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On the Reduction of the Regular Representation of the Poincaré Group

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Abstract. The decomposition of the regular representation of the Poincaré group into irreducible representations is given.

I

We denote by (a, Λ) any element of the Poincaré group $\mathscr{P}$, where a is a 4-Translation and Λ an element of the Lorentz group G. In the following, we shall not distinguish between G and its universal covering $SL(2, C)$. The multiplication law in $\mathscr{P}$ is given by:

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) . \tag{1}$$

We consider the Hilbert space $\mathscr{H}$, the elements of which are functions with square modulus integrable with respect to Haar measure. The mapping

$$f(a, \Lambda) \xrightarrow{(a_0, \Lambda_0)} f(a + \Lambda a_0, \Lambda \Lambda_0) \tag{2}$$

defines a unitary representation of $\mathscr{P}$, the so-called right regular representation. In this work, we shall explicitly decompose this representation into irreducible components.

We set:

$$\hat{f}(\hat{a}, \Lambda) = \int f(a, \Lambda)\, e^{-i\Lambda^{-1}a \cdot \hat{a}}\, da \tag{3}$$

where $a \cdot b$ is the Lorentzian scalar product. Now:

$$f(a, \Lambda) = \frac{1}{(2\pi)^4} \int \hat{f}(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}a \cdot \hat{a}}\, d\hat{a} \tag{4}$$

$$\int |f(a, \Lambda)|^2 da\, d\Lambda = \frac{1}{(2\pi)^4} \int |\hat{f}(\hat{a}, \Lambda)|^2\, d\hat{a}\, d\Lambda . \tag{5}$$

Therefore, equation (3) defines an isometric mapping of $\mathscr{H}$ into $\hat{\mathscr{H}}$, Hilbert space, the elements of which are functions with square modulus integrable with respect to measure $d\hat{a}\, d\Lambda$.

Transformation (2) induces in $\hat{\mathscr{H}}$:

$$\hat{f}(\hat{a}, \Lambda) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}}\, \hat{f}(\Lambda_0^{-1}\hat{a}, \Lambda\Lambda_0) . \tag{6}$$

Let, generally, Ω_m be the hyperboloid:

$$\hat{a} \cdot \hat{a} = m^2$$

and, if $m^2 > 0$, let Ω_m^+, Ω_m^- be superior and inferior sheets of Ω_m.

We set:

$$\left.\begin{aligned} \hat{f}_{m^2}(\hat{a}, \Lambda) &= \hat{f}(\hat{a}, \Lambda) \quad \text{for} \quad \hat{a} \in \Omega_m,\ m^2 < 0 \\ \hat{f}^{\pm}_{m^2}(a, \Lambda) &= \hat{f}(\hat{a}, \Lambda) \quad \text{for} \quad \hat{a} \in \Omega_m^{\pm},\ m^2 > 0\,. \end{aligned}\right\} \tag{7}$$

Now, taking into account equation (6), we have:

$$\left.\begin{aligned} \hat{f}_{m^2}(\hat{a}, \Lambda) &\xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} \hat{f}_{m^2}(\Lambda_0^{-1} \hat{a}, \Lambda \Lambda_0) \\ \hat{f}^{\pm}_{m^2}(\hat{a}, \Lambda) &\xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} \hat{f}^{\pm}_{m^2}(\Lambda_0^{-1} \hat{a}, \Lambda \Lambda_0) \end{aligned}\right\} \tag{8}$$

and, obviously:

$$\begin{aligned} \int |\hat{f}(\hat{a}, \Lambda)|^2\, d\hat{a}\, d\Lambda &= \frac{1}{2} \int_0^\infty dm^2 \Big[\int |\hat{f}^{+}_{m^2}(\hat{a}, \Lambda)|^2 d\sigma_m^+(\hat{a})\, d\Lambda + \\ &+ \int |\hat{f}^{-}_{m^2}(\hat{a}, \Lambda)|^2\, d\sigma_m^-(\hat{a})\, d\Lambda \Big] + \frac{1}{2} \int_{-\infty}^{0} dm^2 \int |\hat{f}_{m^2}(\hat{a}, \Lambda)|^2\, d\sigma_m(\hat{a})\, d\Lambda \end{aligned} \tag{9}$$

where $d\sigma_m^+(\hat{a}), d\sigma_m^-(\hat{a}), d\sigma_m(\hat{a})$ are invariant measures for Ω_m^+, Ω_m^- and Ω_m respectively. This shows that the representation of $\mathscr{P}$ defined by (6) is a direct integral of representations defined by (8). Our problem will now be resolved if we reduce these simpler representations.

