Commun. math. Phys. 4, 123—136 (1967)

Can Current — Operators Determine a Complete Theory?

J. LANGERHOLC* and B. SCHROER**

II. Physikalisches Institut der Universität Hamburg

Abstract. For the free field currents $j_\mu(x)$ in the sector of charge zero we prove that one can reconstruct the bilocal fields $\psi(x)\,\bar{\psi}(y)$.

1. Formulation of the problem

Of all the space-time-dependent operators in local quantum field theory, current densities seem to be the most 'physical'. Certain matrix-elements of such operators (form factors) are directly measurable in electromagnetic and weak interactions. Attempts have been made to formulate relativistic dynamics directly in terms of current operators [1]. The main theoretical problem is the question whether the knowledge of the current operators[1] (i.e. all their vacuum expectation values) completely determines a theory. How, for example, do we compute processes involving charged or baryonic particles if the local operators to be used do not create a charged or baryonic one particle state from the vacuum.

As another application we mention that an answer to this sort of problem would be a prerequisite for a better understanding of quantum electrodynamics within the framework of general quantum field theory. It is well known that there exists no covariant gauge in which the KÄLLÉN-LEHMANN [2] spectral function of the spinors is positive definite; the indefinite metric of the $A_\mu^{(0)}(x)$ field creeps into the spinor — two point function in higher orders of perturbation theory. Therefore in order to obtain a physical (positive definite) Hilbert space one has to consider the vacuum-expectation values of the currents $j_\mu(x)$ (resp. the closely related electromagnetic field strength $F_{\mu\nu}(x)$) only. Hence one runs into the physical completeness problem mentioned before.

It seemed to us worthwhile before trying to understand this problem in the interacting case, to get a complete solution for the free field current. In this paper we show that the free field bilocal operators $\varphi^*(x)\,\varphi(y)$ can be obtained from the free field current operator $j_\mu(x)$ by a certain large

* Supported by the National Science Foundation.

** On leave from the University of Pittsburgh.

[1] Here the currents for all space time points (not only for equal times) are needed.

distance limiting procedure. This technique is in a way opposite to the short distance limiting technique defining the current in terms of the free field [3]. We discuss this in the case of the charged scalar and charged spinor fields (section 3). In the next section we explain the basic idea of the method in the simpler case of an even-odd-superselection rule as defined by the second power of a neutral scalar free field before going over to the slightly more complicated charge superselection rule where in addition to new algebraic complication caused by the presence of indices, one is forced to make use of a charge testing operator to separate two of the terms appearing in the limit.

2. The limiting process for the neutral scalar field

In order to obtain a preliminary connection between the bilocal algebras $R_{\mathfrak{B}_1,\mathfrak{B}_2}(A)$ (defined in Appendix 1) for the free Bose field A and the local algebras $R_{\mathfrak{B}}(j)$ of the 'current' $j = {:}A^2{:}$, we will use a limiting procedure in which certain points are moved to ∞ in a lightlike direction while remaining timelike separated from another fixed cluster of points. For notational convenience, we introduce certain definitions. If $\mathfrak{B}$ is a bounded region in Minkowski space, then $V_+(\mathfrak{B})$ will designate the union of the forward light cones $V_+(p)$ for all p whose backward cones contain $\mathfrak{B}$ (Fig. 1).

In this language, the first result, which we will show how to sharpen later, can be expressed as follows:

$$R_{\mathfrak{B}_1,\mathfrak{B}_2}(A) \subseteqq R_{\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{Z}}(j) \quad [\mathfrak{Z} \subseteqq V_+(\mathfrak{B}_1 \cup \mathfrak{B}_2)]\,. \tag{2.1}$$

The proofs will always consist in showing the inverse inclusion for the commutants; one proceeds by taking p a member of $R_{\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{Z}}(j)'$ i.e.

$$P\overline{j(f)} \subseteq \overline{j(f)}\,P$$

for $\operatorname{supp}(f) \subseteqq \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{Z}$ and shows that $P \in R_{\mathfrak{B}_1,\mathfrak{B}_2}(A)'$. The connection between A and j which is needed for such a manoeuvre comes from the structure of the commutator.

$$[j(x), j(y)] = 4i\Delta(x-y)\, A(x)\, A(y) \tag{2.2}$$

If this distribution is smeared with $f(x)\, g(y)$ with $f \in \mathscr{D}_{(\mathfrak{B})}$, $g \in \mathscr{D}_{(\mathfrak{Z})}$, then the left hand side becomes $[j(f), j(g)]$ which commutes with P according to (2.2). The result is then

$$\begin{aligned} 0 = \Delta(x-y)\,\Delta(x'-y')\,\{(\Psi, P A(x)\, A(y)\, A(x')\, A(y')\Phi) - \\ - (A(y')\, A(x')\, A(y)\, A(x)\,\Psi, P\Phi)\} \end{aligned} \tag{2.3}$$

for $\Psi, \Phi \in \mathfrak{D}_p$. Here one always has in mind smearing with $f(x)\, g(y) \times \times f'(x')\, g'(x')$ where $f \in \mathscr{D}_{(\mathfrak{B}_1)}$, $f' \in \mathscr{D}_{(\mathfrak{B}_2)}$, $g, g' \in \mathscr{D}_{(\mathfrak{Z})}$. In this formula, one takes Φ, Ψ to have the special form $B\Omega = A(f_1) \ldots A(f_n)\Omega$ where

$f_i \in \mathfrak{C}$ which is a space-time region spacelike to $\mathfrak{Z}$ as well as $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (fig. 1). The states obtained this way will be dense in the space $\mathfrak{H}$; and because of the continuity of distributions[2] the results obtained may be extended to all vectors in the polynomial domain of $\mathfrak{C}$ on which the field operators have the same closures as on the basic domain. This restriction on the form of the states will be useful in the proofs of appendix 2.

