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A Theorem on Canonical Commutation and Anticommutation Relations

G.-F. Dell'Antonio*
Institut des Hautes Etudes Scientifiques

S. Doplicher
Istituto di Fisica dell'Università, Roma
and Institut des Hautes Etudes Scientifiques

D. Ruelle
Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, (S. et O.) France

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Abstract. The aim of this note is to characterize representations of the canonical commutation or anticommutation relations which, on a subspace of the "space of test-functions", reduce to a sum of copies of the Fock representation.

1. Generalities[1]

Let $\mathscr{L}$ be a real separated prehilbert space. We assume that $\mathscr{L}$ is separable. One may in a standard way construct a complex Hilbert space $\mathscr{H}$ (Fock space) and, for each $f \in \mathscr{L}$, operators $a(f)$, $a^*(f)$ forming the Fock representation of the canonical commutation relations (CCR) or anticommutation relations (CAR) of $\mathscr{L}$.

In the case of the CAR the operators $a(f)$, $a^*(f)$ are bounded and the C^*-algebra $\mathfrak{A}$ associated with the Fock representation of the CAR is defined as the uniform closure of the algebra generated by all operators $a(f), a^*(f)$. In the case of the CCR the operators $\varphi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f))$ and $\pi(f) = \frac{1}{i\sqrt{2}}(a(f) - a^*(f))$ are self-adjoint and one may define the Weyl operators $U(f) = \exp(i\varphi(f))$, $V(f) = \exp(i\pi(f))$. The C^*-algebra $\mathfrak{A}$ associated with the Fock representation of the CCR is defined as the uniform closure of the algebra generated by all operators $U(f)$, $V(f)$. $\mathfrak{A}$ is irreducible and contains the identity operator $\mathbf{1}$ of $\mathscr{H}$.

A (CCR or CAR) representation of $\mathscr{L}$ in a complex Hilbert space $\mathfrak{H}$ is defined by a $*$-homomorphism γ of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}$ such that $\gamma(\mathbf{1})$ is the identity on $\mathfrak{H}$ and, in the case of the CCR the

* Present address: Courant Institute of Mathematical Science, N. Y. U.

[1] For a general description of CCR and CAR see Gårding and Wightman [4]; for CCR see Lew [5] and references given there to earlier work, in particular by Segal; for C^*-algebras see Dixmier [3].

functions $t \to \gamma(\exp(it\varphi(f)))$, $t \to \gamma(\exp(it\pi(f)))$ of the real variable t are strongly continuous for each $f \in \mathscr{L}$. There are then uniquely defined self-adjoint operators $\varphi_\gamma(f)$, $\pi_\gamma(f)$ on $\mathfrak{H}$ such that

$$\gamma(U(f)) = \exp(i\varphi_\gamma(f)), \quad \gamma(V(f)) = \exp(i\pi_\gamma(f)) . \tag{1}$$

A linear functional ϱ on $\mathfrak{A}$ which is positive ($\geqq 0$) and normalized ($\varrho(1) = 1$) is called a *state* on $\mathfrak{A}$. In the case of the CCR we shall always assume that ϱ is *regular* in the sense that for all A, $B \in \mathfrak{A}$ and $f \in \mathscr{L}$, the functions $\varrho(A\,U(tf)\,B)$ and $\varrho(A\,V(tf)\,B)$ of the real variable t are continuous. By the Gel'fand-Segal construction one obtains a complex Hilbert space $\mathfrak{H}$, a vector $\Omega \in \mathfrak{H}$ such that $\|\Omega\| = 1$ and a $*$-homomorphism γ of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}$ satisfying the properties indicated above, Ω is cyclic with respect to $\gamma(\mathfrak{A})$ and for all $A \in \mathfrak{A}$ one has

$$\varrho(A) = (\Omega, \gamma(A)\Omega) . \tag{2}$$

A state ϱ on $\mathfrak{A}$ defines thus a cyclic representation of $\mathscr{L}$. Conversely a cyclic representation of $\mathscr{L}$ in $\mathfrak{H}$, defined by a $*$-homomorphism γ of $\mathfrak{A}$ and a normalized vector Ω cyclic with respect to $\gamma(\mathfrak{A})$, yields a state ϱ by (2), and ϱ determines the representation within unitary equivalence (uniqueness of the Gel'fand construction). Although $\mathscr{L}$ is assumed to be separable $\mathfrak{H}$ will in general not be separable ($\mathfrak{A}$ is in general not norm-separable in the case of the CCR).