II

First, we study, the representation corresponding to $\hat{f}^{+}_{m^2}(\hat{a}, \Lambda)$, denoted now, in short, by $\varphi(\hat{a}, \Lambda)$. These functions are defined on $\Omega_m^+ \times G$ and have square modulus integrable with respect to invariant measure $d\sigma_m^+(\hat{a})\, d\Lambda$.

We can associate to each $\hat{a} \in \Omega_m^+$ the matrix $\begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix}$, $\lambda > 0$, the element in $SL(2, C)$ which transforms the apex Q_0 of Ω_m^+ ([1]) into $\hat{a}$. Now, if $\Lambda = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, we have:

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix}^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ \zeta_1 & \lambda_1^{-1} \end{vmatrix}^{-1} \begin{vmatrix} u & v \\ -\bar{v} & \bar{u} \end{vmatrix}, \quad \lambda_1 > 0,\ |u|^2 + |v|^2 = 1 \tag{10}$$

and we write:

$$F_{\lambda_1, \zeta_1}(\lambda, \zeta, \tilde{u}) = \varphi(\hat{a}, \Lambda)\,, \quad \tilde{u} = \begin{vmatrix} u & v \\ -\bar{v} & \bar{u} \end{vmatrix}. \tag{11}$$

Taking (8) into account, we deduce:

$$F_{\lambda_1 \zeta_1}(\lambda, \zeta, \tilde{u}) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} F_{\lambda_1, \zeta_1}(\lambda', \zeta', \tilde{u}\tilde{u}') \tag{12}$$

where $\lambda', \zeta', \tilde{u}'$ are defined by:

$$\begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix} \begin{vmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{vmatrix} = \begin{vmatrix} u' & v' \\ -\bar{v}' & \bar{u}' \end{vmatrix} \begin{vmatrix} \lambda' & 0 \\ \zeta' & \lambda'^{-1} \end{vmatrix}. \tag{13}$$

On the other side, one etablish easily, if $d\sigma_m^+(\hat{a}) = \frac{d^3\hat{a}}{\hat{a}_0}$:

$$\int |\varphi(\hat{a}, \Lambda)|^2 \, d\sigma_m^+(\hat{a}) \, d\Lambda = m^2 \int \frac{d\lambda_1 \, d\zeta_1}{\lambda_1^3} \int |F_{\lambda_1\zeta_1}(\lambda, \zeta, \tilde{u})|^2 \frac{d\lambda \, d\zeta}{\lambda^3} \, d\tilde{u} \quad (14)$$

where $d\zeta\,(d\zeta_1)$ denotes the surface element in the complex plane of $\zeta\,(\zeta_1)$, and $d\tilde{u}$ is the invariant measure on $SU(2)$. From this result, we deduce that the representation of $\mathscr{P}$, defined by (8) is a direct integral of the representations defined by (12).

Since $F_{\lambda_1,\zeta_1}(\lambda, \zeta, \tilde{u})$ is of square modulus integrable on $SU(2)$ for almost all λ_1, ζ_1, λ, ζ, we shall write ([2]):

$$F_{\lambda_1\zeta_1}(\lambda, \zeta, \tilde{u}) = \sum_s \sum_{j=-s}^{+s} \sum_{j'=-s}^{+s} F^s_{\lambda_1,\zeta_1;j',j}(\lambda, \zeta) \, D^s_{jj'}(\tilde{u}) \quad (15)$$

where s runs over all integers or half-integers and where $D^s_{jj'}(\tilde{u})$ denotes the customary matrix element of the $SU(2)$ representation D^s. From (12), we associate to each (a_0, Λ_0) the transformation:

$$F^s_{\lambda_1,\zeta_1;j'j}(\lambda, \zeta) \xrightarrow{(a_0,\Lambda_0)} e^{i a_0\cdot \hat{a}} \sum_{k'=-s}^{+s} D^s_{j'k'}(\tilde{u}') \, F^s_{\lambda_1,\zeta_1;k'j}(\lambda', \zeta') \quad (16)$$

which is one possible form for the unitary irreducible representation of $\mathscr{P}$ with mass m and spin s ([1]). Taking into account orthogonality relations for the $D^s_{jj'}(\tilde{u})$, we have:

$$\int \frac{d\lambda \, d\zeta}{\lambda^3} \int d\tilde{u} \, |F_{\lambda_1\zeta_1}(\lambda, \zeta, \tilde{u})|^2 = \sum_{s,j} \frac{1}{2s+1} \sum_{j'} \int \frac{d\lambda \, d\zeta}{\lambda^3} |F^s_{\lambda_1,\zeta_1;j'j}(\lambda, \zeta)|^2 \quad (17)$$

and this finishes the reduction into irreducible components for the representation of (8) corresponding to $\hat{f}^+_{m^2}(\hat{a}, \Lambda)$.