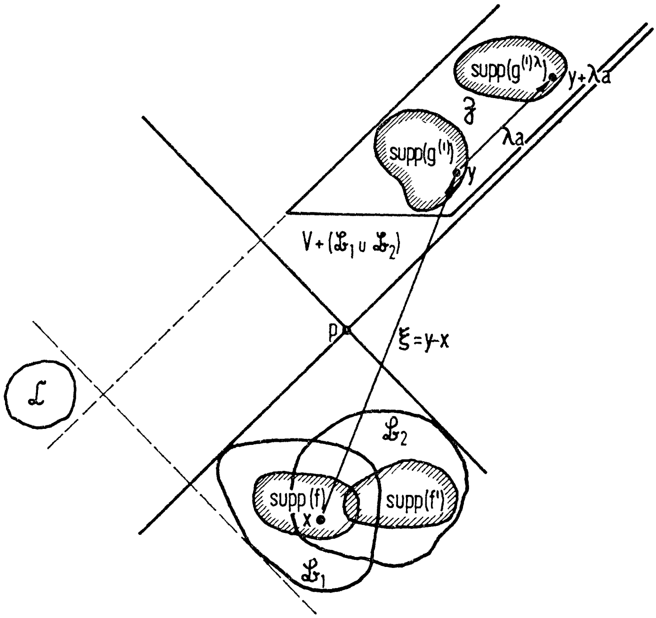

Fig. 1

The problem in (2.3) is the occurence of the Δ functions whose zeros become more closely spaced in lightlike directions and cannot be sidestepped with the support of g as it moves up the cylinder $\mathfrak{Z}$. In the scalar case it is possible to show that the Δ functions may simply be cancelled so that (2.3) is true without these factors as long as $y - x$ and $y' - x'$ are timelike. However, it will not be possible to factor them so simply in the case of the charged fields, so we shall use a somewhat more complicated method that will work in all cases.

Recalling the explicit form of the Δ function and the asymptotic behavior of the Bessel function ([5] page 526) we have with $z = \sqrt{\xi^2}$

$$\Delta(\xi) = \frac{J_1(mz)}{mz} = \sqrt{\frac{2}{\pi(mz)^3}}\left(\cos\left(mz - \frac{3\pi}{4}\right) + O(z^{-1})\right)$$

[2] Even though it is not explicit in the Wightman axioms, both terms in (2.3) are tempered distributions since $A(x_1) \ldots A(x_n)\Omega$ is a vector valued distribution; cf. [4].

and with $2\cos^2\theta = 1 + \cos 2\theta$, the expression

$$\pi (mz)^3 \Delta(\xi)^2 = 1 + \sin(2mz) + O(z^{-1}) .$$

Thus the multiplication of (2.3) by $\pi^2 m^6 (\sqrt{(x-y)^2}\sqrt{(x'-y')^2})^3 \Delta(x-y) \cdot \cdot \Delta(x'-y')$ gives symbolically

$$0 = (1 + \sin 2m\sqrt{(x-y)^2})(1 + \sin 2m\sqrt{(x'-y')^2}) \cdot \\ \cdot 2(\Psi, [P, A(x) A(y) A(x') A(y')]\Phi) + \cdots \tag{2.4}$$

where the terms not mentioned contain factors which are $O(\sqrt{(x^{(\prime)} - y^{(\prime)})^2}^{-1})$. It is fairly easy to see that if the distribution in (2.4) is smeared with $f \otimes g^\lambda \otimes f' \otimes g'^\lambda$ with $g^\lambda(y) = g(y - \lambda a)$ (a being a light-like vector parallel to the elements of $\mathfrak{Z}$), the unmentioned terms drop out for $\lambda \to +\infty$. This is proved in appendix 2 as a by-product of the more difficult considerations needed for the discussions following.

The term in (2.4) containing no sines is the one desired since after smearing it becomes

$$(\Psi, [P, A(f) A(g^\lambda) A(f') A(g'^\lambda)]\Phi) \to (\Psi[P, A(f) A(f')]\Phi) \langle A(g) A(g')\rangle_0 .$$

With $g = g'$, the last factor is the norm of $A(g)\Omega$ which is not zero thus implying that

$$(\Psi, [P, A(f) A(f')]\Phi) = 0$$

i.e. $P \in S_{\mathfrak{B}_1, \mathfrak{B}_2}(A) = R_{\mathfrak{B}_1, \mathfrak{B}_2}(A)'$ which in turn establishes (2.1).

The convergence above is proved by rewriting a typical term in the commutator

$$(\mathscr{P}'(A)\Omega, P\mathscr{P}(A) A(f) A(g^\lambda) A(f') A(g'^\lambda)\Omega)$$
$$= (\mathscr{P}'(A)\Omega, P\mathscr{P}(A) A(f) A(f') U_{(\lambda a)} A(g) A(g')\Omega) +$$
$$+ (\mathscr{P}''(A)\Omega, A(f) U_{(\lambda a)} A(g')\Omega) \langle [A(f'), A(g'^\lambda)]\rangle_0$$

where $\mathscr{P}(A)$ and $\mathscr{P}'(A)$ are polynomials in the field A taken from the region $\mathfrak{C}$ and $\mathscr{P}''(A) = \mathscr{P}(A)^* P\mathscr{P}'(A)$. As $\lambda \to \infty$, the second term approaches 0 because of the two-point function, and the first approaches

$$(\mathscr{P}'(A)\Omega, P\mathscr{P}(A) A(f) A(f') \operatorname{Proj}_{(\Omega)} A(g) A(g')\Omega)$$

which can be rewritten as

$$(\mathscr{P}'(A)\Omega, PA(f) A(f') \mathscr{P}(A)\Omega) \langle A(g) A(g')\rangle_0 .$$

The mirror-image process is performed on the other term, and after factorization of the two-point function, one is left with the limit written above. We will prove in appendix 2 that the oscillating factors give no contribution in the limit.

One would be able to go a step further and remove the cylinder $\mathfrak{Z}$ from the result for a certain class of regions if the duality theorem (in a weak form) were proved for the field j (in its cyclicity subspace) and arrive at:

$$R_{\mathfrak{B},\mathfrak{B}}(A) = R_{\mathfrak{B}}(j)$$

for any diamond $\mathfrak{B}$. A proof follows. For terminological convenience, one denotes by $c\mathfrak{B}$ the space-time region obtained from $\mathfrak{B}$ by taking the interior of $\mathfrak{B}'$, its causal complement.

