Commun. math. Phys. 1, 175—214 (1965)

Tempered Distributions in Infinitely Many Dimensions

I. Canonical Field Operators

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Abstract. The space of testing functions for tempered distributions is characterized in an abstract way as the maximal space in a certain class of locally convex topological vector-spaces. The main characteristic of this class is stability under the differentiation and multiplication operators.

The ensuing characterization of tempered distributions may readily be generalized to the case of infinitely many dimensions, and a certain class of such generalizations is studied. The spaces of testing elements are required to be stable under the action of the canonical field operators of the quantum theory of free fields, and it is shown that extreme spaces of testing elements exist and have simple properties. In fact, the maximal space is a Montel space, and the minimal complete space is a direct sum of such spaces.

The formalism is applied to the problem of extending the canonical field operators, and a number of extension theorems are derived. In a forthcoming paper* the theory of tempered distributions in infinitely many variables will be applied to a structurally simple linear operator equation.

1. Introduction

Quantum theory has motivated the study of families of linear operators, which

(i) are defined in a linear space with a scalar product, and

(ii) are required to satisfy specified algebraic relations (self-adjointness, commutation relations, etc.).

The two-fold canonical family of self-adjoint operators $p_1, p_2, \ldots$ and $q_1, q_2,$, which satisfy the canonical commutation relations

$$\mathrm{fcP}(\gg p_k] = \mathrm{fe}> q_k] = 0\,, \quad fa, \quad q_k] = -i\,\delta_{jk}\,,$$

is perhaps the best known example.

* *Added in proof:* KRISTENSEN, P., L. MEJLBO, and E. THUE POULSEN: Tempered Distributions in Infinitely Many Dimensions. II, Displacement Operators. Math. Scand. 14, 129—150 (1964).

Since a scalar product is required to exist in the underlying linear space — the carrier space — most investigations have naturally been concerned with the situation, where the carrier space is taken as a Hilbert space. It is then easily seen that the conditions stated above do not suffice to determine the canonical family uniquely, and the problem most extensively studied has been how to formulate weak additional conditions which ensure uniqueness.

For the case of a finite number of pairs p and q such conditions have been given by several authors [3, 5, 6, 13, 16, 17, 20].

The case of an infinite (countable) number of pairs p and q has proved to be much more involved. From the early days of the quantum theory of fields one solution — the so-called canonical solution, for which a vacuum element exists — was known. A rigorous mathematical analysis of this solution was given by COOK [2]. To the surprise of most physicists it was shown by VAN HOVE [11], Friedrichs [7], Fuglede [unpubl.], and others, that there exist several sensible solutions which are not unitarily equivalent. A complete characterization of all solutions satisfying the canonical commutation relations (and further weak conditions) was then given by Gårding and Wightman [10] (see also Wightman and Schweber [22]).

A slightly different version of the problem of infinitely many pairs of operators is this: Let x and y denote real variables (or points in a Euclidian space). A pair of operators is called a pair of canonical field operators if

(i) They are defined as distributions from a space of testing functions to a space of linear operators in a carrier space (a linear space with a scalar product).

(ii) They satisfy commutation relations, the symbolic versions of which are

$$[P(x), P(y)] = [Q(x), Q(y)] = 0\,,$$
$$[P(x), Q(y)] = -i\delta(x-y)\,.$$

(iii) They are self-adjoint on real testing functions.

A motivation for the study of such mathematical structures also arises in non-linear functional analysis. Taking for the carrier space some space of (non-linear) functionals, defined on ordinary functions $f(x)$ of a real variable, the operations

$$Q(x)\,\Phi[f] = f(x)\,\Phi[f]\,,$$
$$P(x)\,\Phi[f] = -i\frac{\delta\Phi[f]}{\delta f(x)}\,,$$

constitute, in a formal sense, a representation of the canonical field operators. We have here adopted the notation $\delta/\delta f(x)$ for the first Volterra derivative, viz.

$$\Phi[f+g] - \Phi[f] = \int \frac{\delta\Phi[f]}{\delta f(x)}\, g(x)\,dx + o(g)\,,$$

where these symbols of course do not have a well defined meaning until appropriate topologies are chosen.

The present paper is primarily concerned with describing an alternative approach to the study of such a pair of canonical field operators. One of the main difficulties of the classical theory is that even though the theory of Hilbert spaces is extremely well developed and in most respects very simple, operators satisfying the canonical commutation relations cannot be bounded and everywhere defined, as shown by WIELANDT [21]. Consequently, when several such operators are involved, difficult questions concerning their common domain of definition arise.

Instead of requiring the carrier space to be a Hubert space, we require the operators to be everywhere defined and continuous, and then we analyze the structure of the possible carrier spaces. When we require the operators to be continuous, we imply in particular that the carrier space has a topology, and, in fact, we require the carrier space to be a locally convex vector space.

For applications it is desirable to have a theory which, in the end, can deliver numerical results expressed by means of continuous linear functionals, and it is well known that a topological vector space can be given a locally convex topology such that the continuous linear functionals are the same in the two topologies. In the case considered here, ultimately the relation between theory and physical reality will be established via an interpretation of certain quantities, expressed in terms of bilinear forms, as expectation values. Obviously the topology determined by the totality of all such expectation values is a locally convex topology on the carrier space (the expectation values are semi-norms), and all desired continuity properties hold for this topology. Hence, from the point of view of applications, the assumption of local convexity is no essential restriction. On the other hand, for the present investigation — as well as for the purpose of quantum theory in general — Hilbert space seems to be an uncomfortably wide structure.

It turns out that the choice of a carrier space which in a sense is smaller than Hilbert space, offers an additional advantage to the facilitation of the algebraic manipulation with the operators:

In the formulation of the algebraic properties of the operators p and q, the assumption of self-adjointness is essential. We replace this assumption with the requirement that they be symmetric with respect to the scalar product on the carrier space. This scalar product induces a natural embedding of the carrier space in its dual space, which is larger than Hilbert space. In quantum theory as well as in non-linear functional analysis, representations of a pair of canonical field operators are desired as tools for the investigation of linear operator equations (linear variational equations). The natural way to impose boundary conditions on equations of this nature is to require the solution to be an element of some linear space. To take this space as Hubert space is in most cases so restrictive that only trivial manifolds of solutions are obtained. Thus, to give an example, the structurally extremely simple "gradient" equation: $P(x)\Phi = 0$, possesses no proper solutions with the boundary condition that Φ be an element of Hilbert space, but it does have a solution in the dual of our carrier space.

It is well known from the study of the canonical commutation relations that for technical reasons it is convenient to work with the operators

$$b_i = \frac{1}{\sqrt{2}}(p_i - i q_i)$$

and their adjoints

$$b_i^* = \frac{1}{\sqrt{2}}(p_i + i q_i)\,.$$

Correspondingly, for the case of the field operators, we introduce

$$a = \frac{1}{\sqrt{2}}(P - iQ)$$

$$a^* = \frac{1}{\sqrt{2}}(P + iQ)\,.$$

As α and α* are operator valued distributions, a space of testing functions for these distributions has to be decided upon. In most applications it is requested that differentiation and other 'One-particle operations" can be given a meaning on the field operators. To make such operations possible, we have chosen as a space of testing functions for α and α* a *space of type $\mathscr{S}$*, whereby we understand a space with the following properties:

(i) The space is a locally convex space with a continuous scalar product.

(ii) There exist operators b and b^*, which are continuous linear mappings from the whole of the space into itself, which are adjoint with respect to the scalar product, and which satisfy the canonical commutation relation $[\delta, b^*] = 1$.

(iii) In the space there exists a normed element ψ_0, which verifies the equation $b\psi_0 = 0$, and which is cyclic relative to b and b^*, i.e. $R\psi_0$ is dense in the space, where R denotes the algebra of all polynomials in b and b^*.

As is known from the works quoted above on the finite-dimensional problem in the framework of Hilbert space, the condition (iii) has here been given an unnecessarily strong formulation. As shown by MEJLBO [15], the condition (iii) is in a certain sense also too strong in the framework of locally convex spaces. However, as we are mainly interested in the extent to which the topology of spaces of type $\mathscr{S}$is determined by the required properties, we have tried to make life easy in other respects.

Obviously such a set of requirements comes close to a characterization of a subspace of Hubert space which bears essentially the same relationship to Hubert space as does SCHWARTZ' space ($\mathscr{S}$) of rapidly decreasing infinitely often differentiable functions to L^2. The precise situation is explained below.

