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On the Structure of the von Neumann Algebras Generated by Local Functions of the Free Bose Field

By

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Abstract. It is shown that the von Neumann algebra $R_{\mathfrak{B}}(B)$ generated by any scalar local function $B(x)$ of the free field $A_0(x)$ is equal either to $R_{\mathfrak{B}}(A_0)$ or to $R_{\mathfrak{B}}(:A_0^2:)$. The latter statement holds if the state space $\mathfrak{H}_B$ obtained from the vacuum state by repeated application of $B(x)$ is orthogonal to the one particle subspace. In the proof of these statements, space-time limiting techniques are used.

§ 1. Introduction

Von Neumann algebras of local observables have been introduced into relativistic quantum theory by R. HAAG [1]. A detailed study was made notably by H. ARAKI [2] in a series of papers. ARAKI has shown that most of the rigorous results of general quantum field theory can be obtained in the framework of local observables.

One motivation for introducing these objects is therefore similar to the motivation of introducing algebraic concepts into ordinary quantum mechanics, namely to investigate general structural properties. In quantum mechanics, these algebraic concepts are not of much help in the discussion of a concrete dynamical problem and similarly in relativistic quantum theory, one would expect dynamical laws to be simple only in terms of unbounded field operators associated with (bounded) local observables. From this suggestion of Lagrangian field theory, dynamical equations should have the form of local nonlinear equation of motion in these fields. However, powers in the field operator, viz. $A^3(x)$, cannot be dealt with naively. A well known discussion [3] of two-point functions shows that any Lorentz-covariant field is necessarily an operator-valued distribution. Ignoring this problem, one runs into the well known trouble of ultraviolet divergences. The conjectured remedy [4] for this trouble is a careful treatment of the nonlinear term as a local function obtained by delicate space time limiting procedures. This has indeed turned out to be true in certain soluble models [5]. Little is known about the formulation of such a procedure in the general case. It is in this area that one expects

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the local rings to be of practical use and this was indeed another motivation for their introduction. Using these objects, one obtains a very lucid definition of local functions of a local field: A local field $B(x)$ is called a local function of a local field $A(x)$ if the von Neumann algebras generated by B are always contained in those generated by $A(x)$, i.e.

$$R_{\mathfrak{B}}(B) \subseteq R_{\mathfrak{B}}(A)$$

for arbitrary space time region $\mathfrak{B}$. For example a Wick polynomial in the free field

$$B(x) = \sum_n c_n \ A_0^n : (x)$$

can be shown to be a local function in this sense. It is much less trivial to prove that the converse is also true. The statement

$$R_{\mathfrak{B}}(A_0) \subseteq R_{\mathfrak{B}}(B)$$

can be shown for any Wick polynomial which contains at least one odd power. The space-time limiting techniques for proving this statement are less trivial than those which are involved in the definition of Wick powers [6]. Both statements can be combined and generalized to the following structure theorem on local functions of the free field:

$$R_{\mathfrak{B}}(\ ) = \begin{cases} R_{\mathfrak{B}}(A_0) & \text{if } B \text{ is irreducible in } \mathfrak{H}_{A_0} \\ R_{\mathfrak{B}}(:A_0^2:) & \text{if } B \text{ is reducible in } \mathfrak{H}_{A_0}. \end{cases}$$

Here $\mathfrak{H}_{A_0}$ is the Fock space generated by the free field $A_0(x)$. The second case happens if and only if the Wick polynomial B contains only even powers. In the latter case it would be more appropriate to formulate the equality on the state space $\mathfrak{H}_{:A_0^2:}$ which is cyclically generated from the vacuum by applying the $:A_0^2:$ algebra. Although this structural simplicity of different fields in terms of their von Neumann algebras is surprising from a mathematical point of view, exercises on free fields and their trivial modifications are bound to be physically uninteresting. The only justification for this investigation is our belief that the space-time limiting techniques used in the proof have some bearing on realistic cases i.e. on the problem of definition of currents in field equations.

We are able to prove the mentioned ring-theoretical statements without using self-adjointness for the Wick polynomials of larger than second degree*.

In order to achieve this, we had to study some intricate properties of local von Neumann algebras generated by fields. The relevant theorems (which are of a fairly general nature) are derived in the next section. In the

* In the light of recent investigations of A. S. WIGHTMAN the self adjointness of Wick-powers of higher than second degree seems to be very doubtful (private communication).

third section we briefly outline the space-time limiting techniques and derive the previously indicated ring-theoretical structure theorems. The general idea here is to invert a relationship of the type $B(x) — A_0^3$ with respect to A_0, i.e. to give an operator analogy to the classical cube root inversion $A_0 = \sqrt[3]{B}$. We discuss this problem for all Wick polynomials without derivatives.

The fourth section contains a detailed proof of the convergence of the space-time limiting procedure. In the concluding remarks we will mention some statements which can be obtained by using techniques analogous to the ones outlined in this paper.

§ 2. **Definitions** and preliminary discussions

If A is a field defined on a basic domain $\mathfrak{D}$ [7] on which the operators A (/) are symmetric, one defines $S_{\mathfrak{B}}(A)$ for an open region of space-time 23 as the set of all bounded operators P such that

$$(\Psi,\ PA(f)\Phi) = (A(f)\Psi, P\Phi) \tag{2.1}$$

for all Φ, $\Psi \in \mathfrak{D}$ and for all / $\in \mathscr{D}_{(\mathfrak{B})}$ (the set of all functions of *2* having support in 93). This is not necessarily an algebra although it is a linear space closed in the topology of weak convergence and stable with respect to the operation $P \to P^*$. It contains the identity for every 23, so it is non-empty, and its commutant

$$R_{\mathfrak{B}}(A) = S_{\mathfrak{B}}(A)'$$

is the von Neumann algebra (also called local ring) associated with the region 23. It can be proved, using a technique of Reeh and Schlieder [8] that (2.1) is equivalent to

$$P\overline{A(f)} \subseteqq A(f)^* P\,. \tag{2.2}$$

It is easy to see that (2.2) implies (2.1) conversely, if P satisfies (2.1), then

$$(\{0, P^*\Psi\},\{\Phi, A(f)\Phi\}) = (\{\Psi, A(f)\Psi\},\{0, P\Phi\}) \tag{2.3}$$

where, in the notation of Nagy [9], $(\{\Phi,\ \Psi\},\{\Phi,\ \Psi\}) = (\Phi,\ \Phi) + (\Psi,\ \Psi)$ is the inner product in the space $\mathfrak{H}$ x $\mathfrak{H}$. If we define,

$$B\{\Phi,\ \Psi\} = \{0,\ P^*\Phi\}, \qquad C\{\Phi,\ \Psi\} = \{0,\ P\Phi\},$$

then B and C are bounded and therefore continuous operators, and (2.3) reads

$$(B\psi,\ \phi) = (\psi,\ C\phi)$$

for $\psi = \{\Psi, A(f)\Psi\}$,$\varphi = \{\Phi, A(f)\Phi\}$. This equation is valid for all $\phi,\ \psi \in \mathfrak{B}(A(f))$, the graph of $A(f)$. Since both sides are continuous functions of φ and ψ, the equation holds for all $\varphi,\ \psi \in 93\overline{(A(/))} = 23(\overline{A(/)})$

and thus (2.1) is extended to

$$(\Psi, P\overline{A(f)}\Phi) = (\overline{A(f)}\Psi, P\Phi) \qquad \overline{(2.1)}$$

forall $\Psi, \Phi \in \mathfrak{D}(\overline{A(f)})$. But this is simply the condition that $P\Phi \in$ © $(\overline{A(f)}^*)$

and that $\overline{A(f)}^* P\Phi = P\overline{A(f)}\Phi$. Since this holds for any $\Phi \in \mathfrak{D}(\overline{A(f)})$, we have the relation

$$P\overline{A(f)} \subseteqq \overline{A(f)}^* P$$

which implies (2.2) since $\overline{A(f)}^* = A(f)^{***} = A(f)^*$.

It is seen from (2.2) that if A (/) is essentially self-adjoint on $\mathfrak{D}$, then $\overline{A(f)} = A$ (/)*, which is self adjoint, and P$\overline{A(f)}$ £ $\overline{A(f)}P$. This means that P commutes with all spectral projectors $E_\lambda^{\overline{A}(f)}$ of $\overline{A(f)}$, and that if one defines alternatively the von Neumann algebra $R_{\mathfrak{B}}(\hat{A})$ to be the one generated by all spectral projectors of $\overline{A(f)}$for / $\in \mathscr{D}_{(\mathfrak{B})}$, then $S_{\mathfrak{B}}(A)$ C £ $R'_{\mathfrak{B}}(\hat{A})$. The converse inclusion is trivial so that by taking complements, one obtains

$$R_{\mathfrak{B}}(A) = R_{\mathfrak{B}}(\hat{A}) \,. \qquad (2.4)$$

In this case,

$$S_{\mathfrak{B}}(A) = R_{\mathfrak{B}}(A)', \qquad (2.5)$$

which is an algebra.

If two fields A and B have the same domain $\mathfrak{D}$, then one says A is a local function of B: $A \sqsubset B$ if for every open region 23, $R_{\mathfrak{B}}(A) \subsetneq R_{\mathfrak{B}}(B)$. For use in the following sections, we prove the

Linearity Principle: If A, B, $C$are fields having the same basic domain $\mathfrak{D}$ on which A and B are essentially self-adjoint, and if $A \sqsubset C$, $B \sqsubset C$, then $\lambda A + \mu B \sqsubset C$.

