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On the Interaction Picture*

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Abstract. It is shown that the difficulties connected with Haag's theorem can be bypassed, without losing Euclidean invariance, if the time evolution is considered as being only locally unitarily implementable.

A variant of the conventional interaction picture is defined, and a perturbation expansion derived which is shown to converge absolutely for a class of (non-trivial) Euclidean invariant 2-dimensional models.

1. Introduction

The Hamiltonian formalism, as applied to relativistic quantum field theories, has fallen very much into discredit during the last decade or so. But perhaps the time is now ripe to investigate again some of the basic difficulties of field theories, and this in the light of what has been learned in the so-called "axiomatic" approaches.

Following WIGHTMAN [1], we can label the three main difficulties of the conventional approach by the catchwords — Haag's theorem, — Instability of the vacuum, — Ultraviolet catastrophe. The aim of this paper is to show that the first of these difficulties, namely Haag's theorem, can be bypassed in a systematic way, with a slight alteration of the conventional formalism.

We shall have to define a new picture, closely related to the interaction (or Dirac) picture. The difference will be that the trivial part of the time evolution will be acting on the states (instead of on the operators) and the non trivial part will act on the operators. Haag's theorem says that the non-trivial part of the time evolution cannot be unitarily implemented (if the theory is Euclidian invariant), but we shall remark that it does not forbid us to have the time evolution acting as an algebraic mapping of a certain ring of operators. We shall even prove that this mapping can be locally unitarily implemented if it exists.

We consider the disagreeable feature of having the time evolution being only locally unitarily implementable (instead of globally) as very small in comparison to the advantage of being allowed to use the canonical commutation relations (instead of some inequivalent representation nobody knows how to construct) and to work in the Fock space.

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2. Local rings systems[1]

Let $A(x)$ be a free field operator (in the Heisenberg picture), $\{f(x)\}_{\mathcal{O}}$ the set of all $(s+1)$- (or s-) dimensional test functions with support in a bounded open region of space-time $\mathcal{O}$ (resp. of space $\boldsymbol{\mathcal{O}}$). We denote by $R(\mathcal{O})$ resp. $R(\boldsymbol{\mathcal{O}})$ the von Neumann ring generated by all $A(f)$ (resp. $A(f)$ and $\dot{A}(f)$), $f \in \{f(x)\}_{\mathcal{O}}$. A theorem of ARAKI [6] tells us that $R(\mathcal{O}) = R(\boldsymbol{\mathcal{O}})$, if $\mathcal{O}$ is the double cone spanned by $\boldsymbol{\mathcal{O}}$.

We denote by $\mathfrak{A}$ the norm closure of the set theoretical union of all $R(\mathcal{O})$ for all bounded open $\mathcal{O}: \mathfrak{A} = \overline{\bigcup_{\mathcal{O}\text{ bd.}} R(\mathcal{O})}$. If we now consider the set of all s-dimensional spheres S_r centered at the origin, and denote by $C_r(t)$ the corresponding double cone at time t, we have that $\mathfrak{A} = \mathfrak{A}(t) = \overline{\bigcup_{r<\infty} R(C_r(t))}$.

We shall be interested in isomorphic mappings of $\mathfrak{A}$ into the ring $\mathfrak{B}(\mathfrak{H})$ of all bounded operators on the Hilbert space. An important property to keep in mind, is that, due to the fact that $\mathfrak{A}$ is strictly smaller than $\mathfrak{B}(\mathfrak{H})$, such mappings are not always implemented by unitary operators. An isomorphism β between $\mathfrak{A}$ and a ring $\mathfrak{A}_1$, $\mathfrak{A}_1 = \beta[\mathfrak{A}]$, is called globally unitarily implementable (or spatial) if there exists a unitary operator U such that $\beta[A] = U A U^{-1}$ for all $A \in \mathfrak{A}$. An isomorphism which is not spatial is called algebraic. One can show[2] that in this case it is locally unitarily implementable, that is, that there exists for each bounded region $\mathcal{O}$, a unitary operator $U_{\mathcal{O}}$, such that, for $A \in R(\mathcal{O})$, $\beta[A] = U_{\mathcal{O}} A U_{\mathcal{O}}^{-1}$.