With (2.1) and locality of j, one has

$$R_{\mathfrak{B},\mathfrak{B}}(A)' \supseteqq R_{\mathfrak{B}\cup\mathfrak{Z}}(j)' \supseteqq R_{c(\mathfrak{B}\cup\mathfrak{Z})}(j)$$

for all $\mathfrak{Z} \subseteqq V_+(\mathfrak{B})$. As a consequence

$$R_{\mathfrak{B},\mathfrak{B}}(A)' \supseteqq \bigvee R_{c(\mathfrak{B}\cup\mathfrak{Z})}(j) = R_{(\cup c(\mathfrak{B}\cup\mathfrak{Z}))}(j) \tag{2.5}$$

where the join and union are extended ober all $\mathfrak{Z} \subseteqq V_+(\mathfrak{B})$. One may be easily convinced by fig. 2 that $\cup\, c(\mathfrak{B}\cup\mathfrak{Z}) \supseteqq c\mathfrak{B}$ since if $x \in \operatorname{int}\mathfrak{B}'$, one

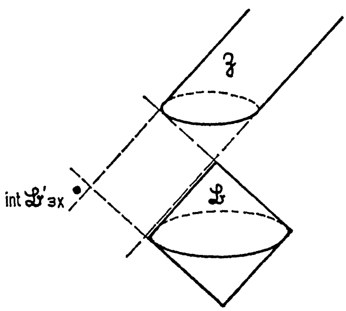

Fig. 2

may always find a cylinder $\mathfrak{Z} \subseteqq V_+(\mathfrak{B})$ such that $x \in \mathfrak{Z}'$. This done, one has

$$x \in (\operatorname{int}\mathfrak{B}') \cap (\operatorname{int}\mathfrak{Z}') = \operatorname{int}(\mathfrak{B}' \cap \mathfrak{Z}') = \operatorname{int}(\mathfrak{B} \cap \mathfrak{Z})' = c(\mathfrak{B} \cup \mathfrak{Z})$$

which completes the proof of the assertion. This assertion, applied to the last term in (2.5) gives the further result

$$R_{\mathfrak{B},\mathfrak{B}}(A)' \supseteqq R_{c\mathfrak{B}}(j) \tag{2.6}$$

to which a weak version of the duality principle may be applied. According to a result of Araki [6] for the free field, one would expect

$$\overline{R}_{\mathfrak{C}}(j)' \subseteqq \overline{R}_{c\mathfrak{C}}(j)$$

for an open diamond $\mathfrak{C}$; here $\overline{R}_{\mathfrak{C}}$ designates the intersection of all algebras $R_{\mathfrak{G}}$ with $\mathfrak{G}$ an open set containing $\operatorname{cl}\mathfrak{C}$.

If $\mathrm{cl}\mathfrak{B} \subseteqq \mathfrak{C}$ and $\mathrm{cl}\mathfrak{C} \subseteqq \mathfrak{C}$ then the definition immediately implies

$$R_{c\mathfrak{B}} \supseteqq \overline{R}_{c\mathfrak{C}} \qquad \overline{R}_{\mathfrak{C}} \subseteqq R_{\mathfrak{C}}\,.$$

Taking commutants in the second of these statements and connecting the results with the duality assumption gives

$$R_{c\mathfrak{B}} \supseteqq \overline{R}_{c\mathfrak{C}} \supseteqq \overline{R}_{\mathfrak{C}} \supseteqq R_{\mathfrak{C}}$$

and with (2.6)

$$R_{\mathfrak{B},\mathfrak{B}}(A)' \supseteqq R_{c\mathfrak{B}}(j) \supseteqq R_{\mathfrak{C}}(j)'$$

if $\mathrm{cl}\mathfrak{B} \subseteqq \mathfrak{C}$. Passage again to commutants gives

$$R_{\mathfrak{C},\mathfrak{C}}(A) = \bigvee_{\mathrm{cl}\mathfrak{B}\subseteqq\mathfrak{C}} R_{\mathfrak{B},\mathfrak{B}}(A) \subseteqq R_{\mathfrak{C}}(j)$$

where the first equality is proved in appendix 1. Since j is obtained from A as a limit of bilocal expressions, the inverse inclusion is trivial and consequently

$$R_{\mathfrak{C},\mathfrak{C}}(A) = R_{\mathfrak{C}}(j)$$

for any diamond $\mathfrak{C}$.

3. Charged fields

In this section we discuss the charged scalar field

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \left(e^{ipx} a^*(p) + e^{-ipx} b(p)\right) d\Omega^{(+)}(p)$$

with the local current

$$j_\mu(x) = i{:}\varphi^*(x)\, \overleftrightarrow{\partial}_\mu\, \varphi(x){:}$$

and the spinor field $\psi_\alpha(x)$ with current

$$j_\mu(x) = i{:}\overline{\psi}(x)\, \gamma_\mu \psi(x){:}$$

and show that similar connections exist between the bilocal rings of the field and the local rings of the respective current density. We thus need a method of recovering the bilocal quantities $\varphi^*(x)\,\varphi(y)$ and $\overline{\psi}_\alpha(x)\,\psi_\beta(y)$ from their currents in a bilocal fashion. For the spinor case, the current commutator has the expression

$$-i(\gamma_\mu S(x-y)\,\gamma_\nu)^{\alpha\beta} : \overline{\psi}_\alpha(x)\,\psi_\beta(y) : + i(\gamma_\nu S(x-y)\,\gamma_\mu)^{\alpha\beta} : \overline{\psi}_\alpha(y)\,\psi_\beta(x) : \quad (3.1)$$

in which $S(\xi) = (\partial^\lambda \gamma_\lambda - m)\,\Delta(\xi)$, and for the scalar case, the form