An analysis of spaces of type $\mathscr{S}$ is given in Section 2, where also the corresponding problem for the case of several operators b and b^* is considered. Apart from some technical material needed in later sections, the main results are:

There exist a minimal space $\tilde{\mathscr{S}}$ *and a maximal space* $\mathscr{S}$, *both of type* $\mathscr{S}$, *such that if* $\mathscr{S}^?$ *is any space of type* $\mathscr{S}$ ', then

$$\tilde{\mathscr{S}} \subseteqq \mathscr{S}^? \subseteq \mathscr{S}$$

algebraically and topologically. The space $\tilde{\mathscr{S}}$ *is dense in* $\mathscr{S}^?$, *and* $\mathscr{S}^?$ *is dense in* $\mathscr{S}$. We further prove that the topology of the maximal space $\mathscr{S}$ is determined by a sequence of increasing norms $\| \ \|_r$, $r = 0, 1, 2, \ldots$, where $\|\varphi\|_r^2 = \langle\varphi, (bb^*)^r\varphi\rangle$. *The maximal space may be identified with that subspace ύ of* l^2 *which consists of all sequences* $c — \{c_n\}$, *which are rapidly decreasing with respect to the index in the sense that all the norms* $\|c\|_r^2 — \sum_n |c_n|^2(n+1)^r$ *arefinite.* Finally, we prove that *the maximal space* $\mathscr{S}$ *can be identified with* SCHWARTZ' *space* ($\mathscr{S}$).

Thus, the space of testing functions for tempered distributions may be characterized uniquely up to unitary equivalence as a subspace of abstract Hilbert space in this way: ($\mathscr{S}$) is a maximal space of type $\mathscr{S}$.

For the case of n pairs of canonical operators we define *spaces of type* $\mathscr{S}^n$in a similar way and obtain corresponding results.

For the investigation of the canonical field operators we have in this work chosen the abstract space *ff** as the space of testing elements. Precisely speaking, we have investigated *spaces of type* $\mathfrak{S}$ which we define as spaces with the following properties:

(i) The space is a locally convex space with a continuous scalar product.

(ii) There exist operator valued distributions a and a^* which are continuous linear mappings from $\mathscr{S}$ into the space of continuous linear mappings from the whole of the space of type $\mathfrak{S}$ into itself. This space of continuous linear mappings is here equipped with the topology of uniform convergence on bounded sets. Further, $a(\varphi^*)$ and $a^*(\varphi)$ are adjoint and satisfy the commutation relations

$$[a(\varphi^*), a(\omega^*)] = [a^*(\varphi), a^*(\omega)] = 0\ ,$$
$$[a(\varphi^*), a^*(\omega)] = \langle\varphi, \omega\rangle$$

for all elements φ, ω of $\mathscr{S}$. Here φ^* denotes the conjugate of the element $\varphi \in \mathscr{S}$ in the sense of the natural conjugation in $\mathscr{S}$.

(iii) There exists an element Ψ_0, called the vacuum element, which verifies the equation $a(\varphi^*)\Psi_0 = 0$ for all $\varphi \in \mathscr{S}$, and which is cyclic relative to a and a^*, i.e. $\boldsymbol{R}\Psi_0$ is dense in the space, where R denotes the algebra of all polynomials in all $a(\varphi^*)$ and all $a^*(\varphi)$.

(iv) To every self-adjoint operator $k \in R$ there exists a self-adjoint continuous mapping K from the space of type $\mathfrak{S}$ into itself, such that

$$[K, a^*(\varphi)] = a^*(k\varphi) .$$

Here, by the condition (iii) we single out the particular (canonical) solution for which a vacuum element exists. This greatly facilitates the analysis and also leads to a case of interest for quantum physics. However, this might not be the only interesting case. The condition (iv) is well motivated in the quantum theory of free fields (K is the bi-quantization of k).

An analysis of spaces of type $\mathfrak{S}$ is given in Section 3. The main results are: *There exist a minimal space $\tilde{\tilde{\mathfrak{S}}}'$, a minimal complete space $\tilde{\tilde{\mathfrak{S}}}$, and a maximal space* $\mathfrak{S}$ *of type* $\mathfrak{S}$. The topology of the maximal space is determined by a sequence of seminorms $|||\ \ |||_r$, where $|||\Psi|||_r^2 = \langle\langle \Psi, H^r\Psi\rangle\rangle$. Here $\langle\langle .,.\rangle\rangle$ denotes the scalar product in $\mathfrak{S}$, and H is the mapping which according to (iv) corresponds to the operator $bb^* \in R$. All spaces of type $\mathfrak{S}$ have so-called FOCK representations [4], in which the elements are represented as $\{\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n)}, \ldots\}$, where the n'th coordinate is an element in the symmetric part of a space of type $\mathscr{S}^n$. In this representation $|||\Psi|||_r^2 = \sum_{n=0}^{\infty} \|\psi^{(n)}\|_r^2$. The FOCK representation of the extreme spaces $\tilde{\tilde{\mathfrak{S}}}'$, $\tilde{\tilde{\mathfrak{S}}}$, and $\mathfrak{S}$ are characterized explicitly, and it is shown that they all have simple topological structures.

In the final section the dual spaces of $\tilde{\tilde{\mathfrak{S}}}$ and $\mathfrak{S}$ are studied. In the FOCK representation $\tilde{\tilde{\mathfrak{S}}}^*$ *consists of all sequences* $T = \{T^{(n)}\}$ *of symmetric tempered distributions, while* $\mathfrak{S}^*$ *consists of sequences* $\boldsymbol{T}$, *which in a certain sense are of at most polynomial growth with respect to n.* Furthermore we have the situation

$$\tilde{\tilde{\mathfrak{S}}} \subset \mathfrak{S} \subset \mathfrak{S}^* \subset \tilde{\tilde{\mathfrak{S}}}^*$$

algebraically and topologically, and each of these spaces is dense in each of the spaces it may be imbedded in.

As $\mathfrak{S}$ is the counterpart of SCHWARTZ' space $\mathscr{S}$ for the case of infinitely many dimensions, elements of $\mathfrak{S}^*$ may be looked upon as tempered distributions in infinitely many dimensions.

Finally it is shown that the canonical field operators have unique continuous extensions to various dual spaces, the main result being: *The* *operator a^* has a unique continuous extension from $\mathcal{S}^*$ into the space of continuous linear mappings from & into $\mathfrak{S}^*$. The operator a has a unique continuous extension from $\mathcal{S}^*$ into the space of continuous linear mappings from $\mathfrak{S}$ into $\mathfrak{S}$.*

Thus, for any tempered distribution $T \in \mathcal{S}^*$, $a^*(T)$ and $a(T)$ have well defined meanings. In particular, if T is the Dirac-measure δ_x, concentrated at the point x, we give a well defined meaning to the field operators at the point x. Observe that by the results above $a^*(T)\,a(T)$ is well defined, but $a(T)a^*(T)$ has no meaning. This situation is well known.

We also consider ordered multiple products of the type $a^*(\varphi_1)\ldots a(\varphi_s)$, so-called normal products (Wick-products), and it is shown that *normal products have unique continuous extensions from $\mathcal{S}^{s*}$ into the space of continuous linear mappings from $\mathfrak{S}$ into $\mathfrak{S}^*$.* Similar results hold when $\mathfrak{S}$ is substituted by $\tilde{\mathfrak{S}}$.

The important question of convergence of series of normal products is just barely touched upon in a final remark of Section 4.

It is our hope that some of the material will be of interest to physicists. With this in mind, some results from the theory of locally convex spaces are compiled in the Appendix A, where also the meaning of various notions used in the text is explained.

The maximal spaces all belong to a general class of spaces studied in Appendix B. We have collected this (well-known) material in an appendix forthe convenience of the reader, and forthe purpose of being able to refer to these results in a forthcoming paper, where a simple linear operator equation will be studied.

2. Spaces of type $\mathcal{S}^n$

Definition. *By a space of type $\mathcal{S}^n$ we understand a locally convex space $\mathcal{S}^?$ with the following properties:*

(2.1) *There exists a continuous scalar product $\langle .,.\rangle$ on $\mathcal{S}^?$.*

(2.2) *There exist transformations b_i, b_i^*, $i = 1, 2, \ldots, n$, in $L(\mathcal{S}^?, \mathcal{S}^?)$, such that b_i and b_i^* are adjoint with respect to the scalar product, and such that*

$$[b_i, b_j^*] = b_i b_j^* - b_j^* b_i = \delta_{ij}, \tag{2.3}$$

$$[b_i, b_j] = [b_i^*, b_j^*] = 0\,. \tag{2.4}$$

(2.5) *There exists an element ψ_0, $\|\psi_0\| = 1$, called the cyclic element, in $\mathcal{S}^?$ such that*

$$b_i\psi_0 = 0 \quad \textit{for} \quad i = 1, \ldots, n, \quad \textit{and}$$

(2.6) *$R\psi_0$ is dense in $\mathcal{S}^?$, where R denotes the subalgebra of $L(\mathcal{S}^?, \mathcal{S}^?)$ generated by all the operators b_i and b_i^*.*

Concerning the concepts involved in (2.1) and (2.2) we refer to Appendix A.