3. The interaction picture

In this section, we introduce a picture closely related to the usual interaction picture, and formally derive a perturbation expansion for the time evolution.

The basic idea of the interaction (or Dirac) picture is to subtract the part of the time evolution which is given by the free Hamiltonian from the total evolution. One, therefore, admits that at time $t = 0$, the Heisenberg and Schrödinger pictures coincide and that the field operators A_D[3] are transformed according to $A_D(t) = e^{iH_0 t} A_D(0)\, e^{-iH_0 t}$, $A_D(0)$

[1] We give here only the essential definitions. For a detailed account see for instance [2], [3], [4] and [5].

[2] Cf. Th. 4.1 and footnote 7.

[3] A subscript D indicates the usual form of the interaction or Dirac picture, a subscript S the Schrödinger picture, and a subscript H the Heisenberg picture. We shall use no subscript for our form of the interaction picture. In other pictures than the Dirac one, $A_D(t)$ means the operator $A_D(t) = e^{iH_0 t} A_D(0)\, e^{-iH_0 t}$ which will be well defined when we shall use it.

$= A_H(0) = A_S$, and the states by

$$|\psi_D(t')\rangle = (e^{iH_0t'} e^{-iH_S(t'-t)} e^{-iH_0t}) \, |\psi_D(t)\rangle = V(t', t) \, |\psi_D(t)\rangle \quad (3.1)$$

where H_S is the total Hamiltonian in the Schrödinger picture, $H_S = H_0 + H_S^I$.

We shall retain the basic idea of the interaction picture, but for reasons which will become clear below, we prefer to have the trivial part acting on the states. We, therefore, define

$$\begin{aligned} A(t) &= e^{-iH_0t} e^{iH_S(t-t')} e^{iH_0t'} A(t') \, e^{-iH_0t'} e^{-iH_S(t-t')} e^{iH_0t} \\ &= U(t, t') \, A(t') \, U^{-1}(t, t') \, . \end{aligned} \quad (3.2)$$

We shall write $U(t, 0) = U(t)$ and $V(t, 0) = V(t)$. In the Dirac picture, we have the differential equation

$$\frac{\partial}{\partial t} V(t) = -iH_D^I(t) \, V(t), \quad V(0) = 1 \, . \quad (3.3)$$

From which we can deduce the integral equation

$$V(t) = 1 - i \int_0^t H_D^I(\tau) \, V(\tau) \, d\tau \quad (3.4)$$

and Schwinger's perturbation expansion:

$$\begin{aligned} V(t) = 1 &+ (-i) \int_0^t d\tau \, H_D^I(\tau) \\ &+ (-i)^2 \int_0^t d\tau_1 \int^{\tau_1} d\tau_2 \, H_D^I(\tau_1) \, H_D^I(\tau_2) \\ &+ \cdots . \end{aligned} \quad (3.5)$$

Again for reasons which will become clear later, we are not interested in equations for $V(t)$ or $U(t)$, but want equations for $U(t) \, A \, U^{-1}(t)$. In our picture, this differential equation is

$$\frac{\partial}{\partial t} U(t) \, A(0) \, U^{-1}(t) = + i \, [H_D^I(-t), \, U(t) \, A(0) \, U^{-1}(t)], \; U(0) = 1 \quad (3.6)$$

and we get the integral equation

$$U(t) \, A(0) \, U^{-1}(t) = A(0) + i \int_0^t d\tau \, [H_D^I(-\tau), \, U(\tau) \, A(0) \, U^{-1}(\tau)]$$

and, therefore, the perturbation expansion

$$\begin{aligned} U(t) \, A(0) \, U^{-1}(t) = A(0) &+ i \int_0^t d\tau \, [H_D^I(-\tau), \, A(0)] \\ &+ i^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \, [H_D^I(-\tau_1), \, [H_D^I(-\tau_2), \, A(0)]] \\ &+ \cdots . \end{aligned} \quad (3.7)$$