$$\begin{aligned}
&({:}\varphi^*(x)\,\varphi(y){:} + {:}\varphi^*(y)\,\varphi(x){:})\, i\partial_\mu\partial_\nu\Delta(x-y) + \\
&\quad + ({:}\varphi^*_\mu(x)\,\varphi_\nu(y){:} + {:}\varphi^*_\nu(y)\,\varphi_\mu(x){:})\, i\Delta(x-y) - \\
&\quad - ({:}\varphi^*_\mu(x)\,\varphi(y){:} + {:}\varphi^*(y)\,\varphi_\mu(x){:})\, i\,\partial_\nu\Delta(x-y) - \\
&\quad - ({:}\varphi^*(x)\,\varphi_\nu(y){:} + {:}\varphi^*_\nu(y)\,\varphi(x){:})\, i\partial_\mu\Delta(x-y)\,,
\end{aligned} \qquad (3.2)$$

in which the derivatives $\varphi_\nu = \partial_\nu \varphi$ of the field appear in such a way as to make the various rates of convergence of the terms difficult to compare. Bearing in mind the Riemann-Lebesgue type of trick, we multiply by $\Delta(x-y)$ and after smearing with a function $F(x, y)$, perform as many partial integrations as are needed to bare the field operators. The result is that the smeared commutator

$$\int F(x, y)\, i\Delta(x-y)\, [j_0(x), j_0(y)]\, d^4x\, d^4y$$

can be written as

$$-\int (:\varphi^*(x)\, \varphi(y): + :\varphi^*(y)\, \varphi(x):) \times \\ \times (F(x, y)\, \Delta(x-y)\, \partial_0^2 \Delta(x-y) + G(x, y))\, d^4x\, d^4y \tag{3.3}$$

where

$$G(x,y) = \Delta(x-y)\, \partial_0 \Delta(x-y) \left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0}\right) F(x,y) + \Delta(x-y)^2 \frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0} F(x,y).$$

To argue away the function $G(x, y)$ as well as to be able to decide asymptotic behaviors of terms appearing later, we compute the relevant derivatives of the Δ function and their asymptotic behaviors. Since $\Delta(\xi)$ is a function only of $z = \sqrt{\xi^2}$, ∂_μ may be replaced by $\xi_\mu \frac{1}{z} \frac{d}{dz}$ and $\partial_\mu \partial_\nu$ by

$$g_{\mu\nu} \frac{1}{z} \frac{d}{dz} + \xi_\mu \xi_\nu \left[\frac{1}{z} \frac{d}{dz}\right]^2.$$

With the formula

$$\Delta(\xi) = \frac{J_1(mz)}{mz} \qquad (\xi \in V_+)$$

and the fact that

$$\frac{1}{z} \frac{d}{dz} \frac{J_k(z)}{z^k} = -\frac{J_{k+1}(z)}{z^{k+1}},$$

one obtains

$$\partial_\mu \Delta(\xi) = -\xi_\mu \frac{J_2(mz)}{z^2}$$

$$\partial_\mu \partial_\nu \Delta(\xi) = -g_{\mu\nu} \frac{J_2(mz)}{z^2} + m\, \xi_\mu \xi_\nu \frac{J_3(mz)}{z^3}$$

so that with the asymptotic formulas on p. 526 of [5] one finds

$$\Delta(x-y)\, \partial_0^2 \Delta(x-y) = -\frac{2}{\pi m} \frac{\xi_0^2}{z^5} \left\{\cos\left(mz - \frac{3\pi}{4}\right) \cos\left(mz - \frac{7\pi}{4}\right) + O(z^{-1})\right\}$$

provided that $\xi_0 = 0(z^2)$, and for the other terms,

$$\Delta(x-y)\, \partial_0 \Delta(x-y) = \frac{2}{\pi m^2} \frac{\xi_0}{z^4} \left\{\cos\left(mz - \frac{3\pi}{4}\right) \cos\left(mz - \frac{5\pi}{4}\right) + O(z^{-1})\right\}$$

$$\Delta^2(x-y) = \frac{2}{\pi m^3} \frac{1}{z^3} \left\{\cos^2\left(mz - \frac{3\pi}{4}\right) + O(z^{-1})\right\}.$$

It is thus clearly desirable to choose

$$F(x, y) = f(x)\, g(y)\, \frac{\pi m}{2} \frac{z^5}{\xi_0^2}$$

so that what will turn out to be the leading term in (3.3) is

$$\varphi^*(f)\,\varphi(g) + \varphi^*(g)\,\varphi(f)\,.$$

The oscillating factor will go to 0 in norm when applied to the vacuum, and it will be sufficient to prove that $G(x, y - \lambda a)$ approaches 0 as $\lambda \to \infty$. We note that if $\xi = x - y + \lambda a$, then both ξ_0 and ξ^2 are asymptotically linear in λ since $a^2 = 0$ so that both ξ_0/z^4 and $1/z^3$ approach 0, and the rest of the proof consists simply in verifying that z^5/z_0^2 and its first two time derivatives do not increase as rapidly as λ, λ, $\lambda^{3/2}$ respectively. Having proved that the smearing functions go to 0 in L_1 norm, we rely on the proof in appendix 2 that this insures the vanishing of the smeared distributions in spite of whatever oscillations the smearing functions may manifest.

A feature which the last expression displays for the scalar case is immediately visible in the spinor case; the bilocal quantities always occur in hermitian combinations that cannot be separated by algebraic manipulations. For this separation, we use the charge testing operator