If $\Phi \in \mathscr{H}$ and $\|\Phi\| = 1$ the state ω_Φ on $\mathfrak{A}$ defined by

$$\omega_\Phi(A) = (\Phi, A\Phi) \quad \text{for all} \quad A \in \mathfrak{A} \tag{3}$$

is called a *vector state*. The Gel'fand representation constructed from ω_Φ is again the Fock representation.

If σ is a *density matrix* (i.e. a positive ($\geqq 0$) operator with trace 1) on $\mathscr{H}$, the state ϱ_σ on $\mathfrak{A}$ defined by

$$\varrho_\sigma(A) = \mathrm{Tr}(\sigma A) \quad \text{for all} \quad A \in \mathfrak{A} \tag{4}$$

is called a *normal state* (with respect to the Fock representation).

Let a (CCR or CAR) representation of $\mathscr{L}$ be defined on a Hilbert space $\mathfrak{H}$ by a $*$-homomorphism δ of $\mathfrak{A}$ into the bounded operators on $\mathfrak{H}$. If we can write $\mathfrak{H}$ as the completed tensor product

$$\mathfrak{H} = \mathfrak{H}_1 \mathbin{\overline{\otimes}} \mathfrak{H}_2 \tag{5}$$

of two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in such a way that

$$\delta = \delta_1 \otimes \mathbf{1}_{\mathfrak{H}_2} \tag{6}$$

and if the representation defined by δ_1 on $\mathfrak{H}_1$ is the Fock representation (i.e. if δ_1 is implemented by an isometry of $\mathscr{H}$ onto $\mathfrak{H}_1$) our original representation will be called *normal*. By choosing an orthonormal basis in $\mathfrak{H}_2$ one sees that *a normal representation is the same thing as a direct sum of copies of the Fock representation.*

Lemma 1. *The Gel'fand representation constructed from a normal state* ϱ_σ *on* $\mathfrak{A}$ *is normal.*

Using the spectral decomposition of σ we may write for all $A \in \mathfrak{A}$

$$\varrho_\sigma(A) = \sum_n c_n \omega_{\Psi_n}(A) \tag{7}$$

where the Ψ_n are orthonormal vectors of $\mathscr{H}$. Let the normalized vector $\Psi \in \mathscr{H} \overline{\otimes} \mathscr{H}$ be defined by

$$\Psi = \sum_n c_n^{\frac{1}{2}} \Psi_n \otimes \Psi_n \tag{8}$$

and let the *-homomorphism δ of $\mathfrak{A}$ into the bounded operators of $\mathscr{H} \overline{\otimes} \mathscr{H}$ be defined by

$$\delta(A) = A \otimes \mathbf{1}\,. \tag{9}$$

We have then for all $A \in \mathfrak{A}$

$$\varrho_\sigma(A) = (\Psi, \delta(A)\Psi)\,. \tag{10}$$

Let $\mathfrak{H}_\Psi$ be the closure of $\delta(\mathfrak{A})\Psi$ in $\mathscr{H} \overline{\otimes} \mathscr{H}$. The projection E on $\mathfrak{H}_\Psi$ commutes with $\delta(\mathfrak{A}) = \mathfrak{A} \otimes \mathbf{1}$ and, since $\mathfrak{A}$ is irreducible, it is of the form $E = \mathbf{1} \otimes E_0$ where E_0 is a projection in $\mathscr{H}$. By the uniqueness of the Gel'fand construction, the Gel'fand representation constructed from ϱ_σ is defined by the restriction δ_0 of δ to $\mathfrak{H}_\Psi$. Lemma 1 follows then from the definition of a normal representation and the relations

$$\mathfrak{H}_\Psi = \mathscr{H} \overline{\otimes} E_0 \mathscr{H} \tag{11}$$

$$\delta_0(A) = A \otimes \mathbf{1}_0 \quad \text{for all} \quad A \in \mathfrak{A} \tag{12}$$

where $\mathbf{1}_0$ is the identity in $E_0\mathscr{H}$.