The developments (3.5) and (3.7) are purely formal ones; however, for the special case where H^I is a bounded operator (and A a quasilocal bounded operator), the above expansions are perfectly legitimate and absolutely convergent with an infinite radius of convergence. If H^I is an unbounded operator, nothing can be said in general, and the series has to be investigated for each particular case on a suitable domain. The development (3.7) is not essentially new; it is contained implicitly in the work of SCHWINGER and of DYSON, although in a different form[4].

The surprising fact will be that the expansion (3.7) is better behaved than the expansion (3.5), at least in some very simple cases (see Section 5).

One can see that the Heisenberg picture is obtained very easily from our interaction picture, simply $A_H(t) = e^{iH_0 t} A(t)\, e^{-iH_0 t}$, and, therefore, one can consider the interaction picture as being merely a trick to obtain the Heisenberg picture by using the canonical commutations relations. We shall use this fact very often in the sequel.

There are many other expansions which may be written in this line of thought and have special virtues for particular problems, we shall deal with them in subsequent publications.

4. Haag's theorem

Haag's theorem (cf. for instance [7]) states that if the theory is Euclidean invariant and irreducible[5], the operator $V(t, t')$ of section 3 exists only if the theory is equivalent to a free field. Things obviously do not go better in the form we have given to the interaction picture, and $U(t, t')$ does not exist either.

If we now look at our interaction picture, we remark that we are not interested in $U(t, t')$ itself, what is relevant is the mapping $A(t') \to A(t)$, which should be implemented by $U(t, t')$. Haag's theorem tells us that $U(t, t')$ does not exist, but this does not mean that the above mentioned mapping cannot be defined and constructed in some other way. Explicitly stated, we shall try to prove that there exists an isomorphic mapping $\beta(t)$ of the algebra $\mathfrak{A}$ into the algebra $\mathfrak{B}(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$, and which corresponds to the mapping described above.

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[4] In the form (5.2) or (5.3), the perturbation expansion is essentially equivalent to the expansion given by F. DYSON [17]; cf. (20) p. 431. We, however, do not use time-ordering, as we want to use locality.

[5] Haag's theorem also applies if the algebra $\mathfrak{A}''$ is countably degenerate, but it does not remain true if $\mathfrak{A}''$ is continuously degenerated and if $V(t', t)$ does not leave the center $\mathfrak{A}' \cap \mathfrak{A}''$ elementwise invariant. However, we do not know of any physically interesting model fitting in this particular category, excepted, of course, classical physics; cf. [8].

In order to prove the existence of $\beta(t)$ $[\mathfrak{A}]$, we propose to apply the following procedure:

a) "Cutt off" the interaction Hamiltonian in such a way that

— $H_D^I(0)_k$ exists (k labeling the various cut offs).

— $H_0 + H_D^I(0)_k$ is essentially self-adjoint on some domain.

— $H_D^I(0)_k$ is not Euclidean invariant (by Haag's th. already necessary for the above point).

b) Construct $U_k(t)$ (which exists if a) is already satisfied).

c) Consider $U_k(t)\, A\, U_k^{-1}(t)$ for $A \in R(\mathcal{O})$, $\mathcal{O}$ bounded, and examine the convergence of $U_k(t)\, A\, U_k^{-1}(t)$ in a suitable topology (strong, in general) whenever one releases some or all "cut-offs".

The above recipe should not be considered as a universal panacea against all kinds of infinities. We believe (and shall show it in some cases), however, that it allows us to eliminate difficulties connected with Haag's theorem, and provide a basis for a completely rigorous study of the other problems of quantum field theories. As we shall see later, the elimination of the troubles connected with Haag's theorem is sufficient to permit the absolute convergence of the perturbative expansion (3.7) in the case of some "cut-off" (but translationally invariant) 2-dimensional models.