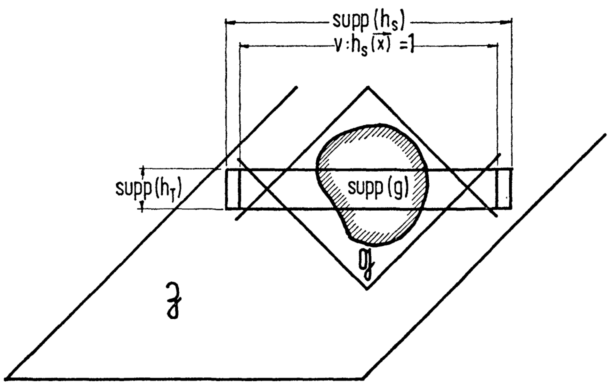

Fig. 3

$Q_h = j_0(h) \approx \int\limits_V j_0(\mathrm{x}, x_0)\, d^3\mathrm{x}$ which approximately tests the amount of charge in the volume V of space at time x_0. In view of the fact that the distribution $j_0(x)$ cannot be restricted to a spacelike surface, we will use the four dimensional test function[3] (Fig. 3)

$$h(x) = h_S(\mathrm{x})\, h_T(x_0)$$

[3] This construction is similar to one used by KASTLER, ROBINSON [7] and SWIECA in a proof of the Goldstone theorem, but we do not envisage passage to a limit in which $\mathfrak{S}$ swells to cover all space-time; the local affiliation of Q_h is essential to our purpose since it should asymptotically commute with $\varphi(x)$. We use here only the infinitessimal operator instead of its exponential to avoid the difficult question of the convergence of the Taylor expansion of the latter.

with $h_S \in \mathscr{D}_{(\mathbb{R}^3)}$ and $h_T \in \mathscr{D}_{(\mathbb{R})}$ chosen so that

$$\int_{-\infty}^{+\infty} h_T(t) = 1 \quad \operatorname{supp}(h) \subseteq \mathfrak{Z}$$

$$h(x) = h_T(x_0) \quad \text{unless } x \text{ is spacelike to } \mathfrak{S}.$$

The last requirement can be insured by setting $h_S(\mathrm{x}) = 1$ when x belongs to a certain region V of space and keeping the support of h_T sufficiently small, as the figure makes clear.

We confine the supports of the functions g, g' to the diamond $\mathfrak{S}$ so that

$$[Q_h, \varphi(g)] = [Q, \varphi(g)] = \varphi(g)$$
$$[Q_h, \varphi(g)] = -\varphi^*(g)\,.$$

Thus commuting Q_h with $\varphi^*(x)\,\varphi(y) + \varphi^*(y)\,\varphi(x)$ gives

$$\varphi^*(x)\,\varphi(y) - \varphi^*(y)\,\varphi(x) + [Q_h, \varphi^*(x)]\,\varphi(y) + \varphi^*(y)\,[Q_h, \varphi(x)]$$

and averaging with the original expression produces

$$\varphi^*(x)\,\varphi(y)$$

plus some terms containing commutators $[Q_h, \varphi^*(x)]$. But we will shift the region $\mathfrak{S}$ up the tube simultaneously with the supports of the g functions and so these commutators will approach 0. Since one has chosen $\operatorname{supp}(h) \subseteq \mathfrak{Z}$, the extra factors of Q_h will not destroy commutativity of the operator P with the expression obtained above.

With these preparations one may now apply the techniques of § 2 to the charged fields as well. With the scalar fields, one obtains

$$0 = (\Psi, [P, \varphi^*(f)\,\varphi(f')]\Phi)\,\langle\varphi(g)\,\varphi^*(g')\rangle_0$$

which leads immediately to the analog of (2.1) upon cancellation of the factor on the extreme right which can be again made nonzero by choosing $g = g'$. With the spinor fields, the result of the limiting process is the commutativity of P with

$$(\gamma_\mu(\gamma a)\,\gamma_\nu)^{\alpha\beta}\,\bar{\psi}_\alpha(f)\,\langle\psi_\beta(g)\,\bar{\psi}_{\beta'}(g')\rangle_0\,\psi_{\alpha'}(f')\,(\gamma_{\mu'}(\gamma\cdot a)\,\gamma_{\nu'})^{\beta'\alpha'} = \bar{\psi}_\alpha(f)\,\psi_{\alpha'}(f')$$
$$\int \left(\gamma_\mu(\gamma\cdot a)\,\gamma_\nu((\gamma\partial) - m)\,\gamma_{\mu'}(\gamma\cdot a)\,\gamma_{\nu'}\right)^{\alpha\alpha'}\, i\Delta^{(+)}(y - y')\,g(y)\,g'(y')\,.$$

By various tricks such as setting $g' = \partial_\lambda g$ etc. one can separate the γ-matrices in the middle term to get

$$0 = (\Psi, [P, \bar{\psi}_\alpha(f)\,\psi_{\alpha'}(f')]\Phi)\,C^{\alpha\alpha'} \tag{3.4}$$

where C is any matrix representable as $\gamma_\mu(\gamma\cdot a)\,\gamma_\nu\Gamma\gamma_{\mu'}(\gamma a)\,\gamma_{\nu'}$ with $\Gamma = \gamma$ or $\Gamma = I$. It is further possible to show that in spite of the singular behaviour of $(\gamma\cdot a)$ — $(\gamma\cdot a)^2 = a^2 = 0$ — linear combinations of such matrices are capable of producing all products of γ-matrices with fewer than 5 factors. With linear combinations of these, however, one can construct any 4×4 matrix — in particular $C^{\alpha\alpha'} = \delta^\alpha_\beta\,\delta^{\alpha'}_{\beta'}$ which when inserted into (3.4) gives

$$0 = (\Psi, [P, \bar{\psi}_\alpha(f)\, \psi_{\alpha'}(f')]\Phi)$$

to complete the proof.

4. Concluding remarks

The ring-theoretical construction of the bilocal algebras makes use of free field properties and cannot be carried over to the interacting case. From the point of view of applications, the most useful thing seems to be the construction of particle states over the current algebras, i.e. the construction of expectation values

$$\langle f_1^{\text{in}} \ldots f_n^{\text{in}} |\, j_{\mu 1}(x_1) \ldots j_{\mu m}(x_m)\, | f_1^{\text{in}} \ldots f_n^{\text{in}} \rangle \,.$$

Using considerations recently proposed by ARAKI and HAAG [8] one can show that all scattering information can be obtained from these expectation values (scattering probabilities for finding a certain number of outgoing charges in a given incoming state). In analogy with the free field treatment in the previous sections, one would expect that one can approximate states $|f_1^{\text{in}} \ldots f_n^{\text{in}} g_{1\lambda}^{\text{in}} \ldots g_{n\lambda}^{\text{in}}\rangle$ where the wave packets g_λ are shifted timelike to the f by a large λ, by using products of commutators

$$[j_\mu(x), j_\nu(y)] + [[j_\mu(x), j_\nu(y)], Q_h]$$

and letting x_0 and y_0 approach ∞ such that $\lambda^2 = (x - y)^2 \to \infty$. Unfortunately, we have not yet found a rigorous argument for the interacting case.