2. Number operators

Let γ be a *-homomorphism of $\mathfrak{A}$ into the bounded operators of a complex Hilbert space $\mathfrak{H}$ defining a (CCR or CAR) representation of $\mathscr{L}$. If $f \in \mathscr{L}$ we write

$$\begin{aligned} b(f) &= \frac{1}{\sqrt{2}}(\varphi_\gamma(f) + i\pi_\gamma(f)) \quad &\text{(CCR)} \\ b(f) &= \gamma(a(f)) \quad &\text{(CAR)} \end{aligned} \tag{13}$$

where $\varphi_\gamma, \pi_\gamma$ are given by (1). If $\|f\| = 1$ a number operator $N(f) = b(f)^* b(f)$ is defined on $\mathfrak{H}$, $N(f)$ is self-adjoint with spectrum constituted by the non-negative integers (CCR) or 0 and 1 (CAR). We note $E_n(f)$ the projection on the subspace corresponding to the eigenvalue n of $N(f)$ so that

$$N(f) = \sum_n n E_n(f)\,, \quad \mathbf{1} = \sum_n E_n(f)\,. \tag{14}$$

Let (f_m) be an orthonormal basis of $\mathscr{L}$. If $\mathbf{n} = (n_m)$ is a family of non-negative integers (CCR) or elements of $\{0, 1\}$ (CAR) such that $|\mathbf{n}|$

$= \sum_m n_m < +\infty$ we define

$$E_{\mathbf{n}} = \prod_m E_{n_m}(f_m) \tag{15}$$

$$E_n = \sum_{|\mathbf{n}| = n} E_{\mathbf{n}} . \tag{16}$$

If the ranges of the orthogonal projections E_n span $\mathfrak{H}$ we define a self-adjoint operator

$$N = \sum_n n E_n \tag{17}$$

and we have in the sense of strong convergence on the domain of N:

$$N = \sum_m N(f_m) . \tag{18}$$

We shall say that *the representation has a total number operator* N if

1. In the case of the CAR, *N exists for one choice of the orthonormal basis* (f_m)

2. In the case of the CCR, *N exists and is the same for every choice*[2] *of the orthonormal basis* (f_m).

Otherwise we shall say that there is no total number operator.

Lemma 2.[3] *A representation of $\mathscr{L}$ on $\mathfrak{H}$ is normal if and only if it has a total number operator N.*

A total number operator is defined for the Fock representation and therefore also for a normal representation.

To prove the converse we first show that, if N exists for one orthonormal basis (f_m) of $\mathscr{L}$, the restriction of the representation to the subspace $\mathscr{L}_0$ of $\mathscr{L}$ generated by finite linear combinations of the f_m is normal. We assume thus that the ranges $\mathfrak{H}_{\mathbf{n}}$ of the projections $E_{\mathbf{n}}$ span $\mathfrak{H}$.

Let $\mathbf{n}$, $\mathbf{n}'$ be such that $n_m - n'_m = \delta_{m m_0}$, then the CCR or CAR show that $b(f_{m_0})\mathfrak{H}_{\mathbf{n}} \subset \mathfrak{H}_{\mathbf{n}'}$, $b(f_{m_0})^* \mathfrak{H}_{\mathbf{n}'} \subset \mathfrak{H}_{\mathbf{n}}$. In fact $\subset$ may be replaced by $=$ in these relations because $n_m^{-1} b(f_{m_0})^* b(f_{m_0})$ reduces to the identity on $\mathfrak{H}_{\mathbf{n}}$ and $n_m^{-1} b(f_{m_0}) b(f_{m_0})^*$ reduces to the identity on $\mathfrak{H}_{\mathbf{n}'}$. In particular, every vector in $\mathfrak{H}_{\mathbf{n}}$ is of the form $M\Psi$ where M is a monomial in the $b(f_m)^*$ and $\Psi \in \mathfrak{H}_{\mathbf{0}}$.

Let (Ψ_α) be an orthonormal basis of $\mathfrak{H}_{\mathbf{0}}$ and let $\mathfrak{H}_\alpha$ be the subspace of $\mathfrak{H}$ spanned by vectors of the form $P\Psi_\alpha$ where P is a polynomial in the $b(f_m)^*$. The spaces $\mathfrak{H}_\alpha$ are orthogonal and the above remarks show that they span $\mathfrak{H}$.

By reference to a standard construction of the Fock space and Fock representation one sees that the representation of $\mathscr{L}_0$ on $\mathfrak{H}$ defined by γ reduces to the Fock representation on each $\mathfrak{H}_\alpha$, it is thus a sum of copies of the Fock representation, i.e. normal.