We conclude this section with a simple theorem, which makes, however, heavy use of the theory of local rings. We denote by $\beta(t)$ $[\mathfrak{A}]$ the limit of the mappings $U_k(t)\, A\, U_k^{-1}(t)$, $A \in \mathfrak{A}$ (whenever it exists), the limit removing some but not necessarily all cut-offs, the others being held fixed (for instance removing only the cut-off which destroys Euclidean invariance).

Theorem 4.1.[6] If $\beta(t)$ $[\mathfrak{A}]$ exists, then it is a (faithful) $*$-isomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$, which is norm-continuous and locally unitarily implementable.

Proof[7]. That it is faithful follows immediately from the fact that $\mathfrak{A}$ is simple [10], and that it is a $*$-isometric mapping follows from this and the fact that it is the limit of $*$, isometric mappings. That it is norm continuous follows from the fact that $\mathfrak{A}$ is a C^*-algebra. If now $R(\mathcal{O})$ is the local von Neumann algebra associated with the bounded region $\mathcal{O}$, then it follows from a theorem of FELDMAN and FELL [11], and from the fact that $R(\mathcal{O})'$ is of infinite type [12], [13], [14], that $\beta(t)$ $[R(\mathcal{O})]$ is again a von Neumann algebra of infinite type, as $\mathfrak{H}$ is separable. One easily sees that both $R(\mathcal{O})$ and $\{\beta(t)\ [R(\mathcal{O})]\}'$ are of infinite type, because $\mathcal{O}$ being bounded there always exists $\mathcal{O}_1$ which is bounded and completely

[6] We have restricted ourself for this theorem to the special case of a scalar bose field in interaction with itself, the generalization to other cases is easy.

[7] This proof is essentially that of a theorem in [9] together with a theorem of [10].

space-like with respect to $\mathcal{O}_1$, $R(\mathcal{O}_1)$ is of infinite type and $R(\mathcal{O}_1) \subset R(\mathcal{O})'$ and $\beta(t)\,[R(\mathcal{O}_1))] \subset \{\beta(t)\,[R(\mathcal{O})]\}'$. We can, therefore, apply a theorem of DIXMIER [15], Cor. 7, p. 321, which implies that the mapping $R(\mathcal{O}) \to \beta(t)\,[R(\mathcal{O})]$ is spatial, that is, unitarily implementable. QED

Corollary 4.2. If $\beta(t)\,[\mathfrak{A}]$ exists, and $\beta(t)$ is permutable with the space translations, then the interacting theory is strictly local at equal times.

We also have the following generalization of a result by ARAKI [16]

Corollary 4.3. If $\beta(t)\,[\mathfrak{A}]$ exists, then in all representations of the interacting theory in separable Hilbert spaces, and in the Heisenberg picture, the local rings $R(\mathcal{O})$ are of type III for the same (bounded) regions $\mathcal{O}$ as this is true for the free field. Furthermore, in all cases where $\mathcal{O}$ is bounded the $R(\mathcal{O})$ of the interacting theory are spatially isomorphic to the corresponding $R(\mathcal{O})$ of the free fields.

We see that although $U(t)$ does not exist[8], we may expect the next best thing to be true, namely that for each bounded region $\mathcal{O}$, there exists $U_{\mathcal{O}}(t)$ such that

$$\beta(t)\,[A] = U_{\mathcal{O}}(t)\, A\, U_{\mathcal{O}}^{-1}(t) \;\forall\, A \in R(\mathcal{O})$$

and we have the automatic insurance that our theory is local at equal times, provided $\beta(t)$ is permutable with translations.