Acknowledgement. The main part of this work was done at the Centre de Physique Théorique de l'Ecole Polytechnique. We thank Prof. L. MICHEL for hospitality and Dr. J. LASCOUX for many discussions.

Note added in proof. It may perhaps be interesting to point out that the free Hamiltonian density and all the other components $t_{\mu\nu}$ of the free matter tensor can be expressed explicitly in terms of a local function of the free current $j_\mu(x)$. This is done by a short distance limiting procedure which in [9] was applied to the $j(x) = {:}A^2(x){:}$ case. This method works with appropriate modifications also for the current of charged bosons or fermions.

Appendix 1. Definition of bilocal algebras and some mathematical results

In analogy with the local algebra $R_{\mathfrak{B}}(A)$ associated to an open region of space time and a neutral scalar field A, one defines bilocal algebras $R_{\mathfrak{B},\mathfrak{C}}(A) = S_{\mathfrak{B},\mathfrak{C}}(A)'$ where $S_{\mathfrak{B},\mathfrak{C}}(A)$ is the set of all bounded operators P such that

$$(\Psi, PA(f)\, A(g)\Phi) = (A(g)\, A(f)\Psi, P\Phi)$$

for all $f \in \mathscr{D}_{(\mathfrak{B})}$, $g \in \mathscr{D}_{(\mathfrak{C})}$ and all $\Phi, \Psi \in \mathfrak{D}$ the basic domain of the field A. For the charged scalar field $\varphi(x)$ and spinor field $\psi(x)$ one uses the operators $\varphi^*(x)\, \varphi(y)$ and $\bar{\psi}^\alpha(x)\, \psi^\beta(y)$ which are invariant under the gauge transformations $\varphi \to e^{i\alpha}\varphi$, $\psi \to e^{i\alpha}\psi$. Symmetric operators may be formed as follows:

$$C(f\otimes g)=\frac{1}{2}(A(f)\,A(g)+A(g)\,A(f)),\quad D(f\otimes g)=\frac{1}{2}i(A(f)\,A(g)-A(g)\,A(f))\,.$$

Since $A(f)\,A(g)=C(f\otimes g)-iD(f\otimes g)$ and the functions $f\otimes g$ span $\mathscr{D}_{(\mathfrak{B}\times\mathfrak{C})}$, a result obtained in a previous paper [3] insures that $S_{\mathfrak{B},\mathfrak{C}}(A)$ is the set of all P such that

$$P\overline{C(h)}\subseteqq C(h)^*P\quad P\overline{D(h)}\subseteqq D(h)^*P$$

for all $h\in\mathscr{D}_{(\mathfrak{B}\times\mathfrak{C})}$; and, if $C(h)$ are essentially self adjoint on $\mathfrak{D}$, the condition reduces to

$$[P,E_\lambda^{\overline{C(h)}}]=0=[P,E_\lambda^{\overline{D(h)}}]$$

for $h\in\mathscr{D}_{(\mathfrak{B}\times\mathfrak{C})}$ and $\lambda\in R$. This last characterization implies that $S_{\mathfrak{B},\mathfrak{C}}(A)$ is an algebra and that $S_{\mathfrak{B},\mathfrak{C}}(A)=R_{\mathfrak{B},\mathfrak{C}}(A)'$; this fact enables one to prove that if $\mathfrak{B}=\bigcup_{\iota\in I}\mathfrak{B}_\iota$, $\mathfrak{C}=\bigcup_{\varkappa\in K}\mathfrak{C}_\varkappa$ then $R_{\mathfrak{B},\mathfrak{C}}(A)=\bigvee_{\iota\in I}\bigvee_{\varkappa\in K}R_{\mathfrak{B}_\iota,\mathfrak{C}_\varkappa}$. A result which is used in the text is that if $\mathfrak{B}=\bigcup_{\iota\in I}\mathfrak{B}_\iota$ and the family $\{\mathfrak{B}_\iota\}_{\iota\in I}$ forms a chain (totally ordered set), then

$$R_{\mathfrak{B},\mathfrak{B}}(A)=\bigvee_{\iota\in I}R_{\mathfrak{B},\mathfrak{B}}(A)\,.$$

This is most conveniently proved in its dual form

$$S_{\mathfrak{B},\mathfrak{B}}(A)=R_{\mathfrak{B},\mathfrak{B}}(A)'=\bigwedge_{\iota\in I}R_{\mathfrak{B}_\iota\mathfrak{B}_\iota,}(A)'=\bigcap_{\iota\in I}S_{\mathfrak{B}_\iota,\mathfrak{B}_\iota}(A)\,.$$

Clearly the left side is included in each term on the right and therefore in their intersection. Conversely, let $P\in S_{\mathfrak{B}_\iota,\mathfrak{B}_\iota}(A)$ for each $1\in I$ and let $f\in\mathscr{D}_{(\mathfrak{B})}$. Since $\operatorname{supp}(f)\subseteqq\bigcup_{\iota\in I}\mathfrak{B}_\iota$ there is a finite subcollection of $\{\mathfrak{B}_\iota\}_{\iota\in I}$ covering the compact set $\operatorname{supp}(f)$. But because this collection is totally ordered, there will be a largest set $\mathfrak{B}_{\iota_*}$, and consequently $\operatorname{supp}(f)\subseteqq\mathfrak{B}_{\iota_*}$, i.e. $f\in\mathscr{D}_{(\mathfrak{B}_{\iota_*})}$. Thus

$$C(f\otimes f)\smile P\smile D(f\otimes f);$$

and since this result is valid for any $f\in\mathscr{D}_{(\mathfrak{B})}$, one has $P\in S_{\mathfrak{B},\mathfrak{B}}(A)$.