Proof: $S_{\mathfrak{B}}(C) \varsubsetneq R_{\mathfrak{B}}(C)'$. Since $R_{\mathfrak{B}}(A) \subsetneq R_{\mathfrak{B}}(C)$, $R_{\mathfrak{B}}(C)' \subsetneq R_{\mathfrak{B}}(A)'$ and since A is essentially self adjoint on the domain $\mathfrak{D}$, $R_{\mathfrak{B}}(A)' = S_{\mathfrak{B}}(A)$. These three inclusions yield $S_{\mathfrak{B}}(C) \subsetneq S_{\mathfrak{B}}(A)$ and similarly $S_{\mathfrak{B}}(C) \subsetneq S_{\mathfrak{B}}(B)$. It is clear that $S_{\mathfrak{B}}(A) \cap S_{\mathfrak{B}}(B)$ C $S_{\mathfrak{B}}(\lambda A + \mu B)$ and therefore

$$S_{\mathfrak{B}}(C) \text{ S } S_{\mathfrak{B}}(A) \cap S_{\mathfrak{B}}(B) \subseteqq S_{\mathfrak{B}}(\lambda A + \mu B)$$

which, on passage to commutants, yields

$$R_{\mathfrak{B}}(C) \supseteqq R_{\mathfrak{B}}(\lambda A + \mu B)$$

or

$$\lambda A + \mu B \sqsubset C \,.$$

This principle will usually be applied in the form of a subtraction, and often in the case that $A \sqsubset B$ where one concludes that $A - B \sqsubset B$. A simple remark that will be usefullater is the following: If $\varkappa' : \mathscr{D} \to \mathscr{D}$ is a continuous linear mapping which leaves $\mathscr{D}_{(\mathfrak{B})}$ invariant for each $\mathfrak{B}$, and one defines a new field $\varkappa$ *Am* terms of A by the formula

$$(\varkappa A)(f) = A(\varkappa' f) \,,$$

then $\varkappa A$ is a local field which is a local function of A. If, in particular, $\varkappa = \mathrm{D} = \partial^\mu \partial_\mu$, then $\varkappa = \varkappa'$, and

$$\Box A \sqsubset A ; \quad (\Box + m^2) A \sqsubset A .$$

In order to consider the situation in which the vacuum is not cyclic (for example $:A^{2n}:$), we prove the following

Theorem: *If $A(f)$ $(f \in \mathscr{D})$ has a self adjoint extension $\hat{A}(f)$ and the vacuum is an analytic vector for $A(f)$ and if E_p is the operator projecting $\mathfrak{H}$ onto the closure of $\mathfrak{D}_p$, the polynomial domain of A, then*

$$E_p \hat{A}(f) \subseteq \hat{A}(f) E_p .$$

Proof: From the work of Borchers and Zimmermann [10] it follows that in $\mathfrak{D}_p$ there is a dense set $\mathfrak{D}_a$ of analytic vectors of $A(f)$. For any n and $\Phi \in \mathfrak{D}_a$,

$$\hat{A}(f)^n \Phi = A(f)^n \Phi \in \mathfrak{D}_p$$

so that $E \hat{A}(f)^n \Phi = 0$ if $E = I - E_p$; and thus for any $\Psi \in \mathfrak{H}$

$$0 = (\Psi, EA(f)^n \Phi) = \int_{-\infty}^{\infty} \lambda^n d(E\Psi, E_\lambda^{\hat{A}(f)} \Phi).$$

As in the next theorem, analyticity of Φ insures the uniqueness of the moment problem involved, and thus

$$0 = (E\Psi, E_\lambda^{\hat{A}(f)} \Phi) = (\Psi, E E_\lambda^{\hat{A}(f)} \Phi)$$

for all $\Psi \in \mathfrak{H}$, $\Phi \in \mathfrak{D}_a$, from which it follows that

$$E E_\lambda^{\hat{A}(f)} \Phi = 0$$

if $\Phi \in \mathfrak{H}_p = \overline{\mathfrak{D}}_p$, i.e. if Φ has the form $E_p \Psi$ for some $\Psi \in \mathfrak{H}$. Consequently,

$$0 = E E_\lambda^{\hat{A}(f)} E_p = E_\lambda^{\hat{A}(f)} E_p - E_p E_\lambda^{\hat{A}(f)} E_p .$$

Because of the self adjointness of the last term on the right, the term preceding it is also equal to its adjoint

$$E_\lambda^{\hat{A}(f)} E_p = (E_\lambda^{\hat{A}(f)} E_p)^* = E_p E_\lambda^{\hat{A}(f)},$$

and thus E_p commutes with $\hat{A}(f)$.

This implies that E_p is a superselecting projector (Wightman [3]) for the field A: $E_p \in S_{\mathfrak{B}}(A) \subseteq R_{\mathfrak{B}}(A)'$ for every region $\mathfrak{B}$. If the field A is essentially self adjoint and each operator is restricted to the coherent subspace $\mathfrak{H}_p$ (in which the vacuum is cyclic) then the new algebra of observables $R_{\mathfrak{B}}(A_p)$ will consist of all operators $P|_{\mathfrak{H}_p}$ where $P \in R_{\mathfrak{B}}(A)$. To prove this, we show that $S_{\mathfrak{B}}(A_p)$ is the set of all operators $E_p Q|_{\mathfrak{H}_p}$ with $Q \in S_{\mathfrak{B}}(A)$, from which the conclusion will follow. It is easy to see that any operator of this form belongs to $S_{\mathfrak{B}}(A_p)$ from the following manipulation: let Ψ, $\Phi \in \mathfrak{D}_p$, $f \in \mathscr{D}_{(\mathfrak{B})}$ then

$$(\Psi, E_p Q|_{\mathfrak{H}_p} A_p(f) \Phi)_p = (E_p \Psi, QA(f)\Phi) = (\Psi, QA(f)\Phi)$$
$$= (E_p A(f) \Psi Q|_{\mathfrak{H}_p} \Phi) = (A_p(f) \Psi E_p Q|_{\mathfrak{H}_p} \Phi)_p .$$

On the other hand, if $Q \in S_{\mathfrak{B}}(A_p)$, one may construct the operator (on $\mathfrak{H}$) $\bar{Q} = Q \circ E_p$. It is clear that $E_p \bar{Q}|_{\mathfrak{H}_p} = Q$, and we show that $\bar{Q} \in S_{\mathfrak{B}}(A)$ as follows:

$$(\Psi, \bar{Q} A(f) \Phi) = (E_p \Psi, Q E_p \hat{A}(f) \Phi)_p = (E_p \Psi, Q \hat{A}_p(f) E_p \Phi)_p .$$

Equation (2.1) holds for Ψ, Φ in the domain of $A_p(f)$ and, as a consequence of the discussion at the beginning of this section, also for Ψ, Φ in the domain of $\overline{A_p(f)}$. But since $A_p(\phi)$ is essentially self-adjoint, $\overline{A_p(f)} = \hat{A}_p(f)$. This means that in the last term on the right, the $\hat{A}_p(f)$ may be shifted to the left

$$(\Psi, \bar{Q} A(f) \Phi) = (\hat{A}_p(f) E_p \Psi, Q E_p \Psi)_p$$
$$= (E_p A(f) \Psi, \bar{Q} \Psi)_p - (A(f) \Psi, E_p \bar{Q} \Psi) = (A(f) \Psi, \bar{Q} \Phi).$$

This completes the proof that $S_{\mathfrak{B}}(A_p) — E_p \circ S_{\mathfrak{B}}(A) \circ 1_{\mathfrak{H}_p}$.

But [11] the fact that $S_{\mathfrak{B}}(A)$ is an algebra stable with respect to adjunction (which follows from essential self-adjointness of $A(f)$) implies that $(E_p \circ S_{\mathfrak{B}}(A) \circ 1_{\mathfrak{H}_p})' = E_p \circ S_{\mathfrak{B}}(A)' \circ 1_{\mathfrak{H}_p}$. According to the general definition, this yields $R_{\mathfrak{B}}(A_p) = E_p \circ R_{\mathfrak{B}}(A) \circ 1_{\mathfrak{H}_p} = R_{\mathfrak{B}}(A) \circ E_p \circ 1_{\mathfrak{H}_p} = R_{\mathfrak{B}}(A) \circ 1_{\mathfrak{H}_p}$ i.e.

$$R_{\mathfrak{B}}(A_p) = \{P \circ 1_{\mathfrak{H}_p} : P \in R_{\mathfrak{B}}(A)\} .$$

All of the assumptions above hold for $A = :A_0^2:$, and thus all results are valid in this case.

To complete the previous theorem and obtain a tool for the study of the Borchers class of a field, we prove the following

Theorem: *If a self-adjoint operator A has a dense set $\mathfrak{D}'$ of analytic vectors and for every $n \in N$ and $\Phi \in \mathfrak{D}'$*

$$(P^* \Phi, A^n \Phi) = (A^n \Phi, P \Phi) ,$$

then P comutes with A.

Proof: In terms of the spectral resolution $A = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda^A$, this equation becomes

$$\int_{-\infty}^{+\infty} \lambda^n d(P\Phi, E_\lambda^A \Phi) = \int_{-\infty}^{+\infty} \lambda^n d(\Phi, E_\lambda^A P \Phi)$$

and thus

$$\int_{-\infty}^{+\infty} \lambda^n df(\lambda) = 0 \tag{2.6}$$

with $f(\lambda) = (\Phi, [P E_\lambda^A - E_\lambda^A P] \Phi)$. The statement that $f(\lambda) - 0$ for all $\lambda \in R$ would lead to the conclusion of the theorem since, with $C = P E_\lambda^A - E_\lambda^A P$, this implies $(\Phi, C\Phi) = 0$ for all $\Phi \in \mathfrak{D}'$. Because C is a bounded operator, this bilinear functional is continuous; and this equation can be extended by continuity to the whole space. This means that $C \geqq 0$, and that C has a positive square root $D: C = D^2$, and hence $0 = (\Phi, C\Phi) = (\Phi, D^2\Phi) = (D\Phi, D\Phi) = \|D\Phi\|^2$ for all $\Phi \in \mathfrak{H}$. Since the norm is definite, $D\Phi = 0$ and $C\Phi = D(D\Phi) = 0$. Thus $PE^A_\lambda = E P^A_\lambda$ for every $\lambda \in R$ and P commutes with A.