The corollary 4.3 is important because (at least for all cases where $\beta(t)$ exists as an automorphism) it forbids a purely algebraical characterization of a field theory. One could indeed have thought that as it is known that there exist non-isomorphic of type III von Neumann algebras, field theories which lead to different S-matrices should have non isomorphic local rings, even in the case when the representations of the Poincaré (or of the translation) group were the same. Cor. 4.3, therefore, means that this is not possible and that the distinction between different theories is not made by the local algebraic structure, which is always the same, but by the different dynamics which is superimposed. This is, in fact, very reminiscent of the situation in the case of problems with a finite number of degrees of freedom, where the algebraic structure is always the same (type I) and only the dynamics differ.

We think that the above approach is superior to the usual way of bypassing Haag's theorem, namely of destroying the Euclidean invariance by enclosing the system in a finite box. To our knowledge, there is no systematic way of getting rid of the box, and we feel contrary to the opinion of many physicists, that discussing scattering and other asympto-

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[8] Among all possible representations of the interacting theory, one could search for the particular one in which the time evolution is globally unitarily implementable (in the Heisenberg picture) and the spectrum condition satisfied. We shall deal with this problem, and related ones, in a coming paper: "On the physical vacuum".

tic properties in a finite box does not make much physical sense. It should be clear that proving that $\beta(t)$ exists is in general a very hard task, but it is an explicitly stated problem, to which one should be able to get an answer in a finite amount of time for each particular case.

5. Applications

We shall now work out explicitly a possible application of the general principles presented above. The intuitive idea is the following: if we suppose that $h_D^I(x)$ is the interaction Hamiltonian density in the Dirac picture (possibly containing some cut offs), we can imagine the particular case in which $\int h_D^I(x)\, f_r(x)\, d^s\mathrm{x} = H_D^I(t)_r$ is well defined (maybe unbounded) operator whenever $f_r(\mathrm{x})$ is a test function with compact support, and such that $f_r(\mathrm{x}) = 1$ for $|\mathrm{x}| < r$. If we can also show that $H_0 + H_D^I(t)_r$ is essentially self-adjoint on some domain, then we can construct $U_r(t)$ and study the behavior of $U_r(t)\, A\, U_r^{-1}(t)$ as $r \to \infty$. If now the Hamiltonian density is local (or quasi-local) with respect to the free field, and A has support inside a double cone spanned by a sphere of radius $r_0 \ll r$, $U_r(t)\, A\, U_r^{-1}(t)$ will no longer depend upon r if r gets very big, A being held fixed. If this idea represents the reality, it means that $\beta(t)$ $[\mathfrak{A}]$ and hence the interaction picture exists in our sense. Because of Theorem 4.1, we can state our idea in the form: "the locality of the interaction implies that locally, we can use the Fock representation for the interacting field".

This intuitive idea is, however, in general very difficult to transform into a mathematically rigorous argument, and we shall take a rather particular case, where everything can be proven quite rigorously, and which, we hope, still contains sufficiently many physically interesting cases. That we shall use the perturbation expansion (3.7) in the sequel, does not at all mean that our formulation of the interaction picture (Section 3) or our recipe for bypassing Haag's theorem (Section 4), or even the intuitive idea above, are in any way dependent upon the success of perturbative explansions, and we hope to show in subsequent publications how to prove the existence of $\beta(t)$ without using them.

We shall consider theories characterized by an interaction Hamiltonian density $h_D^I(x)$ which is a bounded quasilocal self-adjoint operator, i.e.

(i) $\|h_D^I(x)\| = |\lambda| \cdot M < \infty$, $h_D^I(x)^* = h_D^I(x)$

(ii) $[h_D^I(\mathrm{x}, t), h_D^I(\mathrm{y}, t)] = 0$ provided $|\mathrm{x} - \mathrm{y}| \geqq b = \text{const.}$

(iii) $[h_D^I(x), A] = 0$ provided $A \in R(\mathcal{O})$ and the space-like distance from $\mathcal{O}$ to x is greater than b.