To complete this discussion, it is only necessary to point out that for the free field A, the operators $C(f\otimes g)$, $D(f\otimes g)$ are given in the Fock representation by bounded operators connecting the n-particle space to the $n-2$, n, $n+2$ particle spaces and that the norms of these bounded operators do not grow faster than n [3]. This situation has as a consequence that the operators C and D are essentially self adjoint on any domain $\mathfrak{D}\subseteqq\bigoplus_{n=0}^{\infty}\mathfrak{H}^{(n)}$ such that $\mathfrak{D}\cap\mathfrak{H}^{(n)}$ is dense in $\mathfrak{H}^{(n)}$, the n-particle space. This insures that the $S_{\mathfrak{B},\mathfrak{C}}(A)$ are algebras and justifies the above computations.

For the same reason, the operators $j(f)=\,:A^2:(f)$ are essentially self adjoint on their domains and $S_{\mathfrak{B}}(j)=R_{\mathfrak{B}}(j)'$ is an algebra.

Appendix 2. Computation of some light-like limits

The terms whose vanishing we must demonstrate can all be put into the form

$$\begin{aligned}\int (\mathscr{P}'(\varphi)\Omega, [P, :\varphi^*(x)\,\varphi(y)::\varphi^*(y')\,\varphi(x'):]\Omega) \times \\ \times f(x)\,c(x-y)\,g^\lambda(y)\,g'^\lambda(y')\,c'(y'-x')\,f'(x')\,d^4(xx'yy')\end{aligned} \tag{a.1}$$

where $c^{(\prime)}(\xi)$ is either $\sin 2mz$, $O(z^{-1})$, or constant. Here we are permitted to use Wick products for simplicity because the bilinear terms appear in a commutator and the subtraction of scalars does not change the expression. The product of Wick products may be rewritten as a sum of totally Wick ordered products and the scalar term dropped.

We consider each term of the commutator by itself. The polynomial $\mathscr{P}(\varphi)$ may be commuted past the field operators and, along with P, taken to the other side of the scalar product. Designating

$$(P\mathscr{P}(\varphi))^*\,\mathscr{P}'(\varphi)\Omega$$

by χ one may rewrite (a.1) as

$$\begin{aligned}\big(\chi, (:\varphi^*(x)\,\varphi(y)\,\varphi^*(y')\,\varphi(x'): + \langle\varphi^*(x)\,\varphi(x')\rangle_0 :\varphi(y)\,\varphi^*(y'): + \\ + \langle\varphi(y)\,\varphi^*(y')\rangle_0 :\varphi^*(x)\,\varphi(x'):)\Omega\big)\end{aligned}$$

smeared with the same functions that occur in (a.1). By Schwartz's inequality, it will suffice to show that the vectors on the right approach 0 in norm after smearing.

Neglecting symmetrization, the norm will be the integral of the square of the Fourier transform of the test functions over the appropriate mass shells. The first term splits naturally into a product of two factors each of which may be written

$$\begin{aligned}\int e^{i(px+qy)} f(x)\,c(x-y)\,g^\lambda(y)\,d^4x\,d^4y \\ = e^{i\lambda qa}\int e^{i(px+qy)} f(x)\,c(x-y+\lambda a)\,g(y)\,d^4x\,d^4y\,.\end{aligned} \tag{a.2}$$

We will treat the more complicated case of $c(\xi) = \sin 2mz$ and afterwards indicate the simple modifications of the proof which enable the treatment of terms which are $O(z^{-1})$. We want to adapt the usual proof of the Riemann-Lebesgue lemma to the present situation, in particular, integrate by parts to get a negative power of λ so we write

$$\sin 2mz = \frac{z}{2m\xi_0}\,\frac{d}{d\xi_0}\cos 2mz\,.$$

To avoid contributions from the boundary which do not fall off sufficiently, we must cut off this function at the zeros of the cosine. Thus for each $\boldsymbol{\xi}$ and λ an interval $I(\boldsymbol{\xi}, \lambda)$ is chosen for which $\cos 2m\sqrt{(\xi+\lambda a)^2} = 0$ when ξ_0 is one of the endpoints, such that $\operatorname{supp}(f)|_0 - \operatorname{supp}(g)|_0 + \lambda a \subseteq \bigcup_{\boldsymbol{\xi}} I(\boldsymbol{\xi}, \lambda)$ and finally such the lengths of $I(\boldsymbol{\xi}, \lambda)$ are bounded above by a number independent of $\boldsymbol{\xi}$ and λ. We collect the additional factor of $\frac{z}{2m\xi_0}$ into $f \otimes g$ by defining

$$\varphi_\lambda(x, y) = \frac{\sqrt{(x-y+\lambda a)^2}}{2m(x_0-y_0+\lambda)}\,f(x)\,g(y)$$

and have by Fourier's theorem after the promised partial integration,

$$f(x)\sin 2m\sqrt{(x-y+\lambda a)^2}\,g(y)$$

$$= \varphi_\lambda(x,y)\int\limits_{-\infty}^{\infty} dr_0\, e^{ir_0(x-y)_0}\int\limits_{I(\xi,\lambda)} e^{-ir_0\xi_0}\frac{d}{d\xi_0}\cos 2m\sqrt{(\xi+\lambda a)^2}\,d\xi_0 \quad \text{(a.3)}$$

$$= i\,\varphi_\lambda(x,y)\int\limits_{-\infty}^{\infty} dr_0\, e^{ir_0(x-y)_0}\, r_0\int\limits_{I(\xi,\lambda)} e^{-ir_0\xi_0}\cos 2m\sqrt{(\xi+\lambda a)^2}\,d\xi_0\,.$$

In this equation $\boldsymbol{\xi} = \mathbf{x}-\mathbf{y}$ and ξ_0 is simply an integration variable.