Let $\mathcal{S}^?$ be any space of type $\mathcal{S}^n$, and let N^n denote the set of all n-tuples $\nu = \nu_1\nu_2 \ldots \nu_n$ of non-negative integers. For $\nu \in N^n$ we define ψ_ν, called the *Hermite elements* in $\mathcal{S}^?$ as

$$\psi_\nu = \frac{b_1^{*\nu_1} \ldots b_n^{*\nu_n}}{\sqrt{\nu_1! \ldots \nu_n!}} \psi_0, \tag{2.7}$$

and denote by $\tilde{\mathcal{S}}^?$ the linear subspace of $\mathcal{S}^?$ generated by all the elements ψ_ν.

From (2.4) it follows that

$$b_i^* \psi_{\nu_1 \ldots \nu_i \ldots \nu_n} = \sqrt{\nu_i + 1}\, \psi_{\nu_1 \ldots (\nu_i + 1) \ldots \nu_n}, \tag{2.8}$$

and from (2.3) and (2.5) that

$$b_i \psi_{\nu_1 \ldots \nu_i \ldots \nu_n} = \begin{cases} \sqrt{\nu_i}\, \psi_{\nu_1 \ldots (\nu_i - 1) \ldots \nu_n} & \text{if } \nu_i > 0 \\ 0 & \text{if } \nu_i = 0 . \end{cases} \tag{2.9}$$

This shows that $\tilde{\mathcal{S}}^? = R\psi_0$, so that (2.6) can be formulated: $\tilde{\mathcal{S}}^?$ is dense in $\mathcal{S}^?$. Using (2.7), (2.2), (2.9), and the fact that $\|\psi_0\| = 1$, one proves

$$\langle \psi_\nu, \psi_\mu \rangle = \delta_{\nu_1 \mu_1} \delta_{\nu_2 \mu_2} \ldots \delta_{\nu_n \mu_n}, \tag{2.10}$$

so that the elements ψ_ν form an orthonormal basis for $\tilde{\mathcal{S}}^?$.

In particular, then, every element $\varphi \in \tilde{\mathcal{S}}^?$ has a unique representation as a finite linear combination of the elements ψ_ν, viz.

$$\varphi = \sum_{\nu \in N^n}' c_\nu \psi_\nu ,$$

where the prime indicates that the sum is actually finite.

The minimal space $\mathfrak{z}^n$

For each $\nu \in N^n$, let C_ν denote a copy of the complex field, and define

$$\mathfrak{z}^n = \sum_{\nu \in N^n} C_\nu$$

as the direct sum of the spaces C_ν. Thus, algebraically, $\mathfrak{z}^n$ is the space of all multiple sequences

$$c = \{c_\nu\}_{\nu \in N^n}$$

with only a finite number of coordinates c_ν different from zero. We give $\mathfrak{z}^n$ the direct sum topology as explained in Appendix A.

We define a scalar product in $\mathfrak{z}^n$ by

$$\langle c', c'' \rangle = \sum_{\nu \in N^n}' c_\nu' \overline{c_\nu''} \tag{2.11}$$

and operators b_i and b_i^* by

$$(b_i c)_{\nu_1 \ldots \nu_i \ldots \nu_n} = \sqrt{\nu_i + 1}\, c_{\nu_1 \ldots (\nu_i + 1) \ldots \nu_n}, \tag{2.12}$$

$$(b_i^* c)_{\nu_1 \ldots \nu_i \ldots \nu_n} = \begin{cases} 0 & \text{if } \nu_i = 0 \\ \sqrt{\nu_i}\, c_{\nu_1 \ldots (\nu_i - 1) \ldots \nu_n} & \text{if } \nu_i > 0 . \end{cases} \tag{2.13}$$

It is then easily checked that $\mathfrak{z}^n$ is a space of type $\mathcal{S}^n$.

Let $\mathscr{S}^?$ be any space of type $\mathscr{S}^n$and let $\tilde{\mathscr{S}}^?$ be the dense subspace mentioned above. The mapping J defined by

(2.14) $$J : c \to \sum_{\nu \in N^n}{}' c_\nu \psi_\nu$$

is then an algebraic isomorphism of $\mathfrak{z}^n$ onto $\bar{\mathscr{S}}^?$. By the properties of the direct sum topology, J is continuous. Furthermore, J preserves the scalar product and ,,commutes'' with the operators b_i and b_i^* in the respective spaces in the sense that

(2.15) $$b_i J = J b_i\,, \qquad bfJ = J b_i^*\,.$$

Thus J preserves the type-$\mathscr{S}^n$-structure, and hence it is justified to call $\mathfrak{z}^n$ a minimal space of type $\mathscr{S}^n$.

The maximal space $\mathfrak{s}^n$

If $\mathscr{S}^?$ is any space of type $\mathscr{S}^n$,then all semi-norms $\| \quad \|_k$ defined by

$$\|\varphi\|_k = \|k\varphi\|\,, \qquad k \in R\,,$$

are continuous, where E as above denotes the algebra generated by all b_i and b_i^*.

Let $\mathscr{T}$ denote the topology on $\mathscr{S}^?$ determined by the semi-norms $\| \quad \|_k$, $k \in R$. Then $\mathscr{T}$ is the weakest topology on $\mathscr{S}^?$ such that all b_i and bf are continuous.

Before discussing the existence of a maximal space of type $\mathscr{S}^n$ we prove

(2.16) Theorem. *The topology$\mathscr{T}$ on $\mathscr{S}^?$ is determined by the sequence of norms $\| \quad \|_r$ given by*

(2.17) $$\|\varphi\|_r^2 = \langle \varphi, h^r \varphi \rangle,$$

where

$$h = \sum_{i=1}^{n} b_i b_i^*\,.$$

The norms $\| \quad \|_r$ satisfy

(2.18) $$\|\varphi\|_r^2 \geqq n \|\varphi\|_{r-1}^2\,.$$

Proof: First note that (2.18) follows from the identity

$$h = n + \sum_{i=1}^{n} b_i^* b_i\,.$$

Furthermore, since

$$\text{MI?} = \begin{cases} \|h^s \varphi\|^2 & \text{for } r = 2s \\ \sum_{i=1}^{n} \|b_i^* h^s \varphi\|^2 & \text{for } r = 2s + 1\,, \end{cases}$$

all norms $\| \quad \|_r$ are continuous on $\mathscr{S}^?$with respect to the topology $\mathscr{T}$.

In order to prove that the norms $\| \ \|_r$ determine the topology $\mathscr{T}$ on $\mathscr{S}^?$, we must prove that all semi-norms $\| \ \|_k$, $k \in R$, are continuous with respect to the topology determined by the norms $\| \ \|_r$ (lemma A. 2), or, equivalently, that all the operators $k \in R$ are continuous in the topology determined by the norms $\| \ \|_r$. On the other hand, in order to prove this, it is sufficient to prove that b_i and b_i^* are continuous in this topology, and this follows from the identities

$$b_i h^r = (h + 1)^r b_i ,$$
$$b_i^* h^r = (h - 1)^r b_i^* ,$$

which give

$$\sum_i \|b_i \varphi\|_r^2 = \langle \varphi, (h - 1)^r (h - n) \varphi \rangle ,$$
$$\sum_i \|b_i^* \varphi\|_r^2 = \langle \varphi, (h + 1)^r h \varphi \rangle .$$

For certain applications it is sometimes convenient to observe that (2.18) implies that the norms $\| \ \|_r$, r even, determine the topology $\mathscr{T}$ on $\mathscr{S}^?$.

The space $\mathfrak{s}^n$ is now defined as the completion of $\mathfrak{z}^n$ in the topology $\mathscr{T}$.

Then, since the scalar product $\langle .,. \rangle$ and the operators b_i and b_i^* are continuous on $\mathfrak{z}^n$ in the topology $\mathscr{T}$, they have unique continuous extensions defined on $\mathfrak{s}^n$. For these extensions the algebraic relations (2.2), (2.3), and (2.4) hold, and (2.5) is of course also fulfilled in $\mathfrak{s}^n$, the cyclic element of $\mathfrak{z}^n$ being also the cyclic element in $\mathfrak{s}^n$, so that $R\psi_0 = \mathfrak{z}^n$, which is dense in $\mathfrak{s}^n$.

Hence, $\mathfrak{s}^n$ is a space of type $\mathscr{S}^n$.

Before proceeding, we give a concrete representation of the space $\mathfrak{s}^n$ as a space of fast decreasing multiple sequences.

(2.19) Theorem. *The space $\mathfrak{s}^n$ can be identified with the space of those multiple sequences $c = \{c_\nu\}_{\nu \in N^n}$ (with N^n defined as above), for which all the sums*

$$\sum_{\nu \in N^n} (|\nu| + n)^r |c_\nu|^2 = \|c\|_r^2 ,$$

where $|\nu| = \nu_1 + \dots + \nu_n$, are finite. The topology of $\mathfrak{s}^n$ is determined by the norms $\| \ \|_r$ defined above, the scalar product by (2.11), *and the operators b_i and b_i^* by* (2.12) *and* (2.13).