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[2] The authors are thankful to I. Segal for pointing out that Lemma 2 is false if one only assumes the existence of N for *one* basis, counterexamples have been constructed by J. Chaiken (private communication).

[3] See GÅRDING and WIGHTMAN [4] and WIGHTMAN and SCHWEBER [7].

In the case of the CAR, the fact that the representation is normal when restricted to $\mathscr{L}_0$ implies that it is normal because $\mathscr{L}_0$ is dense in $\mathscr{L}$ and $f \rightharpoonup a(f)$ is continuous (see next footnote).

In the case of the CCR, our assumptions imply that for any $f \in \mathscr{L}$, $a(f)^* a(f)$ vanishes on $\mathfrak{H}_0$, hence $a(f)\Psi_\alpha = 0$ for all α. Using the commutation relations shows then that the matrix elements of $a(f)$ are those of the Fock representation.

Remark 1. Since the construction of the Fock representation of $\mathscr{L}$ is independent of the choice of a basis in $\mathscr{L}$, Lemma 2 shows that the existence and definition of a total number operator for the CAR are also independent of the choice of a basis.

Remark 2. For a representation of the CAR without total number operator, let $\mathfrak{H}'$ be the subspace of $\mathfrak{H}$ spanned by the ranges $\mathfrak{H}_n$ of the projections E_n. The above proof shows that the representation leaves $\mathfrak{H}'$ stable and that its restriction to $\mathfrak{H}'$ is normal.

3. Normalcy of an induced representation

Let $\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2$ where $\mathscr{L}_1$ is a real Hilbert space and $\mathscr{L}_2$ a real prehilbert space[4]. We let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be the C^*-algebras associated with the Fock representations of $\mathscr{L}_1$ resp. $\mathscr{L}_2$ in the Fock spaces $\mathscr{H}_1$ resp. $\mathscr{H}_2$ and $\tilde{\mathfrak{A}}_1$, $\tilde{\mathfrak{A}}_2$ be the C^*-subalgebras of $\mathfrak{A}$ generated by the $U(f)$, $V(f)$ (CCR) or the $a(f)$, $a^*(f)$ (CAR) with $f \in \mathscr{L}_1$ resp. $f \in \mathscr{L}_2$. One can identify naturally the Fock space $\mathscr{H}$ of $\mathscr{L}$ with the completed tensor product of $\mathscr{H}_1$ and $\mathscr{H}_2$:

$$\mathscr{H} = \mathscr{H}_1 \mathbin{\overline{\otimes}} \mathscr{H}_2 \tag{19}$$

in such a way that $\mathfrak{A}_1 \otimes \mathbf{1}_2$ is identified with $\tilde{\mathfrak{A}}_1$. In the case of the CCR, $\mathbf{1}_1 \otimes \mathfrak{A}_2$ is also identified with $\tilde{\mathfrak{A}}_2$ (but this is not so for the CAR since $\tilde{\mathfrak{A}}_1$, $\tilde{\mathfrak{A}}_2$ do not commute). Notice that it follows from our definitions that the finite sums $\sum_i A_1^i A_2^i$ with $A_1^i \in \tilde{\mathfrak{A}}_1$, $A_2^i \in \tilde{\mathfrak{A}}_2$ are uniformly dense in $\mathfrak{A}$.

Let a representation of $\mathscr{L}$ be defined by a $*$-homomorphism γ of $\mathfrak{A}$ into the bounded operators in a complex Hilbert space $\mathfrak{H}$, and let γ' be the $*$-homomorphism of $\mathfrak{A}_1$ defined by $\gamma'(A) = \gamma(A \otimes \mathbf{1}_2)$. We say that the representation of $\mathscr{L}_1$ on $\mathfrak{H}$ defined by γ' is the representation *induced* by the above one on $\mathscr{L}_1$.

We consider now to the case where the representation defined by γ is the Gel'fand representation constructed from a state ϱ on $\mathfrak{A}$ (in the case of the CCR we assume as usual that ϱ is regular). There exists then

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[4] In the case of the CAR, we might without loss of generality assume that $\mathscr{L}_2$ (and therefore $\mathscr{L}$) is complete (cf. [2]). We have indeed by virtue of the CAR, $\|a(f)\| = \|a^*(f)\| = \|f\|$; hence $a(f)$ and $a^*(f)$ are continuous in f and the C^*-algebra $\mathfrak{A}_2$ generated by the $a(f)$, $a^*(f)$ with $f \in \mathscr{L}_2$ is identical to the C^*-algebra generated by the $a(f)$, $a^*(f)$ with f in the completion $\bar{\mathscr{L}}_2$ of $\mathscr{L}_2$.