Obviously, we can proceed in the same way for the representation (8) corresponding to $\hat{f}^-_{m^2}(\hat{a}, \Lambda)$. Therefore, we have studied the case $m^2 > 0$ in its entirety.

III

Now, we consider the case $m^2 < 0$. First, we must notice that for almost all elements $\Lambda = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$, we can write:

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{\varepsilon} \begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix}, \; |a|^2 - |b|^2 = 1, \quad \varepsilon = 0, 1, \quad \lambda > 0 \quad (18)$$

which is true for $|\delta|^2 - |\beta|^2 \neq 0$, or:

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{\varepsilon} \begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix} \quad (19)$$

which is true for $|\alpha|^2 - |\beta|^2 \neq 0$ ([6]).

If $Q_0 \in \Omega_m$ has coordinates $(0, 0, 0, m)$ we can associate to each point $\hat{a} \in \Omega_m$, the new coordinates $(\varepsilon, \lambda, \zeta)$ which, from (18), label a right coset of $SL(2, C)$ with respect to $SU(1, 1)$. Writing now:

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix}^{-1} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{-\varepsilon} = \begin{vmatrix} \lambda_1 & 0 \\ \zeta_1 & \lambda_1^{-1} \end{vmatrix}^{-1} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{-\varepsilon_1} \begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}, \varepsilon_1 = 0, 1 \tag{20}$$

we shall define $F_{(\varepsilon_1, \lambda_1, \zeta_1)}(\varepsilon, \lambda, \zeta, \tilde{a})$ by:

$$F_{(\varepsilon_1, \lambda_1, \zeta_1)}(\varepsilon, \lambda, \zeta, \tilde{a}) = \varphi(\hat{a}, \Lambda), \quad \Lambda = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \tag{21}$$

where $\tilde{a}$ is the matrix $\begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}$ in the right member of (20) and where $(\varepsilon, \lambda, \zeta)$ corresponds to $\hat{a}$. To simplify, we omit the index m^2.

Transformation (8) gives:

$$F_{(\varepsilon_1, \lambda_1, \zeta_1)}(\varepsilon, \lambda, \zeta, \hat{a}) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} F_{(\varepsilon_1, \lambda_1, \zeta_1)}(\varepsilon', \lambda', \zeta', \tilde{a}\tilde{a}') \tag{22}$$

where $\varepsilon', \lambda', \zeta', \tilde{a}'$ are defined by:

$$\begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{\varepsilon} \begin{vmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{vmatrix} = \begin{vmatrix} a' & b' \\ \bar{b}' & \bar{a}' \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}^{\varepsilon'} \begin{vmatrix} \lambda' & 0 \\ \zeta' & \lambda^{-1} \end{vmatrix}$$

On the other hand, one established immediately with $d\sigma_m(\hat{a}) = \frac{d^3\hat{a}}{|\hat{a}_0|}$:

$$\int |\varphi(\hat{a}, \Lambda)|^2 d\sigma_m(\hat{a})\, d\Lambda = m^2 \sum_{\varepsilon_1 = 0,1} \int \frac{d\lambda_1\, d\zeta_1}{\lambda_3} \sum_{\varepsilon = 0,1} \int \tag{23}$$

$$|F_{\varepsilon_1, \lambda_1, \zeta_1}(\varepsilon, \lambda, \zeta, \tilde{a})|^2 \frac{d\lambda\, d\zeta}{\lambda^3}\, d\tilde{a}$$

where $d\tilde{a}$ is Haar measure for $SU(1, 1)$. Therefore, the representation of $\mathcal{P}$ defined by (8) is a direct integral of the representations defined by (22).

Now, from (23), $F_{\varepsilon_1, \lambda_1, \zeta_1}(\varepsilon, \lambda, \zeta, \tilde{a})$ has square modulus integrable on $SU(1, 1)$ for almost all $\lambda, \zeta, \lambda_1, \zeta_1$. We can thus write (cf. Appendix for the notations):

$$F_{(\varepsilon_1, \lambda_1, \zeta_1)}(\varepsilon, \lambda, \zeta, \tilde{a}) = \sum_{\eta = 0,1} \sum_{n,m = -\infty}^{+\infty} \int_0^{\infty} d\varrho\, F^{(m,n)}_{\varepsilon_1, \lambda_1, \zeta_1}(\varepsilon, \lambda, \zeta; \varrho, \eta) \times$$