(iv) $h_D^I(\mathrm{x}, t)$ is a C^∞ op.-function in x (5.1)

(this last postulate is not necessary, it only makes the demonstration easier. The reader can easily remove it; he will only have to add a good deal of epsilontics in the proof.)

It is clear that everything is well defined if one is willing to put the system into a finite box, but then there is no way of showing that it is possible to remove the box from the solution. If one does not take a box, the $H^I_D(t)$ is not defined, so that a perturbation expansion of the type (3.5) does not even make sense term by term.

Theorem 5.1. Let a theory be characterized by interaction Hamiltonian density which is bounded and quasi-local (as defined above). Then the perturbative expansion (3.7) is defined term by term in any number of space dimensions, and is absolutely convergent, with finite radius of convergence in the case of one space-dimension.

Proof. In the Heisenberg picture, the perturbation expansion (3.7) can be written as[4]

$$\begin{aligned} A_H(t) &= A_D(t) + i\int_0^t d\tau\,[H^I_D(t-\tau), A_D(t)] + \\ &\quad + i^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [H^I_D(t-\tau_1), [H^I_D(t-\tau_2), A_D(t)]] \qquad (5.2) \\ &\quad + \cdots \\ &= A_D(t) + (-i)\int_0^t d\tau\,[H^I_D(\tau), A_D(t)] + \\ &\quad + (-i)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 [H^I_D(\tau_1), [H^I_D(\tau_1+\tau_2), A_D(t)]] + \qquad (5.3) \\ &\quad + \cdots . \end{aligned}$$

We first remark that if we define $H^I_D(\tau)_r = \int h^I_D(\mathrm{x},\tau)\,\chi_r(\mathrm{x})\,d^s\mathrm{x}$, with $\chi_r(\mathrm{x})$ being the characteristic function equal to 1 on a (s-dim.) sphere of radius r centered at the origin, and equal to 0 outside, then the above perturbation expansion (with $H^I_D(\tau)_r$) is perfectly well defined, and has an infinite radius of convergence. We can take the following estimate of the norm of $H^I_D(\tau)_r$:

$$\|H^I_D(\tau)_r\| \leqq L\cdot|\lambda|\cdot r^s\cdot M \qquad (5.4)$$

where $L = \begin{cases} 2 \\ \pi \\ 4/3\,\pi \end{cases}$ for $s = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$, λ being the coupling constant and M a constant. We shall call $K = \|A_D(0)\|$ the norm of $A_D(0)$, and assume that the support of $A_D(0)$ is contained in the double cone spanned by a sphere of radius a, centered at the origin.

We shall now take the limit $r\to\infty$ in each term of the expansion, show that each term remains well defined, and obtain an estimate of its norm.

Consider the first commutator: $[H_D^I(t-\tau)_r, A_D(t)]$. Because of the quasi-locality, this commutator no longer depends upon r when

$$r > (a + b + \tau) .$$

We can, therefore, estimate its norm to be $< 2KL|\lambda| M(a+b+\tau)^s$, and the norm of the first term of the expansion, will be majorized by

$$c_1 = 2KL|\lambda| M \int_0^t d\tau (a + b + \tau)^s \tag{5.5}$$

with a similar discussion, we get that

$$c_2 = 2^2 K L^2 \cdot |\lambda|^2 \cdot M^2 \int_0^t d\tau_1 (a + 2b + \tau_1)^s \int_0^{\tau_1} d\tau_2 (a + b + \tau_2)^s \tag{5.6}$$

and in general

$$c_n = 2L \cdot |\lambda| \cdot M \int_0^t d\tau (a + nb + \tau)^s c_{n-1}(\tau) . \tag{5.7}$$

It is then clear that every term of the perturbation expansion is well defined. We shall now study its convergence.