The task now remaining is to show that analyticity of Φ implies the identical vanishing of f. This question can be reduced to one in the problem of moments. To do this, we set $f(\lambda) = g(\lambda) + ih(\lambda)$ and decompose $g = g^+ - g^-$ and $h = h^+ - h^-$ into differences of monotone increasing functions. Equation (2.6) yields, in this case, the two equations

$$s_n \equiv \int_{-\infty}^{+\infty} \lambda^n dg^+(\lambda) = \int_{-\infty}^{+\infty} \lambda^n dg^-(\lambda)$$

$$t_n \equiv \int_{-\infty}^{+\infty} \lambda^n dh^+(\lambda) = \int_{-\infty}^{+\infty} \lambda^n dh^-(\lambda)$$

for all $n \geqq 0$ in each of which, two monotone increasing, right-continuous functions are seen to have the same moments. In order to be able to conclude that $g^+(\lambda) = g^-(\lambda)$, $h^+(\lambda) = h^-(\lambda)$, and thus $f(\lambda) = 0$ for all $\lambda \in R$, it is necessary to have some properties of the sequences (s_n) and (t_n) since the moment problem is not in general determinate. For this, we determine upper bounds for the measures given by $g^\pm$ and $h^\pm$. If $\gamma = [\mu, \nu]$ is a nonempty closed interval, then we define $f[\gamma] = f(\nu) - f(\mu)$. From the definition of the total variation g^+, we have the formula for the measure of γ induced by g^+

$$g^+[\gamma] = \sup\left\{\sum_i |g[\gamma_i]| : \gamma = \bigcup_i \gamma_i\right\} \leqq \sup\left\{\sum_i |f[\gamma_i]| : \gamma = \bigcup_i \gamma_i\right\}$$

But from the definition of f, we have the inequalities

$$\begin{aligned} |f[\gamma]| &\leqq |(P^*\Phi, E^A_{[\gamma]}\Phi)| + |(E^A_{[\gamma]}\Phi, P\Phi)| \\ &= |(E^A_{[\gamma]}P^*\Phi, E^A_{[\gamma]}\Phi)| + |(E^A_{[\gamma]}\Phi, E^A_{[\gamma]}P\Phi)| \leqq \\ &\leqq \|E^A_{[\gamma]}P^*\Phi\| \; \|E^A_{[\gamma]}\Phi\| + \|E^A_{[\gamma]}P\Phi\| \; \|E^A_{[\gamma]}\Phi\| \end{aligned}$$

and the sum in the last supremum is majorized by

$$\begin{aligned} &\sum_i \|E^A_{[\gamma_i]}P^*\Phi\| \cdot \|E^A_{[\gamma_i]}\Phi\| + \sum_i \|E^A_{[\gamma_i]}P\Phi\| \cdot \|E^A_{[\gamma_i]}\Phi\| \leqq \\ &\leqq \left(\sum_i \|E^A_{[\gamma_i]}P^*\Phi\|^2\right)^{1/2} \left(\sum_i \|E^A_{[\gamma_i]}\Phi\|^2\right)^{1/2} + \\ &+ \left(\sum_i \|E^A_{[\gamma_i]}P\Phi\|^2\right)^{1/2} \left(\sum_i \|E^A_{[\gamma_i]}\Phi\|^2\right)^{1/2} \\ &= \|E^A_{[\gamma]}P^*\Phi\| \; \|E^A_{[\gamma]}\Phi\| + \|E^A_{[\gamma]}P\Phi\| \; \|E^A_{[\gamma]}\Phi\|, \end{aligned}$$

where we have used HOLDER'S inequality and the law of Pythagoras. From these inequalities, it follows that

$$\int_{-\infty}^{\infty} \phi(\lambda)\, dg^+(\lambda) \leqq (\|P^*\Phi\| + \|P\Phi\|) \left[\int_{-\infty}^{\infty} \phi(\lambda)^2 d\|E^A_\lambda \Phi\|^2\right]^{1/2}. \quad (2.7)$$

For the proof, we consider a particular integral sum $(\lambda_i \in \gamma_i)$

$$|\sum_i \phi(\lambda_i)\, g^+[\gamma_i]| \leqq \sum_i |\phi(\lambda_i)|\, \|E^A_{[\gamma_i]}\Phi\| \cdot \|E^A_{[\gamma_i]} P^* \Phi\| + $$
$$+ \sum_i |\phi(\lambda_i)|\, \|E^A_{[\gamma_i]}\Phi\| \cdot \|E^A_{[\gamma_i]} P \Phi\|$$

for which HÖLDER's inequality gives an upper bound of

$$\left[\sum_i \phi(\lambda_i)^2 \|E^A_{[\lambda_i]}\Phi\|^2\right]^{1/2} \left[\left(\sum_i \|E^A_{[\gamma_i]} P^*\Phi\|^2\right)^{1/2} + \left(\sum_i \|E^A_{[\gamma_i]} P\Phi\|^2\right)^{1/2}\right] \leqq$$
$$\leqq \left[\sum_i \Phi(\lambda_i)^2 \|E^A_{[\gamma_i]}\|^2\right]^{1/2} (\|P^*\Phi\| + \|P\Phi\|)\,.$$

Equation (2.7) is obtained by passage to the appropriate limits. The same result holds for h^+, and with $\phi(\lambda) = \lambda^n$, one derives the inequalities

$$\begin{aligned} |s_n| &= \left|\int_{-\infty}^{+\infty} \lambda^n dg^+(\lambda)\right| \leqq \left(\int_{-\infty}^{\infty} \lambda^{2n} d\|E^A_\lambda \Phi\|^2\right)^{1/2} (\|P^*\Phi\| + \|P\Phi\|) \\ &= \|A^n\Phi\| \cdot (\|P^*\Phi\| + \|P\Phi\|) \\ |t_n| &\leqq \|A^n\Phi\| \cdot (\|P^*\Phi\| + \|P\Phi\|)\,. \end{aligned} \tag{2.8}$$

We may now apply the following criterion [12] for the uniqueness of the integrating function g^+ in the problem of moments: if

$$\limsup_{n\to\infty} |s_n/n!|^{1/n} = \frac{1}{R} < +\infty\,,$$

then the function g^+ is unique. But this hypothesis is equivalent to the statement that the series

$$\sum_{n=0}^{\infty} \frac{s_n}{n!} z^n$$

has radius of convergence $R > 0$. Because of the analyticity of Φ for A, the series

$$\sum_{n=0}^{\infty} \frac{\|A^n\Phi\|}{n!} z^n$$

has a nonzero radius of convergence; and with the inequalities of (2.8), this implies that the moment problems generated by (s_n) and (t_n) are definitive. Thus $g^+ = g^-$, $h^+ = h^-$, and $f(\lambda) = 0$ for *all* $\lambda \in E$.

If P is an unbounded operator such that $\Phi \in \mathfrak{D}(P) \cap \mathfrak{D}(P^*)$, the proof is still valid up to the conclusion

$$(P^*\Psi, E^{\hat{A}(f)}_\lambda \Phi) = (E^{\hat{A}(f)}_\lambda \Psi, P\Phi)\,. \tag{2.9}$$

(The equality for $\Phi = \psi$ implies the equality for $\varphi \neq \Psi$.) This can be used to prove that *if B is in the Borchers class of a dual field A, and every vector in the basic domain of B is an analytic vector of A* (as in the case of the free field) *then*

$$R_{\mathfrak{B}'}(A) \subsetneq S_{\mathfrak{B}}(B)'' = R_{\mathfrak{B}}(B)'\,.$$

This may be obtained by taking $P = B(g)$ with $g \in \mathscr{D}_{(\mathfrak{B})}$ and taking / £ $\mathscr{D}_{(\mathfrak{B}')}$; equation (2.9) yields

$$\{E_\lambda^{\hat{A}(f)}: \lambda \in R,\ / \in \mathscr{D}_{(\mathfrak{B}')}\} \subseteqq S_{\mathfrak{B}}(B). \tag{2.10}$$

Because of the dense set of analytic vectors, A (/) is essentially self-adjoint on the domain of B and thus $R_{\mathfrak{B}'}(A)$ is the von Neumann algebra generated by the set of all spectral projectors in the set on the left of (2.10). But the von Neumann algebra generated by any set $\mathfrak{M}$ is its double corn-mutant $\mathfrak{M}''$, so

$$R_{\mathfrak{B}'}(A) = \{E_\lambda^{\hat{A}(f)}: \lambda \in R,\ / \in \mathscr{D}_{(\mathfrak{B}')}\}'' \subseteqq S_{\mathfrak{B}''}(B)'' = R_{\mathfrak{B}}(B)'.$$

If A satisfies the duality principle* $R_{\mathfrak{B}'}(A) — R_{\mathfrak{B}''}(A)'$, then passage to the commutants once more yields

$$R_{\mathfrak{B}}(B) \subseteqq R_{\mathfrak{B}'}(A') = R_{\mathfrak{B}''}(A), \tag{2.11}$$

and with 93 replaced by $\mathfrak{B}'$,

$$R_{\mathfrak{B}'}(B) \subseteqq R_{\mathfrak{B}'}(AY \qquad R_{\mathfrak{B}'}(A)\ ς\ R_{\mathfrak{B}'}(B)'.$$

Combining these two yields

$$R_{\mathfrak{B}}(B) \subseteqq R_{\mathfrak{B}'}(B)' \qquad R_{\mathfrak{B}'}(B)\ ς\ R_{\mathfrak{B}}(B)',$$

i.e. the rings generated by B are local; and replacing 93 by 93" in (2.11) gives

$$R_{\mathfrak{B}''}(B) \subseteqq R_{\mathfrak{B}''}(A).$$

This shows that the functional relationship holds for diamond shaped regions: $B \underset{\Delta}{\sqsubset} A$. It should be mentioned that the locality of the algebras is not a trivial consequence of locality of B unless the vacuum is analytic for the operators $B(f)$.If it is known that $S_{\mathfrak{B}}(B)$ is an algebra (and thus $S_{\mathfrak{B}}(B) = R_{\mathfrak{B}}(B)'$), then one can prove that

$$R_{\mathfrak{B}}(B) = (\cup\{R_\Delta(B): \Delta = \Delta'' \subseteqq \mathfrak{B}\})''$$

from the fact that the set of all diamonds is a basis for the open sets of Minkowski space, and from this it follows that $B \sqsubset A$.

As a final result, we mention the invertibility of the functional relationship. If B is local in the sense that $R_{\mathfrak{B}'}(B) \subseteq R_{\mathfrak{B}}(B)'$for every $\mathfrak{B}$, and A is a self-dual field, then

$$A \sqsubset B \Rightarrow B \underset{\Delta}{\sqsubset} A.$$

From $A \sqsubset B$, it follows that

$$R_{\mathfrak{B}'}(A)\ g\ R_{\mathfrak{B}'}(B) \qquad R_{\mathfrak{B}'}(A) \subseteq R_{\mathfrak{B}''}(B).$$

Passing to the commutant in the first and using duality of A (for $\mathfrak{B}'$, 93") we obtain

$$R_{\mathfrak{B}'}(BY \subseteqq R_{\mathfrak{B}'}(AY = R_{\mathfrak{B}''}(A) \subseteq R_{\mathfrak{B}'}(B). \tag{2.12}$$

* This discussion is similar to the one found in [13].

Locality of B (for $\mathfrak{B}'$, $\mathfrak{B}''$) expresses the inverse inclusion $R_{\mathfrak{B}'}(B) \subseteq R_{\mathfrak{B}'}(B)'$ and so

$$R_{\mathfrak{B}'}(B)' = R_{\mathfrak{B}''}(B) .$$

Combined with (2.12), this gives

$$R_{\mathfrak{B}''}(B) = R_{\mathfrak{B}''}(A) : B \underset{\Delta}{\sqsubset} A .$$

In the case we will be discussing, we see that all vectors in $\mathfrak{D}$ are analytic vectors of $A_0(f)$ for any $f \in \mathscr{S}$ (cf. appendix 1) so that fields in the Borchers class of A_0 are local functions of A_0 (considering only diamonds) and that the associated algebras are local. From the proof in the following sections that $A_0 \sqsubset B$ if B contains an odd power, it would immediately follow that the converse relationship $B \underset{\Delta}{\sqsubset} A_0$ holds. However, it is possible to derive the stronger result $B \sqsubset A_0$ showing that the algebras even satisfy the relation

$$R_{\bigcup_{\iota \in I} \mathfrak{B}_\iota}(B) = \bigvee_{\iota \in I} R_{\mathfrak{B}_\iota}(B) = \Big(\bigcup_{\iota \in I} R_{\mathfrak{B}_\iota}(B)\Big)''$$

since this is, in fact, true for fields which are essentially self-adjoint on their domains, in particular $\alpha :A_0^2: + \beta A_0$.