Proof: Trivial, since $h\psi_\nu = (|\nu| + n)\psi_\nu$ in $\mathfrak{z}^n$.

We shall now prove that $\mathfrak{s}^n$ is a maximal space of type $\mathscr{S}^n$.

If $\mathscr{S}^?$ is any space of type $\mathscr{S}^n$, and J is the mapping defined by (2.14), it follows from (2.15) and the fact that J preserves the scalar product that J^{-1} is a continuous mapping of $\tilde{\mathscr{S}}^?$ into $\mathfrak{z}^n$, when $\mathfrak{z}^n$ is given the topology $\mathscr{T}$. Since $\mathfrak{s}^n$ is complete and $\tilde{\mathscr{S}}^?$ is dense in $\mathscr{S}$, the mapping J^{-1} has a unique continuous extension J' which maps $\mathscr{S}^?$ into $\mathfrak{s}^n$.

Clearly J' maps the normalized cyclic element in $\mathscr{S}^?$ into the normalized cyclic element in $\mathfrak{s}^n$, it preserves the scalar product and by continuity it follows from (2.15) that

$$b_i J' = J' b_i, \qquad b_i^* J' = J' b_i^* .$$

Hence J' preserves the type-$\mathscr{S}^n$-structure.

In the sequel we shall use the symbols $\tilde{\mathscr{S}}^n$ and $\mathscr{S}^n$ to denote an arbitrary minimal resp. maximal space of type $\mathscr{S}^n$— any two spaces $\bar{\mathscr{S}}^n$ or $\mathscr{S}^n$ having of course isomorphic type-$\mathscr{S}^n$-structures.

With this convention, if $\mathscr{S}^?$ is any space of type $\mathscr{S}^n$, we may write

$$\tilde{\mathscr{S}}^n \subseteq \mathscr{S}^? \subseteq \mathscr{S}^n$$

algebraically and topologically, where $\tilde{\mathscr{S}}^n$ is the space formerly denoted $\tilde{\mathscr{S}}^?$, but provided with the topology of $\tilde{\mathfrak{s}}^n$, while $\mathscr{S}^n$ denotes the completion of $\mathscr{S}^?$ in the topology $\mathscr{T}$.

If the element $\varphi \in \mathscr{S}^n$ corresponds to the multiple sequence $\{c_\nu\}$ in the copy $\mathfrak{s}^n$ of $\mathscr{S}^n$, we shall often find it convenient to write

$$\varphi = \sum_{\nu \in N^n} c_\nu \psi_\nu ,$$

where ψ_ν is the Hermite element with index ν in $\mathscr{S}^n$. It is clear, from the definition of the topology, that this sequence converges unconditionally to φ in $\mathscr{S}^n$.

Representation of $\mathscr{S}^n$ as SCHWARTZ' *space* $(\mathscr{S})$ *in n dimensions*

SCHWARTZ' space $(\mathscr{S})$ over the n-dimensional space E^n is the space of those infinitely often differentiable functions on E^n for which all the semi-norms

$$\sup_{t \in E^n} |t^\alpha D^\beta \varphi(t)|$$

are finite, where

$$t^\alpha D^\beta \varphi(t) = t_1^{\alpha_1} \dots t_n^{\alpha_n} \frac{\partial^{\beta_1}}{\partial t_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial t_n^{\beta_n}} \varphi(t) .$$

The space $(\mathscr{S})$ is given the topology determined by these semi-norms. It is clear that if we put

$$q_i = t_i, \qquad p_i = -i \frac{\partial}{\partial t_i}$$

$$b_i = \frac{1}{\sqrt{2}} (p_i - i q_i) , \qquad b_i^* = \frac{1}{\sqrt{2}} (p_i + i q_i) ,$$

then the operators b_i and b_i^* satisfy (2.2), (2.3), and (2.4), the semi-norms

$$\|\varphi\|_k^{(\mathscr{S})} = \sup_{t \in E^n} |k \varphi|$$

are continuous in the topology of $(\mathscr{S})$ for all operators $k \in R$, and the topology of $(\mathscr{S})$ is determined by these semi-norms.

It is well known that $(\mathscr{S}) \subset L^2$ (with respect to Lebesgue measure) and that the L^2-norm $\| \ \|$ can be estimated by

$$\|\varphi\| \leqq C\|\varphi\|_k^{(\mathscr{S})},$$

where

$$k = (1 + \sum_{i=1}^{n} q_i^2)^s$$

with $4s > n$.

On the other hand we have

(2.20) Sobolev's lemma ([19]). *If s is an integer with $s > \frac{n}{2}$, then there exists a constant K suck that*

$$\sup_{t \in E^n} |\varphi(t)| \leqq K(\|\varphi\| + \sum_{\beta_1 + \cdots + \beta_n = s} \|p_1^{\beta_1} \quad p_n^{\beta_n} \varphi\|)$$

for all $\varphi \in (\mathscr{S})$.

It now follows from the lemma (A. 2) that the topology of $(\mathscr{S})$is determined by the system of semi-norms

$$\|\varphi\|_k = \|k\varphi\|, \quad k \in R,$$

so that the topology of $(\mathscr{S})$ is in fact the topology $\mathscr{T}$.

It is well known that the Hermite functions

$$\psi_\nu = \frac{b_1^{*\nu_1} \ldots b_n^{*\nu_n}}{\sqrt{\nu_1! \ldots \nu_n!}} \psi_0,$$

where

$$\psi_0 = \pi^{-\frac{n}{4}} e^{-\frac{1}{2}\Sigma t_i^2}$$

are elements of $(\mathscr{S})$, that they constitute a complete orthonormal system in L^2, and that (2.5) holds. Let $(\bar{\mathscr{S}})$ denote the linear subspace of $(\mathscr{S})$ spanned by the Hermite functions. Evidently, $(\bar{\mathscr{S}})$ can be identified algebraically with $\bar{\mathscr{S}}^n$. Since $(\mathscr{S})$has the topology $\mathscr{T}$ of the maximal space $\mathscr{S}^n$and is complete, it follows that $(\mathscr{S})$ contains a maximal space $\mathscr{S}^n$.

On the other hand, for any element $\varphi \in (\mathscr{S})$, the norm $\|h^r\varphi\| = \|\varphi\|_{2r}$ is finite, since $h^r\varphi$ is an element of $(\mathscr{S})$. Now, if

$$\varphi = \sum_{\nu \in N^n} c_\nu \psi_\nu$$

in L^2, then

$$\langle \psi_\nu, h^r\varphi \rangle = \langle h^r\psi_\nu, \varphi \rangle = (|\nu| + n)^r c_\nu,$$

and hence, by Parsevals formula,

$$\sum_{\nu \in N^n} (|\nu| + n)^{2r} |c_\nu|^2 = \|\varphi\|_{2r}^2 < \infty.$$

Thus, $(\mathscr{S})$may be identified with a subspace of the maximal space $\mathscr{S}^n$.

Hence we have the result: SCHWARTZ' space $(\mathscr{S})$ in n dimensions is a copy of the maximal space $\mathscr{S}^n$.

We remark that the algebraic and topological isomorphism of SCHWARTZ' space $(\mathscr{S})$ with the space $\mathfrak{s}^n$ of fast decreasing sequences is well known (cf., for instance, SCHWARTZ [18]).

The remaining part of this section contains material needed in the sequel.

A conjugation in the space $\mathscr{S}$

In the sequel *SP* will always denote the maximal space $\mathscr{S} = \mathscr{S}^1$, which, as discussed above can be represented as a space of sequences or as SCHWARTZ' space *(SP)* over the real line.

If we interpret *SP* as SCHWARTZ' space of rapidly decreasing testing functions on the real line, then there is defined a natural conjugation $\varphi \to \varphi^*$ in $\mathscr{S}$ by $\varphi^*(t) = \overline{\varphi(t)}$.

It is easily verified that $\psi_0^* = \psi_0$ and $(b^*\varphi)^* = -\, b^*\varphi^*$, and hence, in the sequence representation,

$$(\sum c_n \psi_n)^* = \sum (-1)^n \bar{c}_n \psi_n ,$$

ψ_n being the Hermite elements in $\mathscr{S}$.

The tensor product $\mathscr{S}^{n\otimes} = \mathscr{S} \otimes \mathscr{S} \otimes$ ® $\mathscr{S}$ as a space of type $\mathscr{S}^n$

Let $\mathscr{S}^{n\otimes}$ denote the n-fold algebraic tensor product

$$\mathscr{S}^{n\otimes} = \mathscr{S} \otimes \cdots \otimes \mathscr{S} ,$$

i.e. $\mathscr{S}^{n\otimes}$ is a vector space having the family of ordered n-ics of the form

$$\varphi = \varphi_1 \ldots \varphi_n , \; \varphi_i \in \mathscr{S} \text{ for } i = 1, \ldots, n ,$$

as generators.