To show that the function in (a.2) converges to 0 in norm in $L_2(d\Omega^{(+)}(p)\,d\Omega^{(+)}(q))$ it is sufficient to show that after multiplication by $|p_0+q_0|^n$ it can be majorized on the mass shells by a number M_λ which is independent of p and q and approaches 0 as $\lambda\to\infty$. Multiplying (a.2) by $|p_0+q_0|^n$ and substituting (a.3) gives an expression bounded above by ($\mathbf{r}=0$ by definition)

$$\int\limits_{-\infty}^{\infty} dr_0|r_0|\,|p_0+q_0|^n\,\Big|\int e^{i[(p+r)x+(q-r)y]}\,\varphi_\lambda(x,y)\,\times$$

$$\times\int\limits_{I(\xi,\lambda)} e^{-ir_0\xi_0}\cos 2m\sqrt{(\xi+\lambda a)^2}\,d\xi_0\Big|\,.$$

For the (future) convergence of the r_0 integral, we multiply and divide by $1+|r_0|^2$. Since $p_0\geqq 0\leqq q_0$, one has $|r_0|\leqq \max\{|p_0+r_0|, |q_0-r_0|\}$ and since $p_0+q_0=(p_0+r_0)+(q_0-r_0)$, one only has to worry about terms of the form

$$\int\frac{dr_0}{1+|r_0|^2}\,|p_0+r_0|^\alpha\,|q_0-r_0|^\beta\int e^{i[(p+r)x+(q-r)y]}\,\varphi_\lambda(x,y)\,\times$$

$$\times\int\limits_{I(\xi,\lambda)} e^{-ir_0\xi_0}\cos 2m\sqrt{(\xi+\lambda a)^2}\,d\xi_0\,.$$

The powers $(p_0+r_0)^\alpha$ and $(q_0-r_0)^\beta$ multiplying the Fourier transform can be replaced by derivatives $\left(\frac{\partial}{\partial x_0}\right)^\alpha$ and $\left(\frac{\partial}{\partial y^0}\right)^\beta$ acting on φ_λ since the last integral is independent of x_0 and y_0. The absolute value bars can be brought inside eliminating the exponentials and giving

$$\left[\int\frac{dr_0}{1+|r_0|^2}\right]\int d^4x\,d^4y\left|\left(\frac{\partial}{\partial x_0}\right)^\alpha\left(\frac{\partial}{\partial y_0}\right)^\beta\varphi_\lambda(x,y)\right|\int\limits_{I(\xi,\lambda)} d\xi_0\,.$$

Using the fact that the last integral is the length of $I(\boldsymbol{\xi},\lambda)$ which is bounded above by a constant independent of $\boldsymbol{\xi}$ and λ, we are led to investigate the behaviour of

$$\left(\frac{\partial}{\partial\xi_0}\right)^\alpha\frac{\sqrt{(\xi+\lambda a)^2}}{\xi_0+\lambda}\,.$$

Because it multiplies a function in $\mathscr{D}_{(x,y)}$ it will be enough to show that it approaches 0 for any value of ξ. One may show by induction on α that this quantity is expressible as

$$\sum_{k=0}^{\alpha}((\xi+\lambda a)^2)^{1/2-k}\,R_\alpha^{(k)}(\xi_0+\lambda)$$

where $R_\alpha^{(k)}$ is a series of (possibly negative) powers of $(\xi_0 + \lambda)$ with exponents not exceeding $k - 1$. It is clearly true for $\alpha = 0$ and the passage from α to $\alpha + 1$ can be checked by differentiating the series above and regrouping terms. Finally since $(\xi + \lambda a)^2 = \xi^2 + (2\xi a)\lambda$, each term is asymptotically proportional to $\lambda^{-1/2-K}$ with $K \geqq 0$.

In the case where $C(z)$ is $O(z^{-1})$ it is not necessary to perform the first integration by parts, and $\varphi_\lambda(x, y)$ will simply be $f(x)\,g(y)$. The last integral in (a.3) can be arranged to be

$$\int_{I(\xi,\lambda)} d\xi_0 \sup_\xi |c(\xi)| \leqq M \sup_\xi |c(\xi + \lambda a)|$$

where, since one need not be so meticulous in the choice of the interval, the supremum may be taken in a suitable cell containing supp(f) — — supp(g).

Next we show how the terms involving one contraction may be reduced to quantities for which the preceeding method is used.

As a function of p and p' the (unsymmetrized) wave function of the vector:

$$\int :\varphi(x)\,\varphi^*(y):\,\Omega g(y)\,c(y - x)\,f(x)\,\langle\varphi^*(x)\,\varphi(x')\rangle \times \\ \times f'(x')\,c'(y' - x')\,g'(y')\,d^4(xy\,x'y')$$

can be written

$$\int e^{i(py + p'y')}\,g(y)\,c(x - y)\,f(x) \times \\ \times \int e^{iq(x-x')}\,d\Omega^{(+)}(q)\,f'(x')\,c'(x' - y')\,g'(y')\,d^4(xy\,x'y') \\ = \int d\Omega^{(+)}(q)\,\Phi(p, q)\,\Phi'(p' - q)$$

with

$$\Phi(p, q) = \int e^{i(py + qx)}\,g(y)\,c(y - x)\,f(x)\,d^4x\,d^4y\,.$$

The negative sign of q can be compensated by making the change of variables $x' \to x_* - x'$ where x_* is chosen in the middle of supp(f). For this expression one may use the preceding method.

References

[1] GELL-MANN, M.: Physics **1**, 63 (1964).
[2] KÄLLEN, G.: Dan. Math. Fys. Medd. **27**, 12 (1953). — LEHMANN, H.: Nuovo cimento **11**, 342 (1955)
[3] LANGERHOLC, J., and B. SCHROER: Commun. Math. Phys. **1**, 215 (1965); see, in particular, the remarks in the last section.
[4] JOST, R.: The general theory of quantized fields. Am. Math. Soc. Providence R.I. (1965).
[5] COURANT, R., and D. HILBERT: Methods of mathematical physics. New York: Interscience 1953.
[6] ARAKI, H.: J. Math. Phys. **1**, 5 (1964).
[7] KASTLER, D., D. W. ROBINSON, and A. SWIECA: To be published in Commun. Math. Phys.
[8] HAAG, R., and H. ARAKI: Univ. of Ill. manuscript and private communication.
[9] HAAG, R., and B. SCHROER: J. Math. Phys. **3**, 248 (1962).