§ 3. General features of the space-time limiting procedure

The aim of this section is to outline the proof that if B is a Wick polynomial in the free field A_0, then $R_{\mathfrak{B}}(A) \subseteq R_{\mathfrak{B}}(B)$ for any open space-time region $\mathfrak{B}$ where $A = \alpha :A_0^2: + \beta A_0$ for some $\alpha, \beta \in E$ depending on B. Further reduction will occur at the end of § 4 in which it is seen that the inclusion holds with $\alpha = 0$ if B contains any odd power, and with $\beta = 0$ in the opposite case.

The result actually obtained is that $S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(A)$ from which the first mentioned result follows upon passage to the commutants. Let $P \in S_{\mathfrak{B}}(B)$: for each Φ, $\Psi \in \mathfrak{D}$ and $f_1 \in \mathscr{D}_{(\mathfrak{B})}$,

$$(\Psi, PB(f_1)\Phi) = (B(f_1)\Psi P\Phi) . \tag{3.1}$$

Replacing Φ by $B(f_2) \ldots B(f_m)\Phi \in \mathfrak{D}$ and repeatedly applying (3.1) we obtain

$$(\Psi, PB(f_1) \ldots B(f_m)\Phi) = (B(f_m) \ldots B(f_1)\Psi, P\Phi) \tag{3.2}$$

as long as $f_i \in \mathscr{D}_{(\mathfrak{B})}$. Using a method of a previous paper [14], we form the operator

$$C(f_\lambda) = \int B(x_1) \ldots B(x_m) f\Big(m^{-1} \sum_{i=1}^{m} x_i\Big) \times$$
$$\times\, g((x_1 - x_2)/\lambda) \ldots g((x_{m-1} - x_m)/\lambda)\, d^4x_1 \ldots d^4x_m$$

which, after subtraction of multiples of the identity and division by a polynomial in λ, converges in a dense domain to $A(f)$. To obtain an equation such as (3.1) for $G(f_\lambda)$, one defines $C(f_1 \otimes \quad \otimes f_n) = B(f_1) \ldots B(f_n)$ and rewrites (3.2) as

$$(\Psi, PC(f_1 \otimes \quad \otimes f_n)\Phi) = (C(f_n \otimes \quad \otimes f_1)\Psi, P\Phi) . \quad (3.3)$$

This equation may be extended by linearity to the linear span of $\mathcal{D}_{(\mathfrak{B})}^{\otimes n}$ and then by continuity to $\mathcal{D}_{(\mathfrak{B}^n)}$. The first extension is trivial; for the second, one takes a sequence $\{h_n\}_{n \in N}$ of elements of the linear span of $\mathcal{D}_{(\mathfrak{B})}^{\otimes n}$ converging to f_λ in the topology of $\mathcal{D}_{(\mathfrak{B}^n)}$. It is shown in Appendix 2 that if $\psi \in \mathcal{D}$, h_n can be chosen so that it is a sum of terms $f_1 \otimes \quad \otimes f_m$ with $f_i \in \mathcal{D}_{(\mathfrak{B})}$ so that (3.3) holds for each h_n. By the methods of § 5 it can be seen that $C(h_n)\Phi \to C(f_\lambda)\Phi$ (in norm) if $\Phi \in \mathfrak{H}$ so that

$$(\Psi, PC(f_\lambda)\Phi) = (C(f_\lambda')\Psi, P\Phi) \quad (3.4)$$

where $f_\lambda'(x_1, \ldots, x_n) = f_\lambda(x_n, \ldots, x_1)$. It will be seen in § 4 that $C(f_\lambda) = C_\lambda + \phi(\lambda) I$ where $\phi(\lambda)$ is a numerical function of λ and C_λ is an operator such that $c^{-1}\lambda^p C_\lambda \Phi \to A(f)\Phi$ for a suitable choice of c and p whenever $\Phi \in \mathfrak{H}$. If g is taken symmetric about the origin, then $f_\lambda' = f_\lambda$; and substituting $C_\lambda + \varphi(\lambda) I$ into (3.4), one sees that the multiples of the identity drop out, and one is left with

$$(\Psi, PC_\lambda\Phi) = (C_\lambda\Psi, P\Phi) .$$

If the terms of this equation are multiplied by $c^{-1}\lambda^p$ and λ is allowed to approach 0, the equation

$$(\Psi, PA(f)\Phi) = (A(f)\Psi, P\Phi)$$

results.

This means that $P \in S_{\mathfrak{B}}(A)$ and thus $S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(A)$, which was to be proved.

§ 4. Decomposition of $C(f_\lambda)$

Let

$$B(x) = \sum_{i=0}^{n} \alpha_i :A_0^i: (x) \qquad (\alpha_n \neq 0)$$

and for n even, consider the product

$$B(x)\, B(y) = \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j :A_0^i: (x) :A_0^j: (y)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j \sum_{k=0}^{i \wedge j} k! \binom{i}{k} k! \binom{j}{k} [i\Delta^{(+)}(x-y)]^k :A_0^{i-k}(x) A_0^{j-k}(y):$$

where $k!\binom{i}{k}$ is the number of ways one may select $A_0(x)$ k times from $:A_0^i: (x)$ and $i \wedge j$ designates the minimum of i and j. If $i = k = j$, the contributions to the sum are simply multiples of the identity which are to be dropped as mentioned in § 3. Of the remaining, it will be seen that the only terms which survive the limiting process are those for which k

is largest, i.e. equal to $n - 1$. This leaves the possibilities $i = n = j$, $i+1 = n = j$, and $i = n = j+1$ for which the corresponding terms are

$$\alpha_n^2 (n-1)!^2 n^2 i^{n-1} \triangle^{(+)}(x-y)^{n-1} :A_0(x) A_0(y):$$

$$\alpha_n \alpha_{n-1} (n-1)!^2 n i^{n-1} \triangle^{(+)}(x-y)^{n-1} [A_0(x) + A_0(y)]$$

The terms $c^{-1}\lambda^p C_\lambda$ of § 3 will then approach

$$n :A_0^2:(f) + 2\alpha_{n-1} A_0(f)$$

if the "smearing" function $f_\lambda(x,y) = f((x+y)/2) g((x-y)/\lambda)$ is applied to these terms and λ is allowed to approach 0. This will be investigated in more detail below and the convergence proved in § 5.

If n is odd, we consider

$$B(x)\, B(y)\, B(z) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_i \alpha_j \alpha_k :A_0^i(x): :A_0^j(y): :A_0^k(z):$$

The greatest number of contractions possible is $\frac{1}{2}(3n-1)$; these can only occur when $i = j = k = n$, $i+1 = j = k = n$, $i = j+1 = k = n$, or $i = j = k+1 = n$. In any of the last three cases, $i+j+k$ is even $(= 3n-1)$ and the resultant after contractions is a multiple of the identity which disappears from equation (3.4). The only terms that need be considered come from the case $i = j = k = n$. Each term will fall into one of three classes depending on whether $A_0(x)$, $A_0(y)$, or $A_0(z)$ is left over. The contribution from the first class will be (to within a non-zero factor depending on n)

$$\triangle^{(+)}(x-y)^{\frac{1}{2}(n+1)} \triangle^{(+)}(x-z)^{\frac{1}{2}(n-1)} \triangle^{(+)}(y-z)^{\frac{1}{2}(n+1)} A_0(x)$$

and those of the other two will be similar. Thus $c^{-1}\lambda^p C_\lambda \Phi \to A_0(f)\Phi$ in this case. With the results below, we will have proved that

$$R_{\mathfrak{B}}(A) \subseteq R_{\mathfrak{B}}(B)$$

where $A = A_0$ for odd n and $n :A_0^2: + \alpha_{n-1} A_0$ for n even.

All of the operators obtained in the decomposition of the products above have the form

$$:A_0^{\alpha_1}(x_1) \ldots A_0^{\alpha_m}(x_m): \prod_{\substack{i<j \\ i,j=1}}^{m} \triangle^{(+)}(x_i - x_j)^{\beta_{ij}}$$

We apply this to f_λ using the Fourier transform of the operator:

$$\int d^4x_1 \ldots d^4x_m f\left(m^{-1} \sum_{i=1}^{m} x_i\right) \prod_{j=1}^{m-1} g(\lambda^{-1}(x_j - x_{j+1})) :\tilde{A}^{\alpha_1}(p_1) \ldots \tilde{A}^{\alpha_m}(p_m): \times \tag{4.1}$$

$$\times \prod_{\substack{i<j \\ i,j=1}}^{m} \prod_{k=1}^{\beta_{ij}} d\Omega^{(+)}(q_{ij}^{(k)}) \exp i\left\{\sum_{i=1}^{m} p_i x_i + \sum_{\substack{i<j \\ i,j=1}}^{m} \sum_{k=1}^{\beta_{ij}} q_{ij}^{(k)} (x_i - x_j)\right\} d^4p_1 \ldots d^4p_m.$$

To get rid of the x-integrations, we make the coordinate transformation

$$\xi_i = x_i - x_{i+1}, \quad \tau = m^{-1} \sum_{i=1}^{n} x_i$$

which can be seen to have a Jacobian of $+1$. To invert this transformation, we note that $\sum_{i=1}^{j-1} \xi_i = x_1 - x_j$ and thus $\sum_{j=2}^{m} \sum_{i=1}^{j-1} \xi_i = \sum_{j=2}^{m} (x_1 - x_j)$ $= (m-1)x_1 - \sum_{j=2}^{n} x_j = mx_1 - m\tau$ Since the number of choices of j for which $i \leqq j-1, j \leqq m$ is equal to $m - i$, the first sum is $\sum_{i=1}^{m-1} (m-i)\xi_i$ so that $x_1 = \tau + \sum_{i=1}^{m-1} (1 - i/m)\xi_i$ This is the first step in an induction proof that

$$x_j = \tau + \sum_{i=1}^{m-1} (1 - \theta(j-i) - i/m)\xi_i \underset{df}{=} x_j' .$$