We define a scalar product on $\mathscr{S}^{n\otimes}$ by putting

$$\langle \varphi_1 \ldots \varphi_n , \omega_1 \ldots \omega_n \rangle = \prod_{i=1}^{n} \langle \varphi_i, \omega_i \rangle$$

for the generating *n-ics* and then extending by linearity.

Let b and b^* denote the operators b_1 and b_1^* in $\mathscr{S}$, and define

$$b_i(\varphi_1 \ldots \varphi_i \ldots \varphi_n) = \varphi_1 \ldots (b\varphi_i) \ldots \varphi_n ,$$

$$6?\,(\varphi_1 \ldots \varphi_i \ldots \varphi_n) = \varphi_1 \ldots (b^*\varphi_i) \ldots \varphi_n .$$

It is clear that if we determine a topology on $\mathscr{S}^{n\otimes}$ by means of the norms \\ $\|_r$ defined by (2.17), then $\mathscr{S}^{n\otimes}$ is a space of type $\mathscr{S}^n$, and if we think of $\mathscr{S}^{n\otimes}$ as a subspace of $\mathscr{S}^n$ (which as we know is permissible), then the topology of $\mathscr{S}^{n\otimes}$ is exactly the topology induced from $\mathscr{S}^n$. Observe that $\mathscr{S}^{n\otimes}$ is not a complete space for $n > 1$.

The symmetric spaces $\mathscr{S}_+^{n\otimes}$ *and* $\mathscr{S}_+^n$

We add a few remarks on the symmetric parts of the n-fold algebraic tensor product $\mathscr{S}^{n\otimes}$ and of $\mathscr{S}^n$.

On the generating elements of $\mathscr{S}^{n\otimes}$ we define an operator sym by

$$\operatorname{sym}(\varphi_1 \ldots \varphi_n) = \frac{1}{n!} \sum_{\pi \in S_n} \varphi_{\pi(1)} \quad \varphi_{\pi(n)},$$

where S_n denotes the symmetric group of degree n, and extend sym by linearity to the whole of $\mathscr{S}^{n\otimes}$. It is easily verified that sym is an orthogonal projection in $\mathscr{S}^{n\otimes}$ with respect to the scalar product in $\mathscr{S}^{n\otimes}$.

Furthermore, if k is any linear operator in $\mathscr{S}$, and if we define k_i by

$$k_i(\varphi_1 \ldots \varphi_i \ldots \varphi_n) = \varphi_1 \ldots (k\varphi_i) \ldots \varphi_n$$

on the generating elements of $\mathscr{S}^{n\otimes}$ and extend by linearity, then

$$\operatorname{sym} k_i = \frac{1}{n} k^{(n)} \operatorname{sym} ,$$

where

(2.21) $$k^{(n)} = k_1 + \cdots + k_n .$$

Hence

$$\operatorname{sym} k^{(n)} = k^{(n)} \operatorname{sym} ,$$

and if we define

$$\mathscr{S}_+^{n\otimes} = \operatorname{sym}(\mathscr{S}^{n\otimes}) ,$$

then $\mathscr{S}_+^{n\otimes}$ is invariant under $k^{(n)}$ for any operator k in $\mathscr{S}$.

In particular, $\mathscr{S}_+^{n\otimes}$ is invariant under the operator $h^{(n)}$, which we earlier denoted h, and which determines the topology of $\mathscr{S}^n$ and $\mathscr{S}^{n\otimes}$.

It follows that for $\varphi \in \mathscr{S}^{n\otimes}$ and $r = 2s$

$$\|\operatorname{sym}\varphi\|_r = \|h^{(n)s}\operatorname{sym}\varphi\| \leqq \|h^{(n)}\varphi\| = \|\varphi\|_r,$$

which shows that sym is continuous on $\mathscr{S}^{n\otimes}$ in the topology of $\mathscr{S}^n$.

Consequently, sym has a unique continuous extension to $\mathscr{S}^n$ we shall also denote this extension by sym — it is of course a projection in $\mathscr{S}^n$, and its effect in any of the two standard representations of $\mathscr{S}^n$ is exactly what one would expect.

Finally, let us prove the following useful lemma.

(2.22) Lemma. *If T is a linear transformation from $\mathscr{S}^{n\otimes}$ into some vector space V, and if*

$$T(\varphi^n) = T(\varphi \ldots \varphi) = 0 \quad \textit{for all } \varphi \in \mathscr{S} ,$$

then $T(\omega) = 0$ *for all* $\omega \in \mathscr{S}_+^{n\otimes}$.

Proof: It is sufficient to prove that $T(\operatorname{sym}(\varphi_1 \ldots \varphi_n)) = 0$ for all generating elements $\varphi_1 \ldots \varphi_n$ in $\mathscr{S}^{n\otimes}$. By assumption we have for all complex numbers $c_1, \ldots, c_n$

$$T((c_1\varphi_1 + \cdots + c_n\varphi_n)^n) = 0 .$$

The left hand side is a polynomial in $c_1, \ldots, c_n$, and since it is identically 0, all coefficients must be 0. In particular, the coefficient to the term $c_1 c_2 \ldots c_n$ must be 0, and this coefficient is $n!\, T((\mathrm{sym}(\varphi_1 \ldots \varphi_n))$.

(2.23) Corollary. *The elements of the form φ^n, $\varphi \in \mathscr{S}$, generate $\mathscr{S}^{n\otimes}_{+}$.*

3. Spaces of type $\mathfrak{S}$

Definition and analysis

Definition. *By a space of type $\mathfrak{S}$ we understand a locally convex space $\mathfrak{S}^?$ with the following properties:*

(3.1) *There exists a continuous scalarproduct $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathfrak{S}^?$ (the corresponding norm is denoted by $|||\ \ |||$).*

(3.2) *There exist continuous linear mappings a and a^* from $\mathscr{S}$ into $L(\mathfrak{S}^?, \mathfrak{S}^?)$, such that $a(\varphi^*)$ and $a^*(\varphi)$ are adjoint with respect to the scalar product in & for all $\varphi \in \mathscr{S}$, and such that*

$$(3.3)\qquad [a(\varphi^*), a^*(\omega)] = \langle \varphi, \omega \rangle ,$$

$$(3.4)\qquad [a(\varphi^*), a(\omega^*)] = [a^*(\varphi), a^*(\omega)] = 0 .$$

(3.5) *There exists an element Ψ_0 of norm one, called the vacuum element, such that $a(\varphi^*)\Psi_0 = 0$ for all $\varphi \in \mathscr{S}$, and*

(3.6) *$\boldsymbol{R}\Psi_0$ is dense in $\mathfrak{S}^?$, where R denotes the subalgebra of $L(\mathfrak{S}^?, \mathfrak{S}^?)$ generated by all operators $a(\varphi^*)$ and $a^*(\omega)$.*

(3.7) *To every self-adjoint operator $k \in R \subset L(\mathscr{S}, \mathscr{S})$ (cf. (2.6)), there exists a self-adjointoperator $K \in L(\mathfrak{S}^?, \mathfrak{S}^?)$ satisfying*

$$(3.8)\qquad [a(\varphi^*), K] = a((k^*\varphi)^*) ,$$

$$(3.9)\qquad [K, a^*(\varphi)] = a^*(k\varphi) .$$

The space $L(\mathfrak{S}^?, \mathfrak{S}^?)$ is provided with the topology of uniform convergence on bounded sets as explained in Appendix A.

In this section we shall prove results completely analogous to the results of Section 2 concerning the existence of a minimal space $\tilde{\mathfrak{S}}'$, a minimal complete space $\bar{0}$, and a maximal space $\mathfrak{S}$ of type $\mathfrak{S}$.

Let $\mathfrak{S}^?$ be any space of type $\mathfrak{S}$, and let $\Psi_n[\varphi_1 \ldots \varphi_n]$ denote the element

$$\Psi_n[\varphi_1 \ldots \varphi_n] = \frac{1}{\sqrt{n!}} a^*(\varphi_1) \ldots a^*(\varphi_n)\Psi_0$$

in $\mathfrak{S}^?$, where $\varphi_i \in \mathscr{S}$ for $i = 1, \ldots, n$.