$$\begin{aligned} c_n &= 2L \cdot |\lambda| \cdot M (a + nb + t)^s \int_0^t c_{n-1}(\tau)\, d\tau - \\ &- 2L|\lambda| M \int_0^t d\tau_1 (a + nb + \tau_1)^{s-1} s \int_0^{\tau_1} d\tau_2 c_{n-1}(\tau_2) \leqq \\ &\leqq 2L|\lambda| M (a + nb + t)^s \int_0^t c_{n-1}(\tau)\, d\tau \end{aligned} \tag{5.8}$$

the second term being positive. $c_{n-1}(\tau)$ is a polynomial in τ of degree contained between $n-1$ and $(n-1)$ $(s+1)$, all coefficients being positive. Therefore,

$$\int_0^t c_{n-1}(\tau)\, d\tau \leqq \frac{t}{n} c_{n-1}(t) \tag{5.9}$$

and

$$c_n \leqq 2L \cdot |\lambda| \cdot M (a + nb + t)^s \cdot \frac{t}{n} c_{n-1} \tag{5.10}$$

if $s = 1$, the series converges absolutely for $t < 1/(2L|\lambda| Mb)$. QED

Therefore, at least in the case of 2-dimensional models, we are able in principle to compute exactly $\beta(t)$, i.e. the time evolution for times which are small enough. As it stands, the theory is still incomplete; we have not given any prescription to compute the transition probabilities or other observable quantities. One difficulty is the fact that the vacuum state of the interacting theory is not a normal state in the Fock space, that is, it cannot be expressed by vectors of our Hilbert space, but has to be considered only as a linear functional on the algebra $\mathfrak{A}$ (or rather

$\mathfrak{A} = \overline{\bigcup_{|t| < \infty} \beta(t)\,[\mathfrak{A}]}$). We shall deal with this and other problems in a subsequent publication (cf. footnote 8).

It is perhaps useful to end this section by showing how one can construct an arbitrary number of Hamiltonian densities satisfying our conditions (5.1). We give the following examples:

a) Let $\{R(\mathcal{O})\}$ be the ring system associated with a free field, choose $\mathcal{O}$ to be the double cone spanned by a sphere of radius $b/2$, take an $A \in R(\mathcal{O})$, let $u(x)$ be the (free) translation in the direction x, and put $h^I_D(x) = u(x)\,A\,u^{-1}(x)$.

b) Let $\phi(x)$ be a free Fermi field. Let $f(\mathrm{y})$ be a s-dimensional test function with support inside a sphere of radius $b/2$ centered at the origin. Then $\int \phi(\mathrm{y}, t)\, f(\mathrm{x} - \mathrm{y})\, d^s\mathrm{y} = \phi_f(\mathrm{x}, t)$ is a bounded operator, and any local polynomial of such smeared out fields will satisfy (5.1), for instance $h^I_D(x) = \bar{\phi}_f(x)\,\Gamma\,\phi_f(x)\,\bar{\phi}_f(x)\cdot\Gamma'\,\phi_f(x)\,.+h\,.\,c.$

c) If now $a(x)$ is a scalar field, we can define as above $a_f(x)$, but it is no longer a bounded operator. We have to introduce a second regularization. As $a_f(x)$ is essentially self-adjoint, we can write $a_f(x) = \int_{-\infty}^{+\infty} \lambda\, dE(\lambda)$. We then can $a_{fR}(x) = \int_{-R}^{+R} \lambda\, dE(\lambda)$, and any local polynomial of $a_{fR}(x)$ will again satisfy (5.1).

d) We can take any combination of a), b) and c) and get for instance a Yukawa type interaction: $h^I_D(x) = a_{fR}(x)\,\bar{\phi}_f(x)\,\phi_f(x) + h\,.\,c.$

The "cut-offs" introduced above have been chosen so that (5.1) could be satisfied. They are as unphysical as the usual cut-offs, but no more; however, if one is interested in the exact analytical form of the solution, they are very impractical.

As another application, we shall show in a coming publication with G. Velo, that the perturbation expansion (3.7) and other perturbation expansions along the same line, are perfectly well defined and convergent, in the case of a quadratic Hamiltonian, and this in any number of space dimensions.