The second consists in showing that $x'_{j+1} - x_j = \xi_j = x_{j+1} - x_j$ With this expression for x_j, the exponent above becomes

$$\tau \sum_{i=1}^{m} p_i + \sum_{i=1}^{m-1} \xi_i \left[\sum_{j=1}^{m} p_j(1 - \theta(j-i) - i/m) + \sum_{s=1}^{i} \sum_{t=i+1}^{m} \sum_{k=1}^{\beta_{st}} q_{st}^{(k)} \right]$$

and the operator (4.1) becomes

$$\int d^4\tau f(\tau) e^{i\tau \sum_{i=1}^{m} p_i} d^4\xi_1 \ldots d^4\xi_{m-1} \prod_{j=1}^{m-1} g(\xi_j/\lambda) \exp i\xi_j \left\{ \sum_{i=1}^{m} p_i c_{ij} + \sum_{stk} q_{st}^{(k)} \right\} \times$$
$$\times \prod_{\substack{i<j \\ i,j=1}}^{m} \prod_{k=1}^{\beta_{ij}} d\Omega^{(+)}(q_{ij}^{(k)}) : \tilde{A}^{a_1}(p_1) \ldots \tilde{A}^{a_m}(p_m) : d^4p_1 \ldots d^4p_m .$$

With a change of variables $\zeta_j = \xi_j/\lambda$ and $r_{st}^{(k)} = \lambda q_{st}^{(k)}$ the differentials $d^4\xi_j$ evolve a factor of $\lambda^{4(m-1)}$. The τ, ξ integrals may be evaluated and the result is

$$\lambda^{4(m-1)} \tilde{f}\left(\sum_{i=1}^{m} p_i\right) \prod_{i=1}^{m-1} \hat{g}\left(\sum_{j=1}^{m} c_{ij} p_j + \sum_{stk} r_{st}^{(k)}\right) \prod_{\substack{i<j \\ i,j=1}}^{m} \prod_{k=1}^{\beta_{ij}} d\Omega^{(+)}(r_{ij}^{(k)}/\lambda) : \tag{4.2}$$
$$: \tilde{A}^{a_1}(p_1) \ldots \tilde{A}^{a_m}(p_m) : d^4p_1 \ldots d^4p_m .$$

If $d\Omega_{(m)}^{(+)}$ refers to the measure having support on the mass hyperboloid $\{p: p^2 = m^2\}$, then $d\Omega_{(m)}^{(+)}(r/\lambda) = \lambda^{-2} d\Omega_{(\lambda m)}^{(+)}(r)$ so that the integral (4.2) finally becomes

$$\lambda^{\{4(m-1) - \sum_{\substack{i<j \\ i,j=1}}^{m} 2\beta_{ij}\}} \int : \tilde{A}^{a_1}(p_1) \ldots \tilde{A}^{a_m}(p_m) : h_\lambda(p_1, \ldots, p_m) d^4p_1 \ldots d^4p_m$$

with

$$\tilde{h}_\lambda(p_1, \ldots, p_m) = f\left(\sum_{i=1}^{m} p_i\right) \int \prod_{\substack{i<j \\ i,j=1}}^{m} \prod_{k=1}^{\beta_{ij}} d\Omega^{(+)}_{(\lambda m)}(r_{ij}^{(k)}) \prod_{j=1}^{m-1} \tilde{g}\left(\sum_{stk} r_{st}^{(k)} + \sum_{i=1}^{m} \lambda c_{ij} p_i\right).$$

Apart from the combinatorical factors obtained in the decomposition of the products, the constant c will be the coefficient of f in $\lim_{\lambda\to 0} \tilde{h}_\lambda$:

$$c \propto \int \prod_{\substack{i<j \\ i,j=1}}^{m} \prod_{k=1}^{\beta_{ij}} d\Omega^{(+)}_{(0)}(r_{ij}^{(k)}) \prod_{j=1}^{m-1} \tilde{g}\left(\sum_{stk} r_{st}^{(k)}\right).$$

This manipulation and the considerations in § 5 show that each of the terms in the decomposition of a product converges, after multiplication by a power of λ, when operating on a vector Φ in a suitably restricted class. The one with the lowest power of λ will then appear alone in the result. But the power of λ is lowest when the sum $\sum_{\substack{i<j \\ i,j=1}}^{n} \beta_{ij}$ is greatest, i. e. in the terms with the most $\Delta^{(+)}$ functions. This justifies our neglect of the remaining terms in the two calculations at the beginning of this section.

Since g was taken symmetric, $\tilde{g}$ is real, and so, therefore, is c. To show that c can be taken not to vanish, one takes a sequence $\{g_n\}_{n=1}^{\infty} \subsetneq \mathscr{D}$ such that $g_n \xrightarrow[\mathscr{S}]{} g_0$, a Gaussian function: $g_0(x_\mu) = e^{-\sum_{\mu=1}^{4} x_\mu^2}$. Because the Fourier transform is a continuous automorphism on $\mathscr{S}$, $\tilde{g}_n \xrightarrow[\mathscr{S}]{} \tilde{g}_0$. But $\tilde{g}_0$ is also a Gaussian function and thus strictly positive. One can show from LEBESGUE's bounded convergence criterion and the properties of convergence in $\mathscr{S}$ that the c obtained from g_n converges to that obtained from g_0 so that for some n, this number is nonzero.

By somewhat simpler limiting procedures (used in the definition of the Wick polynomials*), one can show conversely that $R_{\mathfrak{B}}(B) \subsetneq R_{\mathfrak{B}}(A_0)$ for any polynomial B, and $R_{\mathfrak{B}}(B) \subset R_{\mathfrak{B}}(:A_0^2:)$ if B contains only terms with even powers**.

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* For $n = 3$, the definition of the Wick power can be formally written (without smearing functions) as

$$:A^3:(x) = \lim_{x_1 x_2 x_3 \to x} \left\{A(x_1)A(x_2)A(x_3) - \sum_{\substack{\text{perm}\, ij \\ i<j}} \langle A(x_i)A(x_j)\rangle A(x_k)\right\}.$$

The generalization to arbitrary n is straightforward and can be found in [6].

** This statement belongs logically at the beginning of this paper. However since the connection between Wick polynomials and ring-theoretical local functions involves a limiting procedure, it is most conveniently discussed in the present context.

If the highest and next highest powers appearing in B are even, then the considerations at the beginning of this section show that

$$S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(:A_0^2:)\,.$$

This may be strengthened if B contains an odd power. Let C be the sum of all the monomials in B with even powers and $D = B - C$. Then from the remark of the previous paragraph, $S_{\mathfrak{B}}(:A_0^2:) \subseteq S_{\mathfrak{B}}(C)$ so that $S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(C)$. But this last inclusion implies that $S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(D)$, since if $P \in S_{\mathfrak{B}}(B)$, then $P \in S_{\mathfrak{B}}(C)$, and for any $f \in \mathscr{D}_{(\mathfrak{B})}$,

$$\begin{aligned}(\Psi,\ PD(f)\Phi) &= (\Psi,\ PB(f)\Phi) - (\Psi,\ PC(f)\Phi)\\ &= (B(f)\Psi, P\Phi) - (C(f)\Psi, P\Phi) = (D(f)\Psi, P\Phi)\end{aligned}$$

Thus $P \in S_{\mathfrak{B}}(D)$. But the highest power of D is odd and so $S_{\mathfrak{B}}(D) \subseteq S_{\mathfrak{B}}(A)$ for $A = \alpha :A_0^2: + \beta A_0$ with $\beta \neq 0$. So far it has been proved that if B has no odd powers, then $R_{\mathfrak{B}}(B) = R_{\mathfrak{B}}(:A_0^2:)$ and that if B has an odd power, then $S_{\mathfrak{B}}(A_0) \subseteq S_{\mathfrak{B}}(B) \subseteq S_{\mathfrak{B}}(A)$ for $A = \alpha :A_0^2: + \beta A_0 (\beta \neq 0)$. It remains to be seen that $S_{\mathfrak{B}}(A) \subseteq S_{\mathfrak{B}}(A_0)$ in order to conclude that $R_{\mathfrak{B}}(A_0) = R_{\mathfrak{B}}(B)$. We drop the subscript o to designate the free field and consider the field

$$\frac{1}{2}\Box\,[:A^2: + \alpha A] = :\partial^\mu A\,\partial_\mu A: - m^2\left[:A^2: + \frac{1}{2}\alpha A\right]. \tag{4.3}$$

It will be shown that

$$:\partial^\mu A\,\partial_\mu A: \sqsubset \Box\,[:A^2: + \alpha A] \tag{4.4}$$

and thus, by the subtraction principle,

$$:A^2: + \frac{1}{2}\alpha A = m^{-2}\left\{:\partial^\mu A\,\partial_\mu A: - \frac{1}{2}\Box\,[:A^2: + \alpha A]\right\} \sqsubset [:A^2: + \alpha A].$$

But $\Box\,[:A^2: + \alpha A] \sqsubset :A^2: + \alpha A$ so in fact

$$:A^2: + \frac{1}{2}\alpha A \sqsubset :A^2: + \alpha A\,,$$

which, again by subtraction, leads to

$$A = 2\alpha^{-1}\left\{[:A^2: + \alpha A] - \left[:A^2: + \frac{1}{2}\alpha A\right]\right\} \sqsubset :A^2: + \alpha A\,,$$

which is the result to be proved.

In the proof of (4.4), one proceeds in the usual way with

$$\begin{aligned}&\frac{1}{2}\Box\,[:A^2: + \alpha A](x)\,\frac{1}{2}\Box\,[:A^2: + \alpha A](y)\\ &= :\partial^\mu A\,\partial_\mu A:(x)\,:\partial^\nu A\,\partial_\nu A:(y) + \cdots\\ &= 4i\,\partial_x^\mu\partial_y^\nu\Delta^{(+)}(x-y)\,:\partial_\mu A(x)\partial_\nu A(y): + \cdots\end{aligned}$$

in which only the term with two derivatives of the $\Delta^{(+)}$ function are retained because each derivative adds one λ^{-1} to the expression making this term dominant in the limit. As explicitly calculated in the preceeding work [14], the result after passage to the limit is proportional to

$$I^{\mu\nu}:\partial_\mu A\,\partial_\nu A:(f) \tag{4.5}$$

with

$$I^{\mu\nu}=\int r^\mu r^\nu \hat{g}(r)\,d\Omega^{(+)}_{(0)}(r)$$

if the operator above is "smeared with" $f\left(\frac{1}{2}(x+y)\right)g((x-y)/\lambda)$. If g is chosen so that g is symmetric in r_i, then $I^{\mu\nu}=0$ if $\mu\neq\nu$ and $I^{00}=I^{ii}=-3I^{(i)(i)}$. The operator in (4.5) is then $\frac{1}{3}I^{00}\,[2:\partial_0 A\partial_0 A:+:\partial^\mu A\partial_\mu A:]$ and we may write

$$2:\partial_0 A\,\partial_0 A:+:\partial^\mu A\,\partial^\mu A:\sqsubset\Box\,[:A^2:+\alpha A]\,.$$

By subtraction with (4.3), one obtains

$$2:\partial_0 A\,\partial_0 A:+m^2\left[:A^2:+\frac{1}{2}\alpha A\right]\sqsubset \mathrm{D}\,[:A^2:+\alpha A]\,. \tag{4.6}$$

By an entirely similar squaring procedure, one obtains

$$2:\partial_0 A\,\partial_0 A:\sqsubset\Box[:A^2:+\alpha A]$$

and finally, by subtraction from (4.6),

$$:A^2:+\frac{1}{2}\alpha A\sqsubset\Box[:A^2:+\alpha A]$$

which, as we have seen, leads to the result

$$A_0\sqsubset\ :A_0^2:+\alpha A_0$$

for any $\alpha\neq 0$.