It is clear that Ψ_n can be extended to a linear mapping from $\mathscr{S}^{n\otimes}$ into $\mathfrak{S}^?$. Let $\tilde{\mathfrak{S}}^?$ denote the linear subspace of $\mathfrak{S}^?$ generated by Ψ_0 and all elements $\Psi_n[\varphi^{(n)}]$, $\varphi^{(n)} \in \mathscr{S}^{n\otimes}$, $n = 1, 2, \ldots$

Now we have

$$(3.10)\qquad a^*(\varphi)\Psi_n[\varphi_1 \ldots \varphi_n] = \sqrt{n+1}\,\Psi_{n+1}[\varphi\varphi_1 \ldots \varphi_n] ,$$

and from (3.3) and (3.5) we get for $n > 0$

$$(3.11) \quad a(\varphi^*)\, \Psi_n[\varphi_1 \dots \varphi_n] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \langle \varphi, \varphi_i \rangle \Psi_{n-1}[\varphi_1 \dots \varphi_{i-1} \varphi_{i+1} \dots \varphi_n].$$

Just as in Section 2 we conclude from (3.10), (3.11), and (3.6) that

$$\tilde{\mathfrak{S}}^? \text{ is dense in } \mathfrak{S}^?.$$

Before we proceed to discuss the scalar product and topology of $\mathfrak{S}^?$, we note that because of (3.4) the mappings Ψ_n are invariant under permutations of the factors in $\mathscr{S}^{n\otimes}$, *viz.*

$$(3.12) \qquad \Psi_n[\varphi^{(n)}] = \Psi_n[\mathrm{sym}(\varphi^{(n)})] \quad \text{for all} \quad \varphi^{(n)} \in \mathscr{S}^{n\otimes}.$$

By assumption (3.2), the operators $a^*(\varphi)$ and $a(\varphi^*)$ are adjoint. Now, consider two elements in $\mathfrak{S}^?$ of the form $\Psi_n[\varphi^n]$ and $\Psi_m[\omega^m]$, $\varphi, \omega \in \mathscr{S}$. From (3.10) and (3.11) we get

$$\begin{aligned}
\langle\!\langle \Psi_n[\varphi^n], \Psi_m[\omega^m] \rangle\!\rangle &= \\
&= \frac{1}{\sqrt{n}} \langle\!\langle a^*(\varphi)\Psi_{n-1}[\varphi^{n-1}], \Psi_m[\omega^m] \rangle\!\rangle \\
&= \frac{1}{\sqrt{n}} \langle\!\langle \Psi_{n-1}[\varphi^{n-1}], a(\varphi^*)\Psi_m[\omega^m] \rangle\!\rangle \\
&= \sqrt{\frac{m}{n}}\, \langle \varphi, \omega \rangle \langle\!\langle \Psi_{n-1}[\varphi^{n-1}], \Psi_{m-1}[\omega^{m-1}] \rangle\!\rangle.
\end{aligned}$$

As in the proof of (2.10) we conclude that

$$\langle\!\langle \Psi_n[\varphi^n], \Psi_m[\omega^m] \rangle\!\rangle = \begin{cases} 0 & \text{if } m \neq n \\ \langle \varphi, \omega \rangle^n & \text{if } m = n, \end{cases}$$

since $|||\Psi_0||| = 1$.

It follows from the corollary (2.23) that if $\varphi^{(n)} \in \mathscr{S}_+^{n\otimes}$ and $\omega^{(m)} \in \mathscr{S}_+^{m\otimes}$, then

$$(3.13) \qquad \langle\!\langle \Psi_n[\varphi^{(n)}], \Psi_m[\omega^{(m)}] \rangle\!\rangle = \begin{cases} 0 & \text{if } m \neq n \\ \langle \varphi^{(n)}, \omega^{(n)} \rangle & \text{if } m = n, \end{cases}$$

where $\langle \varphi^{(n)}, \omega^{(n)} \rangle$ denotes the scalar product in $\mathscr{S}^{n\otimes}$.

Thus Ψ_n is an isometry from $\mathscr{S}_+^{n\otimes}$ into $\mathfrak{S}^?$, and, furthermore, when (3.12) is taken into account, we have

(3.14) Theorem. *Every element Φ in $\tilde{\mathfrak{S}}^?$ has a unique representation as a finite orthogonal sum*

$$\Phi = \varphi^{(0)} \Psi_0 + \sum_{n=1}^{N} \Psi_n[\varphi^{(n)}],$$

which we shall write in the form

$$\Phi = \sum_{n=0}^{\infty}{}' \Psi_n[\varphi^{(n)}]$$

with

$$\varphi^{(n)} \in \mathcal{S}_+^{n\otimes},$$

where, for convenience, we let $\mathcal{S}_+^{0\otimes}$ *denote the complex field.*

We shall now investigate the consequences of the requirements (3.7)—(3.9), but first we need the following lemmas.

(3.15) Lemma. *The set of equations*

$$a(\varphi^*)\Psi = 0 \ \textit{for all} \quad \varphi \in \mathcal{S}$$

have in $\mathfrak{S}^?$ *the only solutions*

$$\Psi = c\Psi_0, \quad c \in C .$$

Proof: A solution orthogonal to Ψ_0 is easily seen to be orthogonal to all of $\tilde{\mathfrak{S}}^?$, which, however, is dense in $\mathfrak{S}^?$.

(3.16) Lemma. *If k is any linear operator in* $\mathcal{S}$, *then there exists a unique linear operator K in* $\tilde{\mathfrak{S}}^?$ *satisfying the conditions* (3.8), (3.9), *and*

(3.17) $$K\Psi_0 = 0 .$$

The operator K is given by

(3.18) $$K\Psi_n[\varphi^{(n)}] = \Psi_n[k^{(n)}\varphi^{(n)}] \quad \textit{for all} \ \varphi^{(n)} \in \mathcal{S}_+^{n\otimes},\ n > 0 ,$$

where $k^{(n)}$ *is defined in* (2.21).

If there exists a continuous operator in $\mathfrak{S}^?$ *satisfying* (3.8) *and* (3.9), *then there exists a unique one satisfying* (3.17).

Proof: Assume that K satisfies (3.8). Then, by (3.15)

$$K\Psi_0 = c\Psi_0.$$

Since the operator cI commutes with all $a(\varphi^*)$ and all $a^*(\varphi)$, it follows that if there exists an operator satisfying (3.8) and (3.9), then there exists one which in addition satisfies (3.17).

Assume now that K satisfies (3.9) and (3.17), and write (3.9) in the form

$$K a^*(\varphi) = a^*(\varphi) K + a^*(k\varphi) .$$

By successive application of this formula to Ψ_0, $\Psi_1[\varphi], \ldots, \Psi_n[\varphi^n], \ldots$ we get

$$K\Psi_n[\varphi^n] = \Psi_n[k^{(n)}\varphi^n] .$$

From the corollary (2.23) it follows that K is actually given by (3.18), and the statements concerning uniqueness of K follow. Finally, if for a given operator k we define K on $\tilde{\mathfrak{S}}^?$ by (3.17) and (3.18), then it is easily verified that K satisfies (3.8) and (3.9).

For any operator k in $\mathcal{S}$ both the operator K defined on $\tilde{\mathfrak{S}}^?$ by (3.17) and (3.18) and its continuous extension to $\mathfrak{S}^?$ (if it exists) are called *the normalized bi-quantization of k.*

The minimal space $\tilde{\mathfrak{S}}'$ and its FOCK *representation*

We define the space $\tilde{\mathfrak{S}}'$ as the direct sum

$$\tilde{\mathfrak{S}}' = \sum_{n=0}^{\infty} \mathscr{S}_{+}^{n\otimes},$$

and we shall write elements $\Phi \in \tilde{\mathfrak{S}}'$ in one of the forms

$$\Phi = \{\varphi^{(n)}\} = \{\varphi^{(0)}, \varphi^{(1)}, \ldots\}.$$

The theorem (3.14) can now be formulated:

(3.19) *The mapping* $J: \{\varphi^{(n)}\} \to \sum_{n=0}^{\infty}{}' \Psi_n[\varphi^{(n)}]$ *is an algebraic isomorphism of* $\tilde{\mathfrak{S}}'$ *onto the dense subspace* $\tilde{\mathfrak{S}}^{?}$ *of any space* & *of type* $\mathfrak{S}$.

Furthermore, if we define a scalar product in $\tilde{\mathfrak{S}}'$ by

$$\langle\!\langle \Omega, \Phi \rangle\!\rangle = \sum_{n=0}^{\infty}{}' \langle \omega^{(n)}, \varphi^{(n)} \rangle,$$

then this scalar product is continuous (lemma (A. 13)), and by (3.13) the mapping J preserves scalar products.

We now show that $\tilde{\mathfrak{S}}'$ can be organized as a space of type $\mathfrak{S}$.

As we want the operators $a^*(\varphi)$ and $a(\varphi^*)$ to "commute" with J, we find from (3.10) and (3.11) how they must necessarily behave. It is of course sufficient to define these operators on elements $\chi \in \mathscr{S}_{+}^{n\otimes} \subset \tilde{\mathfrak{S}}'$. From (3.10) we find

(3.20) $$a^*(\varphi)\chi = \sqrt{n+1}\,\mathrm{sym}(\varphi\ \chi), \quad \chi \in \mathscr{S}_{+}^{n\otimes},$$

which we now take as the definition of the linear operator $a^*(\varphi)$ on $\tilde{\mathfrak{S}}'$. Similarly, (3.11) leads to the definition

(3.21) $$a(\varphi^*)\chi = \sqrt{n}\,\langle \varphi, \chi \rangle_{(1)}, \quad \chi \in \mathscr{S}_{+}^{n\otimes},$$

where $\langle \varphi, . \rangle_{(1)}$ denotes that linear mapping from $\mathscr{S}^{n\otimes}$ into $\mathscr{S}^{(n-1)\otimes}$, which on the generating elements of $\mathscr{S}^{n\otimes}$ is given by

$$\langle \varphi, \varphi_1 \ldots \varphi_n \rangle_{(1)} = \langle \varphi, \varphi_1 \rangle \varphi_2 \ldots \varphi_n.$$

Finally, the normalized bi-quantization K of an operator $k \in R$ must be defined by

$$K\chi = k^{(n)}\chi, \quad \chi \in \mathscr{S}_{+}^{n\otimes},$$

as follows from (3.18).