$$\times D_{nm}(\tilde{a}; \varrho, \eta) + \sum_{+,-} \sum_{s=2}^{\infty} \sum_{m,n=0}^{\infty} F^{\pm(m,n)}_{\varepsilon_1, \lambda_1, \zeta_1}\left(\varepsilon, \lambda, \zeta; \frac{s}{2}\right) D^{\pm}_{n,m}\left(\tilde{a}; \frac{s}{2}\right)$$

and from (2.2) obtain the transformations:

$$F^{(m,n)}_{\varepsilon_1, \lambda_1, \zeta_1}(\varepsilon, \lambda, \zeta; \varrho, \eta) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} \sum_{p = -\infty}^{+\infty} D_{mp}(\tilde{a}'; \varrho, \eta) \times$$

$$\times F^{(p,n)}_{\varepsilon_1, \lambda_1, \zeta_1}(\varepsilon', \lambda', \zeta'; \varrho, \eta) \tag{24}$$

$$F^{\pm(m,n)}_{\varepsilon_1, \lambda_1, \zeta_1}\left(\varepsilon, \lambda, \zeta; \frac{s}{2}\right) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} \sum_{p=0}^{\infty} D^{\pm}_{m,p}\left(\tilde{a}'; \frac{s}{2}\right) F^{\pm(p,n)}_{\varepsilon_1, \lambda_1, \zeta_1}\left(\varepsilon', \lambda', \zeta'; \frac{s}{2}\right). \tag{25}$$

Taking account of equation (A.6) in the Appendix, it is obvious that our study of the case $m^2 < 0$ is complete, because, in (24) and (25), we recognize one possible form for the unitary irreducible representation of $\mathcal{P}$ with imaginary mass, induced by the representation $D(\tilde{a}, \varrho, \eta)$ and $D^{\pm}\left(\tilde{a}, \frac{s}{2}\right)$ of the little group $SU(1, 1)$.

IV

In the following $f(a, \Lambda)$ is an infinitely often differentiable function with compact support. If $T^{+(s,m)}(a, \Lambda)$ denotes the operators of the unitary irreducible representation of $\mathscr{P}$ with mass m, spin s, corresponding to Ω_m^+, we consider the operator:

$$\int da\, d\Lambda\, T^{+(s,m)}(a, \Lambda)^{-1} f(a, \Lambda) . \tag{26}$$

The $T^{+(s,m)}(a, \Lambda)$ acts on Hilbert space of functions $h_i(\hat{a})$, $\hat{a} \in \Omega_m^+$, $-s \leqq i \leqq s$, such that:

$$\sum_i \int |h_i(\hat{a})|^2\, d\sigma_m^+(\hat{a}) < \infty .$$

As transformation law, we have:

$$h_i(\hat{a}) \xrightarrow{(a_0, \Lambda_0)} e^{i a_0 \cdot \hat{a}} \sum_i D^s_{i,i'}(\tilde{u}')\, h_{i'}(\hat{a}')$$

where $\tilde{u}'$ and $\hat{a}'$ are defined by:

$$\Lambda_{\hat{a}} \Lambda_0 = \tilde{u}' \Lambda_{\hat{a}'} .$$

Here, $\Lambda_{\hat{a}}$ denotes the matrix $\begin{vmatrix} \lambda & 0 \\ \zeta & \lambda^{-1} \end{vmatrix}$ which transforms Q_0 into $\hat{a}$. We have now:

$$\begin{aligned} &\int da\, d\Lambda\, f(a, \Lambda)\, T^{+(s,m)}(a, \Lambda)^{-1} h_j(\hat{a}) \\ &= \int da\, d\Lambda\, f(a, \Lambda)\, e^{-i\Lambda^{-1} a \cdot \hat{a}} \sum_{j'} D^s_{jj'}(u_1^{-1})\, h_{j'}(\hat{a}_1) \end{aligned} \tag{27}$$

with

$$\Lambda_{\hat{a}} \Lambda^{-1} = \tilde{u}_1^{-1} \Lambda_{\hat{a}_1} .$$

If we choose as a new variable in the right member of (27):

$$\Lambda_0 = \Lambda \Lambda_{\hat{a}}^{-1} = \Lambda_{\hat{a}_1}^{-1} \tilde{u}_1$$

and notice that:

$$d\Lambda_0 = \frac{1}{m^2}\, d\sigma_m^+(\hat{a}_1)\, d\tilde{u}_1$$

we can write

$$\begin{aligned} &\int da\, d\Lambda\, f(a, \Lambda)\, T^{+(s,m)}(a, \Lambda)^{-1} h_j(\hat{a}) \\ &= \sum_{j'} \int d\sigma_m^+(\hat{a}_1) \left[\int d\tilde{u}\, \frac{1}{m^2}\, \hat{f}^+_{m_2}(\hat{a}, \Lambda_{\hat{a}_1}^{-1} \tilde{u} \Lambda_{\hat{a}})\, D^s_{j'j}(\tilde{u}) \right] \times h_{j'}(\hat{a}_1) \\ &= \sum_{j'} \int d\sigma_m^+(\hat{a}_1)\, K_f^{+(jj')}(\hat{a}, \hat{a}_1; m, s)\, h_{j'}(\hat{a}_1) . \end{aligned} \tag{28}$$