§ 5. Convergence

It has to be shown that for $\Phi\in\mathfrak{M}^{(n)}$, $c^{-1}\lambda^p C_\lambda\Phi\to A(f)\Phi$ as $\lambda\to 0$. The decomposition of C_λ into totally Wick-ordered products gives rise to terms of the form: $A_0^{k_1}\ldots A_0^{k_m}:(h_\lambda)$ with

$$\hat{h}_\lambda(p_i)=\hat{f}\left(\sum_{i=1}^{m}p_i\right)\int\prod_{j=1}^{l}d\Omega^{(+)}_{(\lambda m)}(r_j)\,\hat{g}\left(\sum_{j\in J_1}r_j+\lambda\sum_{i=1}^{m}c_i^{(1)}p_i\right)\ldots$$
$$\ldots\hat{g}\left(\sum_{j\in J_{m-1}}r_j+\sum_{i=1}^{m}c_i^{(m-1)}p_i\right)$$

where $J_1\cup\ldots\cup J_{m-1}=\{1,\ldots,l\}$. By using LEBESGUE's bounded convergence criterion and the "falloff" properties of g, it can be seen that $\hat{h}_\lambda(p_i)\to c\,\hat{f}\left(\sum_{i=1}^{m}p_i\right)$ (pointwise convergence) where $c=\int\prod_{j=1}^{l}d\Omega^{(+)}_{(0)}\times(r_j)\,\hat{g}\left(\sum_{j\in J_1}r_j\right)\ldots\hat{g}\left(\sum_{j\in J_{m-1}}r_j\right)$. The coefficients $c_i^{(j)}$ all lie between ± 1 (as is seen in the treatment of the general case).

In forming $:A_0^{k_1}\dots A_0^{k_m}:(h_\lambda)\Phi$, one gets a term for each partition of $k_i = k_i' + k_i''$ where k_i' represents the number of particles "destroyed by $A_0^{k_i}$". This term may be written (as a function of p_j, $1 \leqq j \leqq n$) as

$$\int \hat h_\lambda\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right) \times$$
$$\times\ \Phi(q_i^{(r)}, p_j;\ 1 \leqq r \leqq k_i', 1 \leqq i \leqq n, p_j + p_i^{(r)}) \prod_{i=1}^{n} \prod_{r=1}^{k_i'} d\Omega^{(+)}(q_i^{(r)})\,.$$

The corresponding term in $c\,:A_0^{k_1+\dots+k_m}:(f)\,\Phi$ is gotten by replacing $\hat h_\lambda(\dots)$ by $c\hat f\left(\sum_{i=1}^{m}\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\right)$. The norm of the difference of these two vectors is obtained from the integral

$$\int \prod d\Omega^{(+)}(p_j)\left[\int f_\lambda^0\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\Phi(q_i^{(r)}, p_j)\right]^* \times$$
$$\times\left[\int f_\lambda^0\left(\sum_{r=1}^{k_i'} s_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\Phi(s_i^{(r)}, p_j)\right]$$

where $f_\lambda^0(p_i) = \hat h_\lambda(p_i) - c\hat f\left(\sum_{i=1}^{n} p_i\right)$ converges pointwise to 0. Lebesgue's criterion may be applied to infer that the integral (which is the square of

$$\|\,:A_0^{k_1}\dots A_0^{k_m}:(h_\lambda)\Phi - :A_0^{k_1+\dots+k_m}:(f)\Phi\|)$$

approaches 0 for $\lambda \to 0$ provided the integrand can be shown to be majorized by an integrable function which is independent of λ.

To do this, we consider the integral

$$\left|\int \hat h_\lambda\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\Phi(q_i^{(r)}, p_j; p_j +\right.$$
$$\left. + p_i^{(r)}) \prod_{j=1}^{n} d\Omega^{(+)}(p_j) \prod_{i=1}^{m} \prod_{r=1}^{k_i'} d\Omega^{(+)}(q_i^{(r)})\right| \leqq$$
$$\leqq \int \left|\hat g\left(\sum_{j\in J_1} r_j + \lambda \sum_{i=1}^{m} c_i^{(1)}\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\right)\right| \dots$$
$$\cdot\ f\left(\sum_{i=1}^{m}\left(\sum_{r=1}^{k_i'} q_i^{(r)} - \sum_{r=1}^{k_i''} p_i^{(r)}\right)\right)\Phi(q_i^{(r)}, p_j) \prod_{j=1}^{n} d\Omega^{(+)}(p_j) \times$$
$$\times \prod_{i=1}^{m} \prod_{r=1}^{k_i'} d\Omega^{(+)}(q_i^{(r)}) \prod_{j=1}^{m} d\Omega^{(+)}_{(\lambda)}(r_j)$$

and show that the integrand is bounded by a multiple of

$$\prod_{k=1}^{m}\left(1+\left|\sum_{j\in J_k}' r_j^0\right|^{\alpha}\right)^{-1}\cdot\left(1+\left|\sum_{i=1}^{m}'\sum_{r=1}^{k_i'}' q_i^{(r)0}+\sum_{j=1}^{n}' p_j^0\right|^{\alpha}\right)^{-1} \tag{5.1}$$

which is independent of λ and has a finite integral for α suitably large. From the functions / and Φ, one gets a bound proportional to

$$\left(1+\left|\sum_{i=1}^{m}\sum_{r=1}^{k_i'} q_i^{(r)0}-\sum_{i=1}^{m}\sum_{r=1}^{k_i''} p_i^{(r)0}\right|^{\beta}\right)^{-1}\times$$

$$\times\left(1+\left|\sum_{i=1}^{m}\sum_{r=1}^{k_i'} q_i^{(r)0}+\sum_{p_j\neq p_i^{(r)}} p_j^0\right|^{\beta}\right)^{-1}$$

which, according to (4) of appendix 1 is majored by

$$\left(1+2^{-\beta}\left|\sum^{m}\sum_{=1}^{k_i'} q_i^{(r)0}+\sum_{j=1}^{n} p_j^0\right|^{\beta}\right)^{-1}.$$

Since we are considering bounds only up to constant factors, we may drop the factor $2^{-\beta}$. From each g, there is a bound proportional to

$$\left(1+\left|\sum_{j\in J_k} r_j^0+\sum_{i=1}^{m}\lambda c_i^{(k)}\left(\sum_{r=1}^{k_i'} q_i^{(r)0}-\sum_{r=1}^{k_i''} p_i^{(r)0}\right)\right|^{\alpha}\right)^{-1}.$$

We now set $\beta=\alpha(m+1)$, and associate with each of the above factors, the quantity $(1+|\Sigma q^0+\Sigma p^0|^{\alpha})^{-1}$ and save one for the right factor in (5.1). We investigate the product

$$\left(1+\left|\sum_{j\in J_k} r_j^0+\sum_{i=1}^{m}\lambda c_i^{(k)}\left(\sum_{r=1}^{k_i} q_i^{(r)0}-\sum_{r=1}^{k_i''} p_i^{(r)0}\right)\right|^{\alpha}\right)^{-1}\times$$

$$\times\left(1+\left|\sum_{i=1}^{m}\sum_{r=1}^{k_i'} q_i^{(r)0}+\sum_{j=1}^{n} p_j^0\right|^{\alpha}\right)^{-1}$$

In the second factor, all terms under the absolute value bars are positive, so we may drop all those appearing with positive coefficients in the first factor without decreasing the quantity under consideration. In addition, when $\lambda<1$, all coefficients $\lambda c_i^{(k)}$ will have magnitude less than 1, so that the remaining variables in the second factor may be multiplied by the negatives of their coefficients still without decreasing the quantity considered. If p^+ is the sum of all terms with positive coefficients and p^- the sum of those with negative coefficients, then the result may be written

$$\left(1+\left|\sum_{j\in J_k} r_j^0+p^+-p^-\right|^{\alpha}\right)^{-1}(1+|p^-|^{\alpha})^{-1}$$

which, according to (4) of appendix 1 is majored by

$$\left(1 + 2^{-\alpha}\Big|\sum_{j\in J_k} r_j^0 + p^+\Big|^\alpha\right)^{-1} \leq \left(1 + 2^{-\alpha}\Big|\sum_{j\in J_k} r_j^0\Big|^\alpha\right)^{-1}.$$

Again dropping the $2^{-\alpha}$, we obtain the k-th factor of the left side of (5.1). This completes the proof of the convergence

$$c^{-1}\lambda^p C_\lambda \Psi \to A(f)\Psi \qquad (\Psi \in \mathfrak{D})$$

since it shows that each term in the decomposition converges to a vector without its associated power of λ. When each is multiplied by $c^{-1}\lambda^p$, all the limits will vanish except the one corresponding to $A(f)$ which is associated to the power λ^{-p}.

Concluding remarks

We have given an explicit proof of the ring theoretical statements formulated in the introduction for the case of Wick-polynomials. However the most general scalar field $B(x)$ which is a local function of $A_0(x)$, is a Wick polynomial involving invariant (contracted) derivatives. This is a consequence of the statement that any scalar local function of A a fortiori belongs to the Borchers class of A. The form of the general element of the scalar Borchers-class of the free field is however explicitly known [15]. It is a Wick-polynomial involving contracted derivatives. As the Wick-powers in the derivative free case, it can be gotten by a (trivial) limiting procedure, and hence every element of this Borchers class is a scalar local function. The ring theoretical inversion of this local function relation can be worked out along similar lines as given in the third and fourth section of this paper. For the determination of the leading term in the inverse powers of λ, the degree of the power of the $\triangle^{(+)}$ functions as well as the number of derivatives have to be taken into consideration. Since there is no new idea involved, we refrain from giving a detailed account of the computation.