It is obvious that the linear operators a, a^*, and K thus defined satisfy the correct commutation relations. Consequently, all we need verify is the continuity of these operators.

We first consider a^*. To prove that $a^*(\varphi) \in L(\tilde{\mathfrak{S}}', \tilde{\mathfrak{S}}')$ it is sufficient to show that $a^*(\varphi) \in L(\mathscr{S}_{+}^{n\otimes}, \mathscr{S}_{+}^{(n+1)\otimes})$ (cf. (A. 14)). Since sym is continuous, it is enough to prove that the mapping

$$\chi \rightharpoonup \varphi\chi$$

is continuous from $\mathscr{S}^{n\otimes}$into $\mathscr{S}^{(n+1)\otimes}$.That this is the case follows from the identity

0.22)
$$
\begin{aligned}
\|\varphi\chi\|_r^2 &= \langle \varphi\chi, (h_1 + \quad + h_{n+1})^r \varphi\chi\rangle \\
&= \langle \varphi\chi, \sum_0^r \binom{r}{s} h_1^s \, (h_2 + \cdots + h_{n+1})^{r-s} \varphi\chi\rangle \\
&= \sum_0^r \binom{r}{s} \|\varphi\|_s^2 \, \|\chi\|_{r-s}^2 .
\end{aligned}
$$

Next, we prove that the mapping $a^* : \mathscr{S} \rightharpoonup L(\tilde{\mathfrak{S}}', \tilde{\mathfrak{S}}')$ is continuous. As $\mathscr{S}$ is metrizable it is sufficient to prove that for every bounded set A in *SP* and every bounded set B in $\tilde{\mathfrak{S}}'$ the subset $a^*(A)\,B$ of $\tilde{\mathfrak{S}}'$ is bounded (lemma (A. 7) and lemma (A. 10)). If B is a bounded set in $\tilde{\mathfrak{S}}'$, then (lemma (A. 16)) there exists a number N and bounded sets B_0, B_1, . . ., B_N in $\mathscr{S}_+^{0\otimes}$, $\mathscr{S}_+^{1\otimes}$, . . ., $\mathscr{S}_+^{N\otimes}$ respectively, such that

$$B \subset \sum_{i=1}^{N} B_i .$$

Hence, it is sufficient to prove that if B is a bounded set in some space $\mathscr{S}_+^{n\otimes}$ and A a bounded set in $\mathscr{S}$, then $a^*(A)B$ is bounded in $\mathscr{S}_+^{(n+1)\otimes}$, and thisfollows from (3.22).

Consider next the operator a. If SCHWARTZ' representation of $\mathscr{S}^n$is used, one has

$$\|\langle \varphi, \chi\rangle_{(1)}\|^2 = \langle \varphi, G\varphi\rangle ,$$

where G is the non-negative Hilbert-Schmidt operator defined by

$$\beta\varphi(x) = // \chi(x, z)\, \chi^*(y, z)\; dz\, \varphi(y)\, dy .$$

Since $\langle \varphi, G\varphi\rangle \leqq \|\varphi\|^2$ trace G, we have the inequality

$$\|\langle \varphi, \chi\rangle_{(1)}\| \leqq \|\varphi\|\, \|\chi\| .$$

From this we get

$$
\begin{aligned}
& \|\langle \varphi, \chi\rangle_{(1)}\|\|s \\
&= \langle (h_2 + \cdots + h_n)^s \langle \varphi, \chi\rangle_{(1)}, (h_2 + \cdots + h_n)^s \langle \varphi, \chi\rangle_{(1)}\rangle \\
&= \langle \langle \varphi, (h_2 + \cdots + h_n)^s \chi\rangle_{(1)}, \langle \varphi, (h_2 + \cdots + h_n)^s \chi\rangle_{(1)}\rangle \qquad (3.23) \\
&\leqq \|\varphi\|^2 \, \|(h_2 + \cdots + h_n)^s \chi\|^2 \\
&\leqq \|\varphi\|^2 \, \|\chi\|_{2s}^2 .
\end{aligned}
$$

This shows that the mapping $\langle \varphi, \;\rangle_{(1)}$ is continuous from $\mathscr{S}^{n\otimes}$ into $\mathscr{S}^{(n-1)\otimes}$ It is clear that it maps $\mathscr{S}_+^{n\otimes}$ into $\mathscr{S}_+^{(n-1)\otimes}$, and by an argument analogous to that given above, we conclude that a is continuous $\mathscr{S} \rightharpoonup L(\tilde{\mathfrak{S}}', \tilde{\mathfrak{S}})$.

Finally we prove that for any operator $k \in L(\mathscr{S},\mathscr{S})$, the normalized bi-quantization K of k defined on $\tilde{\mathfrak{S}}'$ by

$$K\chi = k^{(n)}\chi \quad \text{for all } \chi \in \mathscr{S}_{+}^{n\otimes},$$

where $k^{(n)}$ is defined by (2.21), is continuous and satisfies (3.8) and (3.9). Furthermore, K is self-adjoint if and only if k is self-adjoint.

The only thing requiring proof is the continuity of K, and as above, it is sufficient to prove that the restriction $k^{(n)}$ of K to each of the spaces $\mathscr{S}_{+}^{n\otimes}$ is continuous.

First, consider the operator k_n on $\mathscr{S}^{n\otimes}$. Every element $\chi \in \mathscr{S}^{n\otimes}$ can be written in the form

$$\chi = \sum_{\nu} \psi_\nu \varphi_\nu,$$

where ψ_ν runs through the Hermite elements of $\mathscr{S}^{(n-1)\otimes}$, while the φ_ν are elements in $\mathscr{S}$. The norms of χ are given by

$$\begin{aligned}\|\chi\|_r^2 &= \langle \sum_{\nu} \psi_\nu \varphi_\nu, (h_1 + \cdots + h_n)^r \sum_{\nu} \psi_\nu \varphi_\nu \rangle \\ &= \langle \sum_{\nu} \psi_\nu \varphi_\nu, \sum_{s=0}^{r} \binom{r}{s} (h^{(n-1)})^s h_n^{r-s} \sum_{\nu} \psi_\nu \varphi_\nu \rangle \\ &= \sum_{s=0}^{r} \binom{r}{s} \sum_{\nu} \|\psi_\nu\|_s^2 \|\varphi_\nu\|_{r-s}^2,\end{aligned}$$

so that also

$$\|k_n \chi\|_r^2 = \sum_{s=0}^{r} \binom{r}{s} \|\psi_\nu\|_s^2 \|k\varphi_\nu\|_{r-s}^2. \tag{3.24}$$

Since $k \in L(\mathscr{S},\mathscr{S})$, there exist constants $C(r)$ and $q(r) \geqq r$ such that

$$\|k\varphi\|_s \leqq C(r) \|\varphi\|_{q(r)}$$

for all $\varphi \in \mathscr{S}$ and all $s \leqq r$. It then follows from (3.24) that

$$\begin{aligned}\|k_n\chi\|_r^2 &\leqq 2^r C(r)^2 \sum_{\nu} \langle \psi_\nu, (h^{(n-1)})^{q(r)} \psi_\nu \rangle \langle \varphi_\nu, h^{q(r)} \varphi_\nu \rangle \\ &\leqq 2^r C(r)^2 \langle \chi, ((h^{(n-1)})^{q(r)} + h_n^{q(r)})\chi \rangle \\ &\leqq 2^r C(r)^2 \|\chi\|_{q(r)}^2.\end{aligned} \tag{3.25}$$

Clearly, for $\chi \in \mathscr{S}_{+}^{n\otimes}$ we have

$$\|k_1\chi\| = \|k_2\chi\| = \cdots = \|k_n\chi\|,$$

and hence we get from (3.25) and (2.18)

$$\begin{aligned}\|k^{(n)}\chi\|_r &\leqq n\|k_n\chi\|_r \\ &\leqq A(r) \|\chi\|_{q(r)} \\ &\leqq A(r) \|\chi\|_{q(r)+2}.\end{aligned} \tag{3.26}$$

Thus, K is continuous, and we conclude that $\tilde{\mathfrak{S}}'$ is a space of type (S.