From orthogonality relations between the $D^s_{jj'}(\tilde{u})$ and taking into account equations (11), (15), we conclude immediately:

$$F^{+s}_{\lambda_1, \zeta_1; j, j'}(\lambda, \zeta) = m^2 (2s+1)\, K_f^{+(jj')}(\hat{a}, \hat{a}_1; m, s)$$

and it is easy to prove that $K_f^{+(jj')}(\hat{a}, \hat{a}_1, m, s)$ is the kernel for a Hilbert-Schmidt operator.

It is obvious that we can treat in the same way unitary irreducible representations corresponding to Ω_m^-. We denote by $K_f^{-(j,j')}(\hat{a}, \hat{a}_1; m, s)$ the corresponding kernel. For the representations with imaginary mass induced by representations $D(\tilde{a}; \varrho, \eta)$ and $D^{\pm}\left(\tilde{a}; \frac{s}{2}\right)$ of the little group, we can apply a similar procedure with the modifications implied by the particular parametrization of Ω_m and the Plancherel measure on $SU(1,1)$. One can repeat word by word the preceding reasoning, as one will easily see if one takes, as point of departure, a form of representations similar to (24) and (25). We always obtain thus kernels for Hilbert-Schmidt operators.

This being said, we wish now to express $f(a, \Lambda)$ in terms of its components. First of all, we have:

$$f(a, \Lambda) = \frac{1}{2} \frac{1}{(2\pi)^4} \int_0^\infty dm^2$$

$$\left[f_{m^2}^+(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}a\cdot\hat{a}}\, d\sigma_m^+(\hat{a}) + \int f_{m^2}^-(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}a\cdot\hat{a}}\, d\sigma_m^-(\hat{a})\right] + \tag{29}$$

$$+ \frac{1}{2} \frac{1}{(2\pi)^4} \int_{-\infty}^0 dm^2 \int f_{m^2}(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}a\cdot\hat{a}}\, d\sigma_m(\hat{a}) .$$

We give detailed calculations for the first term in the right member; the other terms can be treated in the same way. We can write:

$$\int f_{m_2}^+(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}\hat{a}\cdot a}\, d\sigma_m^+(\hat{a}) = \int d\sigma_m^+(\hat{a})\, f_m^+(\hat{a}, \Lambda_{\hat{a}_1}^{-1}\tilde{u}\,\Lambda\hat{a})\, e^{i(\Lambda_{\hat{a}}^{-1}\tilde{u}^{-1}\Lambda_{\hat{a}_1})\,a\cdot\hat{a}}$$

$$= \int d\sigma_m^+(\hat{a})\, f_{m^2}^+(\hat{a}, \Lambda_{\hat{a}_1}^{-1}\,\tilde{u}\Lambda_{\hat{a}})\, e^{ia\cdot\hat{a}_1} \tag{30}$$

by the definition of the $\Lambda_{\hat{a}}$'s. Then:

$$\int d\sigma_m^+(\hat{a})\, f_{m_2}^+(\hat{a}, \Lambda)\, e^{i\Lambda^{-1}a\cdot\hat{a}} = \sum_s (2s+1) \times$$

$$\times \sum_{j,j'} m^2 \int d\sigma_m^+(\hat{a})\, K_f^{+(j,j')}(\hat{a}, \hat{a}_1; m, s)\, D_{j'j}^s(\tilde{u})\, e^{ia\cdot\hat{a}_1}$$

and $\Lambda_{\hat{a}_1}$, $\tilde{u}$, $\Lambda_{\hat{a}}$ are such that

$$\Lambda_{\hat{a}_1}\cdot\Lambda = \tilde{u}\cdot\Lambda_{\hat{a}} .$$

Now, let us consider:

$$T^{+(s,m)}(a, \Lambda) \int da'\, d\Lambda'\, f(a', \Lambda')\, T^{+(s,m)}(a', \Lambda')^{-1} .$$