We would like to mention, that the free fermion current

$$j_\mu(x) = :\bar{\psi}(x)\,\gamma_\mu \psi(x):$$

can be obtained as a ring theoretical local function in the „bilocal" von Neumann ring $R_{\mathfrak{B}}(\bar{\psi}, \psi)$ generated by elements of the form $\bar{\psi}(f_1)\cdot\psi(f_2)$ with $f_1, f_2 \in \mathcal{D}_{(\mathfrak{B})}$.

Since the smeared out fermion fields are bounded operators [16], the ring theoretical discussion would simplify considerably.

Finally we would like to mention, that all limiting procedures occurring in the definition of currents for the known solvable models [17], can be reformulated on a ring theoretical level by using the methods outlined in this paper.

Appendix 1. Definition of the fields

The Hubert space is the usual Fock space [18] $\mathfrak{H} = \bigoplus_{n=0}^{\infty} \mathfrak{H}^{(n)}$ where $\mathfrak{H}^{(n)}$ is the tensor product of n one particle spaces $\mathfrak{H}^{(n)} = Sy[\mathfrak{H}^{(1)} \otimes \cdots \otimes \mathfrak{H}^{(1)}]$. $\mathfrak{H}^{(1)}$ is the space of complex valued functions on R^4 square integrable with respect to the measure $d\Omega^{(+)}(p) = \delta(p^2+m^2)\,\theta(p_0)\,d^4p$ (mod the class of functions of 0 norm with respect to this measure). For the domains of definition of the fields, we form the subsets $\mathfrak{M}^{(n)}$ of functions Φ in $\mathfrak{H}^{(n)}$ for which $|\Phi(p_1 \ldots p_n)| \left(1 + \sum_{i=1}^{n} p_i^{0\,\alpha}\right)$ is bounded for $p^0 \geqq 0$, for each $\alpha \in N$. The basic domain $\mathfrak{D}$ is defined as the linear span of the sets $\mathfrak{M}^{(n)}$ $(n \geqq 0)$. To define the fields $:A_0^l:$, one notes that a vector in $\mathfrak{H}$ is a sequence $\{\Phi^{(n)}\}_{n=0}^{\infty}$ with $\Phi^{(n)} \in \mathfrak{H}^{(n)}$ and $\sum_{n=0}^{\infty} \|\Phi^{(n)}\|^2 < +\infty$. For $\Phi \in \mathfrak{D}$ one defines the component of $:A_0^l:(f)\Phi$ lying in $\mathfrak{H}^{(n)}$ as follows (6)

$$(:A_0^l:(f)\Phi)^{(n)} = \frac{\pi^{l/2}}{(2\pi)^{2(l-1)}} \sum_{j=0}^{l} T_{(l;n,j)}(\tilde{f})\,\Phi^{(n-l+2j)}$$

with the operators $T_{(l;n,j)}(g)\,\mathfrak{M}^{(n-l+2j)} \to \mathfrak{M}^{(n)}$ defined by

$$(T_{(l;n,j)}(g)\Psi)(p_i)_{i=1}^{n} = ((n-l+2j)!\,n!)^{1/2} \sum_{k_1+\cdots+k_{l-j}}{}_1^n \int \prod_{r=1} d\Omega^{(+)}(n_r) \times$$
$$\times\, g\left(\sum_{r=1}^{j} n_r - \sum_{s=1}^{l-j} p_{k_s}\right)$$
$$\Psi(n_r, p_i;\, 1 \leqq r \leqq k,\ 1 \leqq i \leqq n[k_s])\,.$$

With a little patience, the usual formula for $A_0(f)$ can be recognized as the special case when $l = 1$. It can be seen with the use of FUBINI's theorem (which is justified by inequalities derived below) that

$$(T_{(l;n,j)}(g))^* = T_{(l,n-l+2j,l-j)}(g^*) \tag{1}$$

where $g^*(p) = g(-p)$. ($g = g^*$ if it is the Fourier transform of a real function.) In these operators, j represents the number of particles annihilated and $l-j$, the number created. Since we obtain fields $A = \alpha :A_0^2: + \beta A_0$ in the intermediate stages of the proof and want to use the results of § 2, we must prove that they have dense domains of analytic vectors. It is not enough to show that the vacuum is analytic since it is not cyclic in the case $\beta = 0$. However, it is easy to see that any vector in $\mathfrak{M}(n)$ for any $n \geqq 0$ is an analytic vector for any of the operators $(\alpha :A_0^2: + \beta A_0)(f)$. For this it is necessary to investigate the norms of the operators $T_{(l;n,j)}(g)$ for $l = 1, 2$.

For the free field, $T_{(1,n,0)}(g)\Psi_{(p_i)} = (n-1)!n!)^{1/2} \sum_{k=1}^{n-1} ff(P^*)$ X x $\Psi(p_i; 1 \leqq i \leqq n \quad i \neq k)$ and thus

$$\|T_{(1,n,0)}(g)\Psi\| \leqq n^{1/2}\|g\| \quad \|\Psi\| \quad ; \quad \|T_{(1,n,0)}(g)\| = n^{1/2}\|g\|$$

Also, $\|T^*\| = \|T\|$ for any bounded linear transformation T, so

$$\|T_{(1,n-1,1)}(g)\| = n^{1/2}\|g\|$$

because of (1). By similar manipulations, one sees that

$$\|T_{(2,n,0)}(g)\| = \|T_{(2,n-2,2)}(g)\| \leqq$$
$$\leqq \sqrt{n(n-1)} \int d\Omega^{(+)}(q_1)\, d\Omega^{(+)}(q_2)\, |g(-q_1-q_2)|^2 .$$

So far, all the operators have had norms whose growth with $n \to \infty$ is bounded by a multiple of n. It remains to show the same of the operator*

$$T_{(2,n,1)}(g) = [T_g \otimes I \otimes \cdots \otimes I] + [I \otimes T_g \otimes \cdots \otimes I] +$$
$$+ \quad + [I \otimes I \otimes \cdots \otimes T_g]$$

where $T_g: \mathfrak{H}^{(1)} \to \mathfrak{H}^{(1)}$ is defined by

$$(T_g\Phi)(p) = \int g(q-p)\Phi(q)\,d\Omega^{(+)}(q) .$$

Clearly $\|T_{(2,n,1)}(g)\| \leqq n\|T_g \otimes I \otimes \cdots \otimes I\|$, and the desired result will follow from the statements

$$\|T_g \otimes I \otimes \cdots \otimes I\| \leqq \|T_g\| < \infty .$$

The first inequality is essentially proved by Dixmier on page 23.* The restriction to two factors is not essential here. It is, however, essential that this upper bound does not depend on the number of factors, n.

To show that T_g is bounded, we show that the bilinear form it induces is bounded. Thus if $\Phi', \Phi \in \mathfrak{H}^{(1)}$,

$$|(\Phi', T_g\Phi)| \leq \int |\Phi'_{(p)}\|g_{(q-p)}\|\Phi_{(q)}|\, d\Omega^{(+)}_{(p)} d\Omega^{(+)}_{(q)} \leqq$$
$$\leqq \int \Psi'_{(\mathbf{p})} h(\mathbf{q}-\mathbf{p})\,\Psi(\mathbf{q})\, d^3\mathbf{p}\, d^3\mathbf{q} ,$$

where $\Psi(\mathbf{p}) = (\mathbf{p}^2 + m^2)^{-1/2}|\Phi(\mathbf{p}, (\mathbf{p}^2+m^2)^{1/2})|$ and $h(\mathbf{r}) = \sup_{r_0 \in R} |g(\mathbf{r}, r_0)|$. If $\|\Psi\|_{L_2} = \int |\Psi(\mathbf{p})|^2 d^3\mathbf{p}$ is the norm with respect to the Lebesgue measure, then

$$\|\Psi\|_{L_2} = \int (\mathbf{p}^2+m^2)^{-1/2} |\Phi(\mathbf{p}, (\mathbf{p}^2+m^2)^{1/2})|^2 (\mathbf{p}^2+m^2)^{-1/2} d^3\mathbf{p} \leqq$$
$$\leqq m^{-1} \int |\Phi(p)|^2 d\Omega^{(+)}(p) = m^{-1}\|\Phi\|_{\Omega} .$$

If p is any polynomial in $\mathbf{r}$, then $M_p = \sup_{\mathbf{r} \in R^3} |p(\mathbf{r}) h(\mathbf{r})| = \sup_{r \in R^4} |p(\mathbf{r}) g(r)| < \infty$.

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* For a discussion of tensor product spaces, see [11] I § 3 or appendix I of [19].

For $p(\mathbf{r}) = (1 + \mathbf{r}^2)^\alpha$, let $M p$ be designated by M_α. Then

$$h(\mathbf{r}) \leqq M_\alpha (1+\mathbf{r}^2)^{-\alpha} = \mathrm{fc}_\beta(\mathrm{r})$$

and

$$|(\Phi', T_g \Phi)| \leqq \int \Psi'_{(\mathbf{p})} k_{\alpha(\mathbf{q}-\mathbf{p})} \Psi(\mathbf{q})\, d^3\mathbf{p}\, d\, \mathbf{q}\,. \tag{2}$$

If α is chosen suitably large, $k_\alpha \in L_1$ and thus has a Fourier transform:

$$k_\alpha(\mathbf{r}) = \int_{-\infty}^{\infty} e^{i\mathbf{r}\cdot\mathbf{x}} \tilde{k}_\alpha(\mathbf{x}) d^3\mathbf{x}$$

which, because k_α is infinitely differentiable, will fall off faster than the reciprocal of any polynomial in x ([20], p. 46). As a consequence, $\tilde{k}_\alpha$ will be bounded and will belong to L_1. Let us consider the integral in (2) with the ranges of the variables p and q restricted to bounded cells C_p, C_q and with the Fourier integral representation of k_α inserted.

$$\int_{C_p} d^3\mathbf{p} \int_{C_q} d^3\mathbf{q} \int_{R^3} \tilde{k}_\alpha(\mathbf{x}) e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \Psi'_{(\mathbf{p})} \Psi_{(\mathbf{q})} d^3\mathbf{x}$$
$$= \int_{R^3} d^3\mathbf{x}\, \tilde{k}_\alpha(\mathbf{x}) \int_{C_p} d^3\mathbf{p}\, e^{i\mathbf{p}\cdot\mathbf{x}} \Psi'_{(\mathbf{p})} \int_{C_q} d^3\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \Psi_{(\mathbf{q})}$$

where the interchange of integrations is justified by FUBINI's theorem and the absolute integrability ($\tilde{k}_\alpha \in L_1$):

$$\int_{C_p} d^3\mathbf{p}\, \Psi'_{(\mathbf{p})} \int_{C_q} d^3\mathbf{q}\, \Psi_{(\mathbf{q})} \int_{R^3} |\tilde{k}_\alpha(\mathbf{x})|\, d^3x < \infty\,. \tag{3}$$