a normalized vector $\Omega \in \mathfrak{H}$, cyclic with respect to $\gamma(\mathfrak{A})$ and such that

$$\varrho(A) = \omega_\Omega(\gamma(A)) = (\Omega, \gamma(A)\Omega) \tag{20}$$

for all $A \in \mathfrak{A}$. With these notations we prove

Theorem. *The condition*

A. *If the projections E'_n are defined by* (16) *for the representation defined by the $*$-homomorphism γ', then*

$$\sum_n \omega_\Omega(E'_n) = 1 \tag{21}$$

is implied by and, in the case of the CAR, *equivalent to, the following equivalent conditions*

B. *The representation of $\mathscr{L}_1$ on $\mathfrak{H}$ defined by γ' has a total number operator N'.*

C. *The representation of $\mathscr{L}_1$ on $\mathfrak{H}$ defined by γ' is normal.*

D. *There exists a density matrix σ on $\mathscr{H}_1$ such that*

$$\varrho(A \otimes \mathbf{1}_2) = \omega_\Omega(\gamma'(A)) = \mathrm{Tr}(\sigma A) = \varrho_\sigma(A) \tag{22}$$

for all $A \in \mathfrak{A}_1$.

The implication $B \Rightarrow A$ is obvious, and the equivalence $B \Leftrightarrow C$ follows directly from Lemma 2. We prove now successively $A \Rightarrow B$ (for CAR), $C \Rightarrow D$, $D \Rightarrow B$.

$A \Rightarrow B$ (CAR)

Equation (21) expresses that Ω is contained in the subspace $\mathfrak{H}'$ of $\mathfrak{H}$ spanned by the ranges $\mathfrak{H}'_n$ of the projections E'_n. We have to prove that $\mathfrak{H}' = \mathfrak{H}$, or equivalently that $\gamma(\mathfrak{A})\,\Omega \subset \mathfrak{H}'$ or, since the finite sums $\sum_i A_1^i A_2^i$ with $A_1^i \in \tilde{\mathfrak{A}}_1$, $A_2^i \in \tilde{\mathfrak{A}}_2$ are uniformly dense in $\mathfrak{A}$, that $\gamma(\tilde{\mathfrak{A}}_1) \times \gamma(\tilde{\mathfrak{A}}_2)\,\Omega \subset \mathfrak{H}'$. Since $\Omega \in \mathfrak{H}'$ and $\gamma(\tilde{\mathfrak{A}}_2)\,\mathfrak{H}'_n \subset \mathfrak{H}'_n$ it remains to check that $\gamma(\tilde{\mathfrak{A}}_1)\,\mathfrak{H}' = \gamma(\mathfrak{A}_1 \otimes \mathbf{1}_2)\mathfrak{H}' = \gamma'(\mathfrak{A}_1)\,\mathfrak{H}' \subset \mathfrak{H}'$, but this follows from Remark 2 at the end of Section 2.

$C \Rightarrow D$

By definition, condition B means that one can write $\mathfrak{H} = \mathfrak{H}_1 \bar{\otimes} \mathfrak{H}_2$ and $\gamma' = \gamma'_1 \otimes \mathbf{1}_{\mathfrak{H}_2}$ where γ'_1 is implemented by an isometry W of $\mathscr{H}_1$ onto $\mathfrak{H}_1$. Let (Ψ_α) be an orthonormal basis of $\mathfrak{H}_2$, then an isometry W_α of $\mathscr{H}_1$ onto $\mathfrak{H}_1 \otimes \Psi_\alpha$ is defined by $W_\alpha \Psi = W\Psi \otimes \Psi_\alpha$. Let $c_\alpha^{1/2}\Omega_\alpha$ be the component of Ω in $\mathfrak{H}_1 \otimes \Psi_\alpha$, where $\|\Omega_\alpha\| = 1$; we have then

$$\sum_\alpha c_\alpha = 1 \tag{23}$$

and for all $A \in \mathfrak{A}_1$

$$\begin{aligned}\varrho(A \otimes \mathbf{1}_2) = \omega_\Omega(\gamma'(A)) &= \sum_\alpha c_\alpha \omega_{\Omega_\alpha}(W_\alpha A W_\alpha^{-1}) \\ &= \sum_\alpha c_\alpha (W_\alpha^{-1}\Omega, A W_\alpha^{-1}\Omega_\alpha)\,.\end{aligned} \tag{24}$$

If E_α is the projection on $W_\alpha^{-1}\Omega_\alpha$ in $\mathscr{H}_1$, (23) implies that

$$\sigma = \sum_\alpha c_\alpha E_\alpha \tag{25}$$

is a density matrix and D follows from (24) and (25).

