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Fock Representation and Massless Particles

S. Doplicher*
University of Aix-Marseille

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Abstract. We present some remarks on the representations of Canonical Commutation or Anticommutation Relations which describe free massless particles.

In connection with the Goldstone theorem, there has been some interest in the degeneracy of the positive energy representations of a free (scalar) massless field. Namely it has been stressed that:

1. There exist many positive energy (non-vacuum) representations [1], [2].

2. There exists a degeneracy of the vacuum state, given by the (broken) symmetry group $\varphi \to \varphi + \lambda$ [3][1].

We present some elementary comments on this subject, that can be summarized as follows:

α) The vacuum states given by the degeneracy 2. are the only ones for the scalar massless free field.

β) In the case of a free Dirac massless field one has that:

$\beta 1$) There exist many positive energy representations.

$\beta 2$) There exists only one vacuum state.

Proof of α): The intuitive reason is that, in order to make an invariant state with zero energy for the free Hamiltonian, we can only add particles with zero momentum.

Let $a(f)$, $f \in \mathscr{S}^{(3)}$, be the (closed) destruction operators in an irreducible representation of the canonical commutation relations: we specify the free massless case assigning the translation automorphisms

$$f \to f_x, \ \tilde{f}_x(\mathbf{p}) = e^{-i(|\mathbf{p}|x_0 - \mathbf{p}\cdot\mathbf{x})}\tilde{f}(\mathbf{p}) \, . \tag{1}$$

We assume furthermore positivity of the energy; namely there should exist a unitary (strongly continuous) representation $U(x)$ of the

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* On leave from Istituto di Fisica dell'Università, Roma.

Postal Address: Physique Théorique, Faculté des Sciences, Place Victor-Hugo 13 — Marseille — France.

[1] See also [4] for a proof of the Goldstone theorem.

translation group with the property

$$U(x)\, a(f)\, U(x)^{-1} = a(f_x) \qquad U(x) = \int_{V+} e^{ipx}\, dE(p)\,. \tag{1'}$$

Finally there should exist a vacuum state Ω which is in the domain of the operators $a(f)$, such that $U(x)\Omega = \Omega$. It is well known that Ω is the only vector with the last property.

Let us consider the vector

$$a(f)\Omega, \quad f \in \mathscr{S}^{(3)}\,,$$

it follows from eq. (1) and (1′) that its spectrum of energy is

$$\text{spectrum} \quad a(f)\Omega \subseteq \{-|\mathbf{p}| : \mathbf{p} \in \text{Support}\, \tilde{f}\}\,,$$

then the spectrum condition implies

$$a(f)\Omega = \lambda(f)\Omega$$

with $\lambda(f)$ a c-number which is zero if $0 \notin \text{Supp}\tilde{f}$.

But

$$\lambda(f) = \langle \Omega a(f) \Omega \rangle \text{ is a (tempered) distribution}$$

$$\lambda(f) = \lambda(f_x) \text{ because } U(x)\, a(f)\Omega = a(f_x)\Omega\,,$$

these two conditions imply $\lambda(f) = c \int f(\mathbf{x})\, d^3x$, then

$$(a - c)\, (f)\Omega = 0, \forall f$$

i.e.

$$a(f) = a_{\text{Fock}}(f) + \mathrm{c} \int f(\mathbf{x})\, d^3x, \quad \text{q.e.d.}$$

Proof of $\beta 2$)*:* In this case we cannot fill up the zero momentum level with infinitely many particles, because of the exclusion principle, so that the result is intuitively clear.

Let $a(f)$ be the destruction operators in a representation of the CAR.

$$[a(f), a(g)^*]_+ = \int f(\mathbf{x})\, \overline{g(\mathbf{x})}\, d^3x\,, \tag{2}$$

it is known that it follows from (2) that $a(f)$ is a bounded operator and

$$\|a(f)\| = \|f\|_{\mathscr{L}^2} = \{\int |f(\mathbf{x})|^2\, d^3x\}^{1/2}, \tag{3}$$

so that as a test function space we can consider the whole Hilbert space of square-integrable functions. Assume again (1) and (1′) as in the CCR case, and suppose that there is a cyclic vector Ω for the $a(f)$ and $a(f)^*$, which is invariant for $U(x)$.