Recall that if $\Psi \in L_2(R^3)$, then its restriction to C_p belongs to $L_1(C_p)$ since $\Psi \in L_2(C_p)$, $1 \in L_2(C_p)$ and therefore

$$\int_{C_p} |\Psi_{(\mathbf{p})}|\, d^3\mathbf{p} \leq \left[\int_{C_p} |\Psi_{(\mathbf{p})}|^2 d^3\mathbf{p}\right] \left[\int_{C_p} 1\, d^3\mathbf{p}\right] < \infty\,.$$

The integral (3) admits the upper bound

$$\sup_{\mathbf{x}\in R^3} |\tilde{k}_\alpha(\mathbf{x})| \int_{R^3} d^3\mathbf{x}\, |\int_{C_p} d^3\mathbf{p}\, \Psi'(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|\, |\int_{C_q} d^3\mathbf{p}\, \Psi(q) e^{-i\mathbf{q}\cdot\mathbf{x}}| \leqq$$

$$\leqq \sup|\tilde{k}_\alpha| \left[\int_{R^3} d^3\mathbf{x}\, |\int_{C_p} d^3\mathbf{p}\, \Psi'(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}|^2\right]^{1/2} \left[\int_{R^3} d^3\mathbf{x}\, |\int_{C_q} d^3\mathbf{q}\, \Psi(\mathbf{q}) e^{-i\mathbf{q}\cdot\mathbf{x}}|^2\right]^{1/2}$$

in which HÖLDER's inequality is used. But by PARSEVAL's formula ([20], p. 53: 2.8, 12),

$$\int d^3\mathbf{x}\, |\int_{C_p} d^3\mathbf{p}\, \Psi'_{(\mathbf{p})} e^{i\mathbf{p}\cdot\mathbf{x}}|^2 = \int_{C_p} d^3\mathbf{p} |\Psi'_{(\mathbf{p})}|^2 \leqq \|\Psi'\|^2_{L_2}$$

so that finally

$$(\Phi', T_\varphi \Phi) \leqq \sup|\tilde{k}_\alpha| \cdot \|\Psi'\|_{L_2} \cdot \|\Psi\|_{L_2} \leqq m^{-2} \sup|\tilde{k}_\alpha|\, \|\Phi'\|_\Omega \|\Phi\|_\Omega$$

and $\|T_g\| \leqq m^{-2} \sup|\tilde{k}_\alpha|$, a number independent of n.

These bounds on the norms of $T_{(l,n,j)}(g)$ for $l = 1, 2$ are suitable for showing that any vector in $\mathfrak{H}^{(m)}$ is an analytic vector for the operator

$A = (:A_0^2: + A_0)(f)$. If $\|T_{(l,n,j)}(f)\| \leq K$ n and $\Phi \in \mathfrak{H}^{(m)}$, then $\|A\Phi\| \leq \leq 5K(m+2)\|\Phi\|$ since $A\Phi$ has components only in $\mathfrak{H}^{(m)}$, $\mathfrak{H}^{(m\pm 1)}$, and $\mathfrak{H}^{(m\pm 2)}$. Applying A again gives $\|A^2\Phi\| \leq 5K(m+4)$ $\|A\Phi\| \leq \leq (5K)^2(m+4)(m+2)\|\Phi\|$. For $A^n\Phi$, one has the bound

$$\|A^n\Phi\| \leq (5K)^n(m+2n)\ldots(m+2)\|\Phi\| \leq (10K)^n(n+k)!k\mathrm{i}\|\Phi\|$$

where k is the smallest integer greater than or equal to $\frac{1}{2}m$ and $k\mathrm{i} = (k!)^{-1}$. Thus

$$(\|A^n\Phi\| \cdot n\mathrm{i})^{1/n} \leq 10K[(n+k)!n\mathrm{i}]^{1/n}[k\mathrm{i}\|\Phi\|]^{1/n} \to 10K$$

which implies that Φ in analytic for A.

It is still to be shown that $T_{(l,n,j)}(g)$ takes vectors from $\mathfrak{M}^{(n-l+2j)}$, into $\mathfrak{M}^{(n)}$. This is however a consequence of the fact that $d\Phi^{(+)}(p)$ is 0 for $p^0 \geqq 0$ and of the inequality, valid for $p_i^0 \geqq 0$

$$\left[1+2^{-\alpha}\left|\sum_{i=1}^{m} p_i^0\right|^\alpha\right] \cdot \left[1+\left|\sum_{i=1}^{n} p_i^0 + \sum_{j=1}^{l} q_j^0\right|^\alpha\right]^{-1} \times$$
$$\times \left[1+\left|\sum_{i=n+1}^{m} p_i^0 - \sum_{j=1}^{l} q_j^0\right|^\alpha\right]^{-1} \leqq 1 \tag{4}$$

which can be checked in the three cases

$$\Sigma q^0 \leq -\sum_1^n p^0$$

$$-\sum_1^n p^0 \leq \Sigma q^0 \leq \sum_{n+1}^m p^0$$

$$\sum_{n+1}^m p^0 \leq \Sigma q^0,$$

the second case relying on the inequality

$$2^{-\alpha}(a+b)^\alpha \leqq a^\alpha + b^\alpha \quad \text{if } a \geqq 0 \leqq 6\,.$$

With inequality (4), the falloff properties of Ψ and Φ are bestowed on $T(g)\Phi$ which, as a consequence, belongs to $\mathfrak{M}^{(n)}$.

Appendix 2

In order to justify the continued use of the commutativity property (3.2) in § 3, it must be shown that the function h_n can be made from linear combinations of functions $f_1 \otimes \cdots \otimes f_m$ with $f_i \in \mathcal{D}_{(\mathfrak{B})}$. To show this, we consider the carrier and support of the function f_λ:

$$f_\lambda(x_1, \ldots, x_m) = f\left(\frac{1}{m}\sum x_i\right) g\left(\frac{1}{\lambda}(x_1 - x_2)\right) \cdots g\left(\frac{1}{\lambda}(x_{m-1} - x_m)\right).$$

By the carrier of a function, is meant the set of all points at which it does not vanish, and by support, the closure of the carrier. For all topological concepts, the reader is referred to KOWALSKY [21].

If $(x_1, \ldots, x_n)$ is in the carrier of f_λ, then $\frac{1}{m} \sum_{i=1}^{m} x_i \in \operatorname{supp}(f) = \mathfrak{K}$. If the diameter of the support of g is ε, then also the distance from x_i to x_{i+1} is less than $\varepsilon\lambda$: $\delta(x_i, x_{i+1}) < \varepsilon\lambda$, and for any $i, j, \delta(x_i, x_j) < (m-1)\varepsilon\lambda$. The segment $x_i - \frac{1}{m} \sum_{j=1}^{m} x_j$ can be expressed as $\frac{1}{m} \sum_{j=1}^{m} (x_i - x_j)$, and from the usual inequalities,

$$\delta\left(x_i, \frac{1}{m} \sum_{j=1}^{m} x_j\right) \leq \frac{1}{m} \sum_{j=1} \delta(x_i, x_j) < (m-1)\varepsilon\lambda .$$

Thus, when $(m-1)\varepsilon\lambda < \delta$, the distance between x_i and $\mathfrak{K}$ is

$$\delta(x_i, \Lambda) \leqq \delta\left(x_i, \frac{1}{m} \sum_{j=1}^{m} x_j\right) < (m - 1)\varepsilon\lambda < (5.$$

If $\mathfrak{K}_\delta$ is the (compact) set of points having distance from $\mathfrak{K}$ bounded by δ, this result states that:

If $(x_1, \ldots, x_n)$ is in the carrier of f_λ and $(m-1)\varepsilon\lambda <$ (5, then $x_i \in \mathfrak{K}_\delta$ or if $\mathfrak{C}(f_\lambda)$ is the carrier of f_λ, then $\mathfrak{C}(f_\lambda) \subseteq \mathfrak{K}_\delta \times \quad \times \mathfrak{K}_\delta$, and since this set is closed, $\mathfrak{K}(f_\lambda) = \overline{\mathfrak{C}(f_\lambda)} \subseteq \mathfrak{K}_\delta \times \quad \times \mathfrak{K}_\delta$. Since $\mathfrak{K}$ is compact and $\mathbf{R}^4 \setminus 93$ is closed, the distance between the two sets

$$\delta(\mathfrak{K}, \mathbf{R}^4 \setminus 23) = \inf\{\delta(x, y) : x \in \mathfrak{K}, y \in \mathbf{R}^4 \setminus \mathfrak{B}\}$$

is positive. If δ is chosen smaller than this number, then $\mathfrak{K}_\delta \subseteq \mathfrak{B}$ and as a consequence, $\mathfrak{K}(f_\lambda) \subseteq \mathfrak{K}_\delta \times \quad \times \mathfrak{K}_\delta \subseteq 93 \times \quad \times \mathfrak{B}$.

It is now a matter of finding a function $f_* \in \mathscr{D}_{(\mathfrak{B})}$ which takes on the value $f_*(x) = 1$ when $x \in \mathfrak{K}_\delta$, for if such a function exists, we may set $f'(x_1, \ldots, x_n) = f_*(x_1) \ldots f_*(x_n)$ and have

$$f_\lambda = f_\lambda \cdot f' = \lim_n (h_n \cdot f')$$

where $h_n \ f'$ will be composed of linear combinations of elements of the form

$$(f_1 \otimes \cdots \otimes f_n) \cdot f' = (f_1 \cdot f_*) \otimes \cdots \otimes (f_n \cdot f_*)$$

with $f_n \ f_* \in \mathscr{D}_{(\mathfrak{B})}$. To prove the existence of such a function f_*, one may consider an open covering of the set $\mathfrak{K}_\delta$ by taking at each point x, a sphere of radius $2r$ centered at x which is contained in 23 (since 93 is open). The spheres $\mathfrak{S}_r(x)$ will cover the set $\mathfrak{K}_\delta$, and there will be a finite subcollection which also covers $\mathfrak{K}_\delta$ because of its compactness. For each of these spheres, one may take a function which is 0 in $\mathfrak{S}_r(x)$ and 1 outside of

$\mathfrak{S}_{2r}(x)$ and which is infinitely differentiable. The product of these functions will have the value 1 on $\mathbf{R}^4 \setminus 93$ and 0 at every point of $\mathfrak{K}_\delta$ and will be infinitely differentiable. If this function is subtracted from 1, a function f_* with the properties mentioned above is obtained.

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