We have:

$$T^{+(s,m)}(a, \Lambda) \int da'\, d\Lambda'\, f(a', \Lambda')\, T^{+(s,m)}(a', \Lambda')^{-1}\, h_j(\hat{a})$$

$$= T^{+(s,m)}(a, \Lambda) \sum_{j'} \int d\sigma_m^+(\hat{a}_1)\, K_f^{+,(j,j')}(\hat{a}, \hat{a}_1; m, s)\, h_{j'}(\hat{a}_1)$$

$$= e^{ia\cdot\hat{a}} \sum_k D_{jk}^s(\tilde{u}') \int d\sigma_m^+(\hat{a}_1)\, K_f^{+(k,j')}(\hat{a}', \hat{a}_1; m, s)\, h_{j'}(\hat{a}_1)$$

where $\tilde{u}'$, $\hat{a}'$ are defined by:

$$\Lambda_{\hat{a}}\Lambda = \tilde{u}'\Lambda_{\hat{a}'} .$$

From this follows:

$$\sum_{j,j'} \int d\sigma_m^+(\hat{a})\, K_f^{+(j,j')}(\hat{a}, \hat{a}_1; m, s)\, D^s_{j'j}(\tilde{u})\, e^{i a\cdot \hat{a}_1}$$
$$= \operatorname{Tr} T^{+(s,m)}(a, \Lambda) \int da'\, d\Lambda'\, f(a', \Lambda')\, T^{+(s,m)}(a', \Lambda')^{-1}$$
$$= \operatorname{Tr} \int da'\, d\Lambda'\, f_{(a,\Lambda)}(a', \Lambda')\, T^{+(s,m)}(a', \Lambda')^{-1}$$

where $f_{(a,\Lambda)}(a', \Lambda')$ is the right-translated by (a, Λ) of $f(a', \Lambda')$:

$$f_{(a,\Lambda)}(a', \Lambda') = f(a' + \Lambda' a, \Lambda' \Lambda)\,.$$

If we denote by $T^{+(s,m)}(f)$ the quantity:

$$\operatorname{Tr} \int f(a', \Lambda')\, T^{+(s,m)}(a', \Lambda')\, da'\, d\Lambda'$$

we can write finally, taking into account the unitary properties:

$$\int d\sigma_m^+(\hat{a})\, f^+_{m^2}(\hat{a}, \Lambda)\, e^{i\Lambda^{-1} a\cdot \hat{a}} = m^2 \sum_s (2s+1)\, \overline{T^{+(s,m)}(f_{(a,\Lambda)})}\,.$$

With similar calculations for the other terms in the right member of (30), we obtain (cf. Appendix):

$$\begin{aligned} f(a,\Lambda) &= \frac{1}{2}\frac{1}{(2\pi)^4} \sum (2s+1) \int_0^\infty m^2\, dm^2 \left(\overline{T^{+(s,m)}(f_{(a,\Lambda)})} + \overline{T^{-(s,m)}(f_{(a,\Lambda)})}\right) + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_{-\infty}^0 |m^2|\, dm^2 \int_0^\infty d\varrho\, \varrho\, \operatorname{th}\frac{\pi\varrho}{2}\, \overline{T^{\varrho,0,im}(f_{(a,\Lambda)})} + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_{-\infty}^0 |m^2|\, dm^2 \int_0^\infty d\varrho\, \varrho\, \operatorname{cth}\frac{\pi\varrho}{2}\, \overline{T^{\varrho,1,im}(f_{(a,\Lambda)})} + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \sum_{+,-} \sum_{s=1}^\infty (s-1) \int_{-\infty}^0 |m^2|\, dm^2\, \overline{T^{\pm(\frac{s}{2},im)}(f_{(a,\Lambda)})} \end{aligned} \tag{31}$$

with obvious notations.

In particular, for $a = 0$, $\Lambda = e$:

$$\begin{aligned} f(0,e) &= \frac{1}{2}\frac{1}{(2\pi)^4} \sum_s (2s+1) \sum_{+,-} \int_0^\infty m^2\, dm^2 \left(\overline{T^{\pm(s,m)}(f)}\right) + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_\infty^0 |m|^2\, dm^2 \int_0^\infty d\varrho\, \varrho\, \operatorname{th}\frac{\pi\varrho}{2}\, \overline{T^{\varrho,0,im}(f)} + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_{-\infty}^0 |m^2|\, dm^2 \int_0^\infty d\varrho\, \varrho\, \operatorname{cth}\frac{\pi\varrho}{2}\, \overline{T^{\varrho,1,im}(f)} + \\ &+ \frac{1}{2}\frac{1}{(2\pi)^4} \sum_{s=1}^\infty (s-1) \sum_{+,-} \int_{-\infty}^0 |m^2|\, dm^2 \left(\overline{T^{\pm(\frac{s}{2},im)}(f)}\right). \end{aligned} \tag{32}$$

