theorem IV). As already stated in section II, we derive our expression (12) from W^I by dropping all terms which are local in the variables $x_1 - x_2$ or $x_3 - x_4$. But all these terms do not contain a pole neither in the variable $k_1 + k_3$ nor in $k_2 + k_3$. Because the same statement is true for the inhomogeneous part of the differential equation (31) we can (at least for a variety of state vectors Φ_{k_4} and Ψ_{k_1}) use this fact to restrict the class of solutions of the boundary value problem (31), (35)—(36), which we have obtained in section IV. Because the inhomogeneous part of (31) does not contain these poles, we have the condition:

$$[(k^2 - m_i^2) \int d^3\mathbf{u}\, ds\, \delta(q^{(0)^2} - (\mathbf{q} - \mathbf{u})^2 - s)\, \varphi_j(p_1, \mathbf{u}, s, p_2)_n]_{k^2 = m_i^2} = 0 \quad (65)$$

for

$$i = 1, \ldots, N \quad \text{and} \quad k = (p_1 - p_2)$$

or

$$k = q - \frac{p_1 + p_2}{2} - \frac{1}{2}\varepsilon(n)\, p_n \quad [\varepsilon(1) = 1,\ \varepsilon(2) = -1,\ \varepsilon(3) = 0]\,.$$

Now it seems plausible to demand that the distributions $\boldsymbol{\psi}_n$; $\mathfrak{Z}$, $\mathbf{Z}$ do not behave worse than the given inhomogeneous part φ_j with respect to their singular structure on the one-particle mass-shells, i.e. equation (65) should be valid for them. Because of the denominator $s + (\mathbf{u} - \mathbf{a})^2 - a^{(0)^2}$ in equation (57) it can happen that the left hand side of (60) is unequal to zero for ψ_0. For instance $\varphi_j(p_1, \mathbf{u}, s, p_2)_1$ must for certain values of p_1, p_2 contain δ-functions in $\mathbf{u}$ and s, which produce the δ-singularity on the mass-shell in the corresponding matrix element $\langle 0|\,[\mathrm{J}_1(p_1 + k_1), \mathrm{A}(k_2)\,|\Phi_{p_2}\rangle$. These δ-functions then induce via the denominator in (57) poles at the one-particle mass-shells in the commutator $\widetilde{\mathrm{K}}_\mu(p_1, q, p_2)_1$.

Therefore in these cases the equation (65) for ψ_0 leads to proper restrictions for the distributions $\boldsymbol{\psi}$, $\mathfrak{Z}$, $\mathbf{Z}$ and $E(u)$. Unfortunately we were unable to construct the general solutions of these conditions. We could only prove the existence of solutions for the case of the poles at $(p_1 - p_2)^2 = m_i^2$ $(i = 1, 2, \ldots, N)$. This follows immediately from the fact, that these poles do not depend on the variable q.

The same conclusions are true in the case of the Dyson-representation.

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References

[1] FUBINI. S., and G. FURLAN: Physics **1**, 229 (1965).
[2] STREATER, R. F., and A. S. WIGHTMAN: PCT, spin and statistics and all that. New York: W. A. Benjamin 1964.
[3] JOST, R.: The general theory of quantized fields. Providence, Rhode Island: The American Mathematical Society 1965.
[4] SYMANZIK, K.: J. Math. Phys. **1**, 249 (1960).
[5] — Lectures in theoretical physics, Vol. III. Boulder 1960. New York: Interscience Publishers 1961.
[6] ZIMMERMANN, W.: Nuovo Cimento **13**, 503 (1959).
[7] — Nuovo Cimento **16**, 690 (1960).
[8] STREATER, R. F.: Nuovo Cimento **25**, 274 (1962).
[9] STORA, R.: Private Communication
[10] VÖLKEL, A. H.: Commun. Math. Phys. **2**, 176 (1966).
[11] THEIS, W. R.: Nuovo Cimento 45A, 124 (1966).
[12] SCHWARTZ, L.: Theorie des distributions I, II. Paris: Hermann 1957 et 1959.
[13] GELFAND, I., G. E. SCHILOW: Verallgemeinerte Funktionen I. Berlin: Deutscher Verlag der Wissenschaften 1960.
[14] JOST, R., and H. LEHMANN: Nuovo Cimento **5**, 1598 (1957).
[15] DYSON, F. J.: Phys. Rev. **110**, 1460 (1958).
[16] WIGHTMAN, A. S.: In: Dispersion relations and elementary particles. Paris: Hermann 1960.