$D \Rightarrow B$

Let $\mathfrak{H}_\Omega$ be the closure in $\mathfrak{H}$ of $\gamma'(\mathfrak{A}_1)\Omega$. By the uniqueness of the Gel'fand construction, the restriction to $\mathfrak{H}_\Omega$ of the representation defined by γ' in $\mathfrak{H}$ is, by (22), identical to the Gel'fand representation constructed from the normal state ϱ_σ on $\mathfrak{A}_1$ and thus normal by Lemma 1. This restricted representation has thus a total number operator N''. From this follows the existence of a total number operator N' for the representation defined by γ' in $\mathfrak{H}$: if $\Psi \in \mathfrak{H}_\Omega$ and $A_2 \in \mathfrak{A}_2$, then $N'\,\gamma(A_2)\Psi = \gamma(A_2)N''\Psi$.

Remark. In the case of the CCR, C may be rewritten as C'. *One may write* $\mathfrak{H} = \mathfrak{H}_1 \mathbin{\overline{\otimes}} \mathfrak{H}_2$ *and there exist *-homomorphisms* γ_1', γ_2' *of* $\mathfrak{A}_1$, *resp.* $\mathfrak{A}_2$ *into the bounded operators on* $\mathfrak{H}_1$ *resp.* $\mathfrak{H}_2$ *such that*

$$\gamma(A_1 \otimes A_2) = \gamma_1'(A_1) \otimes \gamma_2'(A_2)$$

and γ_1' *is implemented by an isometry of* $\mathscr{H}_1$ *onto* $\mathfrak{H}_1$.

It is clear that $C' \Rightarrow C$. To prove that $C \Rightarrow C'$ we note that C implies the existence of the decomposition $\mathfrak{H} = \mathfrak{H}_1 \mathbin{\overline{\otimes}} \mathfrak{H}_2$ and of γ_1' such that $\gamma' = \gamma_1' \otimes \mathbf{1}_{\mathfrak{H}_2}$ and γ_1' is implemented by an isometry of $\mathscr{H}_1$ onto $\mathfrak{H}_1$. Since $\gamma(\mathbf{1}_1 \otimes \mathfrak{A}_2)$ is in the commutant of $\gamma(\mathfrak{A}_1 \otimes \mathbf{1}_2) = \gamma'(\mathfrak{A}_1) = \gamma_1'(\mathfrak{A}_1) \otimes \mathbf{1}_{\mathfrak{H}_2}$ and since $\gamma_1'(\mathfrak{A}_1)$ is irreducible, every operator $\gamma(\mathbf{1}_1 \otimes A_2)$ with $A_2 \in \mathfrak{A}_2$ is of the form $\mathbf{1}_1 \otimes \gamma_2'(A_2)$, which concludes the proof.

4. Physical interpretation

The mathematical situation described by the theorem of Section 3 is of interest in the study of quantum mechanical systems with an infinite number of degrees of freedom. For instance, if $\mathscr{L}$ is the space of real square-integrable functions with compact support in R^ν, ϱ may be taken to be the expectation value functional describing the state of an infinite system of bosons (CCR) or fermions (CAR) in thermodynamic equilibrium in R^ν (see [1], [2] and [6]). Let then $\mathscr{L}_1$ be the space of real square-integrable functions on a bounded (measurable) subset Λ of R^ν. The restriction of ϱ to $\tilde{\mathfrak{A}}_1 = \mathfrak{A}_1 \otimes \mathbf{1}_2$ will describe the particles contained in the region Λ. Condition B (or A: in the case of CAR they are equivalent) expresses that the probability of finding an infinite number of particles simultaneously in Λ vanishes. This condition is always satisfied for particles with hard cores; in general its violation would correspond to a catastrophic behaviour of the system from the thermodynamic point of view. The theorem tells us then that the restriction of ϱ to the region Λ (i.e. to $\mathfrak{A}_1 \otimes \mathbf{1}_2$) is given by a density matrix. For more details see [6].

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