Finally, denoting by $K_f^+(s, m)$, $K_f^-(s, m)$, $K_f(\varrho, \eta, im)$ and $K_f^\pm\left(\frac{s}{2}, im\right)$ the operator corresponding to kernels connected to representations appearing in (31), (32), we obtain, applying the same calculations to (5) and (9):

$$\int |f(a, \Lambda)|^2 \, da \, d\Lambda \tag{33}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^4} \sum_{+,-} \sum (2s+1) \int_0^\infty m^2 \, dm^2 \operatorname{Tr} K_f^\pm(s, m) \, K_f^\pm(s, m)^* +$$

$$+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_{-\infty}^0 |m^2| \, dm^2 \int_0^\infty d\varrho \, d\varrho \operatorname{th} \frac{\pi\varrho}{2} \operatorname{Tr} K_f(\varrho, 0, im) \, K_f(\varrho, 0, im)^* +$$

$$+ \frac{1}{2}\frac{1}{(2\pi)^4} \int_{-\infty}^0 |m^2| \, dm^2 \int_0^\infty d\varrho \, \varrho \operatorname{cth} \frac{\pi\varrho}{2} \operatorname{Tr} K_f(\varrho, 1, im) \, K_f(\varrho, 1, im)^* +$$

$$+ \frac{1}{2}\frac{1}{(2\pi)^4} \sum_{+,-} \sum_{s=1}^\infty (s-1) \int_{-\infty}^0 |m^2| \, dm^2 \operatorname{Tr} K_f^\pm\left(\frac{s}{2}, im\right) K_f^\pm\left(\frac{s}{2}, im\right) .$$

As the infinitely often differentiable functions with compact support are dense in the Hilbert space of functions with square modulus integrable on $\mathscr{P}$, (33) is still true for all such functions. So, (31) and (33) contain the essential results concerning the Fourier transform on $\mathscr{P}$, this last being understood as in Guelfand's work ([7]).

Appendix

On unitary representation in principal series of $SU(1, 1)$ and Plancherel formula

a) Continuous representations in the principal series. Let $\mathscr{H}$ be the Hilbert space the elements of which are functions $f(\varphi)$ such that:

$$\int_0^{2\pi} |f(\varphi)|^2 \, d\varphi < \infty .$$

Let us associate to each element $\tilde{a} = \begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}$ of $SU(1, 1)$ the transformation

$$f(\varphi) \to (b e^{i\varphi} + \bar{a})^{i\frac{\varrho}{2} + \frac{\eta}{2} - \frac{1}{2}} (\bar{b} e^{-i\varphi} + a)^{i\frac{\varrho}{2} - \frac{\eta}{2} - \frac{1}{2}} f(\varphi') = D(\tilde{a}; \varrho, \eta) f(\varphi) \tag{A.1}$$

where $\varrho > 0$, $\eta = 0$ or 1 and φ' is given by:

$$e^{i\varphi'} = \frac{a e^{i\varphi} + \bar{b}}{b e^{i\varphi} + \bar{a}} .$$

(A.1) defines a unitary irreducible representation of $SU(1, 1)$ for which Casimir's operator has the value:

$$q = \frac{1}{4} + \frac{\varrho^2}{4} .$$

If $\eta = 0$, one obtains representations, isomorphic to representation C_q^0 in Bargmann's work and, if $\eta = 1$, to representations $C_q^{1/2}\left(q > \frac{1}{4}\right)$.

We take as a norm in $\mathscr{H}$, Bargmann's value:

$$\|f\| = \left(\frac{1}{2\pi}\int_0^\infty |f(\varphi)|^2 \, d\varphi\right)^{\frac{1}{2}}.$$

b) Discrete representation in the principal series. Let $\mathscr{H}_s$ be the Hilbert space the elements of which are functions $f(\zeta)$, analytic in the disk $|\zeta| < 1$, and such that:

$$\int_{|\zeta|<1} |f(\zeta)|^2 (1 - |\zeta|^2)^{s-2} \, d\zeta < \infty \quad s \geqq 2 .$$

Let us associate to each element $\tilde{a} = \begin{vmatrix} a & b \\ \bar{b} & \bar{a} \end{vmatrix}$ of $SU(1, 1)$ the transformation

$$f(\zeta) \to (b\zeta + \bar{a})^{-s} f\left(\frac{a\zeta + \bar{b}}{b\zeta + \bar{a}}\right) = D^+\left(\tilde{a}; \frac{s}{2}\right) f(\zeta) \tag{A.2}$$

or the transformation:

$$f(\zeta) \to (\bar{b}\zeta + a)^{-s} f\left(\frac{\bar{a}\zeta + b}{\bar{b}\zeta + a}\right) = D^-\left(\tilde{a}; \frac{s}{2}\right) f(\zeta) . \tag{A.3}$$

We define thus unitary irreducible representations isomorphic to representations $D_{s/2}^{\pm}$ in Bargmann's work.