6. Discussion[9]

We think that equations (3.7) or (5.2) are the form of perturbation expansions that are best suited for a discussion of some basic problems. We shall not aim at mathematical rigor in this discussion, but want merely to outline what are the results we may expect, and the insight which may be gained by using the present formalism.

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[9] In all following discussions, it is no longer necessary to consider $h^I_D(x)$ as a bounded operator, but one can take it as unbounded and then consider the expectation values between suitable states, and the convergence of the series of expectation values.

One fact one can wonder about, is why the radius of convergence in theorem 5.1 has to be finite. We do not think that this is simply due to the weakness of the mathematical techniques used, but that there are underlying physical reasons. Indeed, if we assume the radius of convergence to be infinite, the time evolution of a quasi local operator is then (at fixed time) an entire function of the coupling constant and this means that the theory would also exist for negative interaction Hamiltonian. We can then think of an argument of the type of DYSON's to make this last property at least very suspicious from a physical standpoint.

One can also study the support properties of the interacting field when expressed in term of the free fields. Examining the proof of Theorem 5.1, we see that its support is the entire strip contained between the hyperplanes at time $-(a+t)$ and $+(a+t)$, provided $b \neq 0$. We can ask ourselves, however, what happens when $b = 0$, that is, when the interaction Hamiltonian density is strictly local. Suppose, therefore, that $h_D^I(x)$ is strictly local, and also local with respect to the free fields. Suppose also that the perturbation expansion (5.2) is simply convergent for all finite t. Then $A_H(t)$ has its support in the double cone spanned by a sphere of radius $(a+t)$ centered at the origin. This means that for each time t, the interacting field will commute with the free field, if their spacelike distance is greater than $2t$. If the interacting field is Lorentz invariant, it will be strictly local with respect to the free field, and, therefore, belongs to its Borchers' class. This argument is independent of whether we have the renormalized or the unrenormalized series, but depends only upon the fact that it is convergent and that the conterterms (which may be infinite in number) are again local with respect to the free field and themselves.

If the above argument (of the handwaving class!) is correct, the following theorem holds:

Theorem 6.1. Let a relativistic field theory be characterized by an interaction Hamiltonian density which is local, and local with respect to the free field. If the perturbation series is convergent, then the theory has a trivial S-matrix.

We have no idea whether to expect a similar statement to be true without the hypothesis of a convergent perturbation expansion. Clearly, the example to look at is the Federbush model, the solution of which is exactly known and does not lie in the Borchers' class of the free field (cf. [1]).

7. Conclusions

We have presented here the idea that in order to bypass Haag's theorem, one should consider the time evolution as being only locally unitarily implementable and not globally. We have shown that this idea really works in some particular cases. There are, of course, still a tremendous number of problems left open, we hope, however, to be able to deal with a least some of them in a not too distant future. If one wants to study the existence of theories which are not only translation invariant, but also Lorentz invariant, one has very probably to work without perturbation expansions, because even the relatively well behaved expansion (3.7) no longer makes sense for nontrivial interactions, and the mathematical difficulties are then very great.

One may object that, after all, we have only shown explicitly that Haag's theorem could be bypassed for some translationally invariant models, and that the question could remain open say for fully relativistic models. We challenge this idea. Indeed Haag's theorem arises solely from Euclidean invariance and has, therefore, in our opinion, nothing to do with whether the theory is Lorentz invariant or not.

The idea that the time evolution need not be linked to some unitary operator is in itself not new. Segal [18] (and earlier references given in [18]) for instance, has urged for years to consider the Poincaré group and all symmetries as being simply algebraic automorphisms of some abstract C^*-algebra. To our knowledge, however, he has never presented any program for the computation of these automorphisms, or to prove their existence in some model.

There is also a formal analogy between our treatment of the time evolution and some recent treatments of so called broken symmetries, for instance [19], [20] and [21], but we do not think there is anything deep in this analogy.

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