As above

$$a(f)\Omega = 0 \quad \text{if} \quad 0 \notin \text{Supp}\, \tilde{f}$$

Now the functions whose support is momentum space does not include the origin are dense in $\mathscr{L}^2(\mathbf{R}^3, d^3x)$, so that by eq. (3) it follows

$$a(f)\Omega = 0 \quad \forall f \in \mathscr{L}^2(\mathbf{R}^3, d^3x)$$

i.e.

$$a(f) = a_{\mathrm{Fock}}(f). \quad \text{q.e.d.}^2$$

Proof of $\beta 1$): We observe that the result has to be expected because in each small neighbourhood of the origin in momentum space there are infinitely many degrees of freedom which can be filled each with one particle. Let us in fact proceed as follows:

Let $0_1, 0_2, \ldots$ be an infinite sequence of non overlapping domains in momentum space, such that 0_n is contained in the sphere of radius $\varepsilon \cdot 2^{-n}$ around the origin.

Let $\{f_n\}$ be an orthonormal set of functions in $\mathscr{L}^2(\mathrm{R}^2, d^3x)$ such that $\tilde{f}_n$ has support in 0_n, and let $\{f_n, g_m\}$ be its extension to a complete orthonormal set.

Then every discrete representation [5], which is non-Fock for the degrees of freedom $\tilde{f}_n$ and which is the Fock representation for the degrees of freedom g_m, will provide an example; it is in fact clear that the free Hamiltonian will be convergent, because each wave packet f_n has energy not greater than $\varepsilon/2^n$, and $\sum \varepsilon/2^n = \varepsilon$.

To explain this explicitly, let us call $a(f_n) = a_n$, $a(g_m) = b_m$, and specify an irreducible representation of CAR requiring that it contains a cyclic vector Ω with the property

$$a_n^* \Omega = 0, \; b_m \Omega = 0, \; n, m, = 1, 2, \ldots .$$

To each x in a neighbourhood of zero, we define the translated vector $\Omega(x)$ in the following way. The state Ω is infinitely occupied with particles each in the state f_k, and we want to replace them with particles each in the translated state $f_{k,x}$[3]. To this end consider

$$\Omega_n(x) = a(f_{n,x})^* a_n a(f_{n-1,x})^* a_{n-1} \ldots a(f_{1,x})^* a_1 \Omega ;$$

an elementary computation shows that

$$\|\Omega_n(x)\| = \|\Omega\| ; \tag{4}$$

on the other hand, if we define

$$h_n(x) = f_{nx} - f_n, \; \left(\text{so that } a(f_{k,x})^* a_k = \{1 + a(h_k(x))^* a_k\} a_k^* a_k\right)$$

$$Q_n(x) = \prod_{k=n}^{1} (1 + a(h_k(x))^* a_k) ,$$

we have $Q_n(x)\Omega = \Omega_n(x)$, and $\|h_n(x)\| \leqq \frac{\varepsilon}{2^n} (|x_0| + |\mathrm{x}|)$, so that if $|x_0| + |\mathrm{x}| < \varepsilon^{-1}$, $Q_n(x)$ converges uniformly to a bounded operator $Q(x)$,

[2] After this note was completed, R. F. STREATER informed me that he proved on different lines statements similar to α) and $\beta 2$).

[3] Remark that $f_{k,x}$ is orthogonal to all f_h for $h \neq k$, because of definition (1) and of the support properties of f_k in momentum space.

which satisfies

$$\|1 - Q(x)\| \leqq \frac{\lambda}{1-\lambda}, \text{with } \lambda = \varepsilon(|x_0| + |\mathrm{x}|)\,.$$