We take as a norm in $\mathscr{H}_s$, Bargmann's value:

$$\|f\| = \left(\frac{s-1}{\pi}\int_{|\zeta|<1} (1 - |\zeta|^2)^{s-2} \, |f(\zeta)|^2 \, d\zeta\right)^{+1/2}.$$

The discrepancies between our formulas and those of BARGMANN come from the dissimilar action of the group on homogeneous spaces (unit circle, unit disk): in our representation the group acts from the right.

c) Plancherel formula and regular representation ([4], [5]). Let $D_{n,m}(\tilde{a}; \varrho, \eta)$ be matrix elements of $D(\tilde{a}; \varrho, \eta)$ in the orthonormal basis $e^{in\xi}$, $-\infty \leqq n \leqq +\infty$ and let $D_{n,m}^{\pm}\left(\tilde{a}; \frac{s}{2}\right)$ be matrix elements of $D^{\pm}\left(\tilde{a}; \frac{s}{2}\right)$ in the orthonormal basis $\left(\frac{n!(s-1)!}{(n+s-1)!}\right)^{1/2}\zeta^n$, $n = 0, 1, \ldots$. It result from Bargmann's work that this set of function is a complete system in the Hilbert space whose elements are functions with square modulus integrable on $SU(1, 1)$. Let $\phi(\tilde{a})$ be such a function; for almost all $\tilde{a}$, we can write

$$\begin{aligned} \phi(\tilde{a}) = \sum_{\eta=0,1} \sum_{n,m=-\infty}^{\infty} \int_0^\infty d\varrho \, \phi_{nm}(\varrho, \eta) \, D_{nm}(\tilde{a}; \varrho, \eta) + \\ + \sum_{+,-} \sum_{s=2}^{\infty} \phi_{n,m}^{\pm}\left(\frac{s}{2}\right) D_{n,m}^{\pm}\left(\tilde{a}; \frac{2}{s}\right). \end{aligned} \tag{A.4}$$

According to orthogonality relations between matrix elements we have:

$$\left.\begin{aligned} \varphi_{nm}(\varrho, 0) &= \varrho \operatorname{th} \frac{\pi\varrho}{2} \int d\tilde{a}\, \phi(\tilde{a})\, \overline{D_{n,m}(\tilde{a}, \varrho, 0)} \\ \phi_{nm}(\varrho, 1) &= \varrho \operatorname{cth} \frac{\pi\varrho}{2} \int d\tilde{a}\, \phi(\tilde{a})\, D_{n,m}(\tilde{a}; \varrho, 1) \\ \phi^{\pm}_{n,m}\left(\frac{s}{2}\right) &= (s-1) \int d\tilde{a}\, \phi(\tilde{a})\, D^{\pm}_{n,m}\left(\tilde{a}; \frac{s}{2}\right). \end{aligned}\right\} \tag{A.5}$$

Further

$$\int |\phi(\tilde{a})|^2\, d\tilde{a} =$$

$$= \sum_{n,m=\infty}^{+\infty} \left[\int_0^\infty d\varrho\, \varrho \operatorname{th} \frac{\pi\varrho}{2} |\phi_{n,m}(\varrho, 0)|^2 + \int_0^\infty d\varrho\, \varrho \operatorname{cth} \frac{\pi\varrho}{2} |\phi_{n,m}(\varrho, 1)|^2\right] +$$

$$+ \sum_{s=1}^{\infty} (s-1) \sum_{n,m=0}^{\infty} \left(\left|\phi^{+}_{n,m}\left(\frac{s}{2}\right)\right|^2 + \left|\phi^{-}_{n,m}\left(\frac{s}{2}\right)\right|^2\right). \tag{A.6}$$

If we replace $\phi(\tilde{a})$ by its translated $\phi(\tilde{a}\tilde{a}_0)$, the coefficients in A.5 become, taking into account unitarity and invariance of $d\tilde{a}$:

$$\sum_{p=-\infty}^{+\infty} D_{m,p}(\tilde{a}_0; \varrho, \eta)\, \phi_{n,p}(\varrho, \eta), \quad \sum_{p=0}^{\infty} D^{\pm}_{m,p}\left(\tilde{a}_0; \frac{s}{2}\right) \phi^{\pm}_{n,p}\left(\frac{s}{2}\right).$$

So, for n fixed, vector functions $\phi_{n,m}(\varrho, \eta)$, $\phi^{\pm}_{n,m}\left(\frac{s}{2}\right)$ transform according to irreducible representations of $SU(1, 1)$ and this with A.6, resolves the problem of decomposing the right regular representation of $SU(1, 1)$.

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