If x, x' are such that $x' - x$ is contained in an appropriate neighbourhood $N(0)$ of the origin, define similarly

$$Q(x, x') = \lim_{n\to\infty} a(f_{n,x})^* a(f_{n,x'}) \ldots a(f_{1,x})^* a(f_{1,x'})\,;$$

then one verifies easily that, for $x - x', x' - x'', x - x'' \in N(0)$,

$$Q(x, x')\, Q(x', x'') = Q(x, x'')\,. \tag{5}$$

Then by eq. (4) there exists a vector $\Omega(x)$ such that

$$\lim_{n\to\infty} \|\Omega(x) - \Omega_n(x)\| = 0,\ \|\Omega(x)\| = \|\Omega\|\,;$$

it is easily seen that

$$a(f_{n,x})^*\, \Omega(x) = 0\,,\ a(g_{n,x})\, \Omega(x) = 0\,;$$

from these properties one can conclude that, to each translation $x \in N(0)$, there exists an unitary operator $U(x)$ such that

$$U(x)\, a(f)\, U(x)^{-1} = a(f_x) \tag{6}$$

$$U(x)\Omega = \Omega(x)\,. \tag{7}$$

Equation (6) implies, using the irreducibility of the representation $a(f)$,

$$U(x)\, U(y)\, U(x+y)^{-1} = c(x, y) \quad (c\text{-number})\,;$$

moreover, of course, using (5) and (7), if $x, y \in N(0)$,

$$\begin{aligned} U(x)\, U(y)\Omega &= U(x)\, Q(y)\Omega \\ &= Q(x+y, x)\, Q(x)\Omega = \Omega(x+y) = U(x+y)\Omega\,, \end{aligned}$$

so that $c(x, y) = 1$ and, using the group property, $U(x)$ can be defined for all x, giving a unitary representation, which is evidently weakly continuous.

To verify the spectrum condition, we notice first that, by differentiating eq. (7), we get

$$\begin{aligned} P_\mu \Omega &= i \frac{\partial}{\partial x_\mu} Q(x)\Omega\Big|_{x=0} = i \lim_{n\to\infty} \frac{\partial}{\partial x_\mu} Q_n(x)\Omega\Big|_{x=0} \\ &= \sum_{k=1}^{\infty} a(p_\mu f_k)^* a\Omega \end{aligned} \tag{8}$$

where $f \to p_\mu f$ is the energy-momentum operator in the underlying one-particle space $\mathscr{L}^2(\mathbf{R}^3, d^3x)$ (see eq. (1)).

Denoting by $\mathscr{L}(0_k)$ the subspace of $\mathscr{L}^2(\mathbf{R}^3, d^3x)$ consisting of functions which have Fourier transform vanishing outside 0_k, we have that the vectors

$$a(g)^* a_k \Omega, \quad g \in \mathscr{L}(0_k)$$

form a closed subspace of the representation Hilbert space, unitarily equivalent to $\mathscr{L}(0_k)$; if we define the operators

$$P_\mu^{(k)} a(g)^* \Phi = a(p_\mu g)^* \Phi$$

on vectors $a(g)^* \Phi$ such that $g \in \mathscr{L}(0_k)$,

$$a(f)\Phi = 0 \quad \text{for all} \quad f \in \mathscr{L}(0_k)\,;$$

$$P_\mu^{(k)} = 0 \ \text{on the orthogonal complement},$$

we have that, up to a multiplicity, the mentioned unitary equivalence carries $P_\mu^{(k)}$ onto the restriction of p_μ to $\mathscr{L}(0_k)$.

As the latter operators have spectrum in the forward light cone, the same is true for $P_\mu^{(k)}$. Since by eq. (8) we have

$$P_\mu \Omega = \sum_{k=1}^{\infty} P_\mu^{(k)} \Omega$$

and a similar relation holds for all powers of P_μ, because by construction $P_\mu^{(k)} U(x) = U(x) P_\mu^{(k)}$, we conclude that the energy momentum spectrum of Ω is contained in the forward light cone. On identical lines the same property is proved for all vectors of the type $a_{n1} \ldots a_{ns}\Omega$. Moreover, each creation operator b_m^* adds an energy momentum on the forward light cone; since by application of all monomials in a_n, b_m^* $(n, m = 1, 2, \ldots)$ to Ω we get a complete set, we can conclude that $U(x)$ satisfies the spectrum condition.

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