(3.27) Theorem. *$\tilde{\mathfrak{S}}'$ is a minimal space of type Θ, i.e. $\tilde{\mathfrak{S}}'$ is of type $\mathfrak{S}$, and if $\mathfrak{S}^{?}$ is any space of type $\mathfrak{S}$, then there exists an identification mapping J of $\tilde{\mathfrak{S}}'$ onto a dense subspace $\tilde{\mathfrak{S}}^{?}$ of $\mathfrak{S}^{?}$, which preserves the type-$\mathfrak{S}$-structure.*

By this last statement we mean that J preserves scalar products, that it maps a normalized vacuum element into a normalized vacuum element, and that it "commutes" with all operators $a(\varphi^*)$ and $a^*(\varphi)$ as well as with all normalized bi-quantizations K of self-adjoint operators $k \in E$.

Proof: The identification mapping J of the theorem is of course the mapping defined by (3.19). Obviously, all we need prove is that J is continuous. Here it is sufficient to verify that the mappings Ψ_n from $\mathscr{S}^{n\otimes}$ into $\mathfrak{S}^{?}$ are continuous (lemma (A. 14)). By assumption, a^* is continuous from $\mathscr{S}$ into $L(\mathfrak{S}^{?}, \mathfrak{S}^{?})$. Then it is easily seen that

$$\Psi_n[\varphi_1 \quad \varphi_n] = \frac{1}{\sqrt{n!}} a^*(\varphi_1) \quad . . \; a^*(\varphi_n)\Psi_0$$

is continuous in each variable φ_i separately from $\mathscr{S}$ into $\mathfrak{S}^{?}$. It now follows from results due to GROTHENDIECK [9] that Ψ_n is continuous from $\mathscr{S}^{n\otimes}$ into $\mathfrak{S}^{?}$. However, in the present case, a proof may also be given by elementary means. We here give the proof for the case $n = 2$. The general case presents no new problems.

We first prove that if $U^{?}$ is any neighbourhood of 0 in $\mathfrak{S}^{?}$, then there exists a neighbourhood

$$K_{r,\sigma} = \{\varphi \in \mathscr{S} \quad \|\varphi\|_r < \sigma\}$$

of 0 in $\mathscr{S}$ such that

$$\varphi \in K_{r,\sigma}, \psi \in K_{r,\sigma} \Rightarrow a^*(\varphi)\; a^*(\psi) \;\; \Psi_0 \in U^{?}.$$

Assume the contrary, then there exists a sequence of pairs φ_n, ψ_n, such that

$$\|\varphi_n\|_n < \frac{1}{n}, \qquad \|\psi_n\|_n < \frac{X}{n}$$

and

(3.28) $$a^*(\varphi_n)\, a^*(\psi_n)\, \Psi_0 \notin U^{?}.$$

Now, since $\psi_n \to 0$ in $\mathscr{S}$, the set $\{\psi_n\}$ is bounded in $\mathscr{S}$, and hence the set

$$B = \{\psi \,\mathrm{I}\; \Psi = a^*(\psi_n)\, \Psi_0 \text{for some } n\}$$

is bounded in $\mathfrak{S}^{?}$.

On the other hand, since $\varphi_n \to 0$ in $\mathscr{S}$, $a^*(\varphi_n)\Psi$ tends to 0 in & uniformly on every bounded set, but that contradicts (3.28).

The continuity of the mapping Ψ_2 now follows from

(3.29) Lemma. *Let $K^2_{r,\sigma}$ denote the subset*

$$K^2_{r,\sigma} = \{\chi \,\mathrm{I}\; \chi = \varphi\psi,\; \varphi \in K_{r,\sigma},\; \psi \in K_{r,\sigma}\}$$

of $\mathscr{S}^{2\otimes}$. Then the convex hull $\operatorname{conv}(K^2_{r,\sigma})$ *contains a neighbourhood*

$$V_\varrho = \{\chi \mid \|\chi\|_{2r+8} < \varrho\}$$

of 0 *in $\mathscr{S}^{2\otimes}$.*

Proof: We first remark that $\|\chi\|_{2s}^2 \geqq \langle \chi, h_1^s h_2^s \chi \rangle$ in $\mathcal{S}^{2\otimes}$. Thus, for aU elements χ of V_ϱ,

$$\langle \chi, h_1^{r+4} h_2^{r+4} \chi \rangle < \varrho^2 .$$

As $\chi \in \mathcal{S}^{2\otimes}$, it is of the form

$$X = \sum_{i=1}^{n} \psi_i \psi_{n+i}, \qquad \psi_i \in \mathcal{S}, \quad i = 1, \ldots, 2n,$$

for some integer n. Consider the at most $2n$-dimensional space E spanned by $\psi_1, \ldots, \psi_{2n}$, and let P_E be the projection on E with respect to the scalar product $\langle \cdot, \ \rangle_r = \langle \cdot, h^r \ \rangle$. Further, let $h_E = P_E h^4 P_E$ denote the projection on E of the operator h^4, and let $\varkappa_\mu$, $\mu = 1, \ldots, \dim E \leqq 2n$, be a system of eigenfunctions of h_E, orthonormal with respect to $\langle \cdot, \ \rangle_r$. Thus

$$\langle \varkappa_\mu, h^r \varkappa_\nu \rangle = \delta_{\mu\nu},$$

$$\langle \varkappa_\mu, h^{r+4} \varkappa_\nu \rangle = \lambda_\mu \delta_{\mu\nu}.$$

Obviously, we may assume that the eigenvalues λ_μ do not decrease with μ. By the well known maximum-minimum properties of the eigenvalues of self-adjoint operators, we conclude that

$$\lambda_\mu \geqq \mu^4,$$

the number on the right hand side being the μ'th eigenvalue of the operator h^4. For the application to the present case it is of course essential that the operator which enters the scalar product $\langle \cdot, \ \rangle_r$ commutes with h^4.

If we expand χ in the form

$$\chi = \sum_{\mu,\nu=1}^{\dim E} t_{\mu\nu} \varkappa_\mu \varkappa_\nu,$$

then

$$\langle \chi, h_1^{r+4} h_2^{r+4} \chi \rangle = \sum_{\mu,\nu=1}^{\dim E} |t_{\mu\nu}|^2 \lambda_\mu \lambda_\nu < \varrho^2 .$$

Thus, we have the upper bound

$$|t_{\mu\nu}| < \varrho \mu^{-2} \nu^{-2} .$$

Further, as $\|\varkappa_\mu\|_r = 1$, we have, choosing $\hat\sigma < \sigma$,

$$\% = \sum_{\mu,\nu=1}^{\dim E} t_{\mu\nu} \hat\sigma^{-2} \hat\varkappa_\mu \hat\varkappa_\nu,$$

where $\hat\varkappa_\mu \in K_{r,\sigma}$. The proof of the lemma is now completed by use of the estimate

$$\sum_{\mu,\nu=1}^{\dim E} |t_{\mu\nu}| \hat\sigma^{-2} < \frac{\pi^4 \varrho}{36 \hat\sigma^2}$$

The minimal complete space $\tilde{\tilde{\mathfrak{S}}}$

We define the space $\tilde{\tilde{\mathfrak{S}}}$ as the direct sum

$$\tilde{\tilde{\mathfrak{S}}} = \sum_{n=0}^{\infty} \mathscr{S}_{+}^{n}$$

with the same convention as above: $\mathscr{S}_{+}^{0}$ is interpreted as C. Elements in $\tilde{\tilde{\mathfrak{S}}}$ are represented in the same way as elements in $\tilde{\tilde{\mathfrak{S}}}'$.

It is easily seen that $\tilde{\tilde{\mathfrak{S}}}$ is a complete space — in fact, that it is the completion of $\tilde{\tilde{\mathfrak{S}}}'$. In particular, $\tilde{\tilde{\mathfrak{S}}}'$ is dense in $\mathfrak{S}$, so that every continuous linear transformation from $\tilde{\tilde{\mathfrak{S}}}'$ into some complete locally convex space S has a unique continuous extension from $\tilde{\tilde{\mathfrak{S}}}$ into S.

If we apply this remark with $S = \tilde{\Theta}$, it follows that $\tilde{\tilde{\mathfrak{S}}}$ is of type $\mathfrak{S}$, and if we apply it with $8 = \mathfrak{S}^{?}$, where $\mathfrak{S}^{?}$is any complete space of type $\mathfrak{S}$, we get

(3.30) Theorem. $\tilde{\Theta}$ *is a minimal complete space of type* $\mathfrak{S}$.

This statement is to be interpreted in the way elaborated in the formulation of the theorem (3.27).

The maximal space $\mathfrak{S}$ and its FOCK *representation*

We define the space $\mathfrak{S}$ as the completion of $\tilde{\tilde{\mathfrak{S}}}'$ (or, equivalently, $\tilde{\tilde{\mathfrak{S}}}$ or any space $\mathfrak{S}^{?}$ of type $\mathfrak{S}$) in the topology $\mathscr{T}$ determined by all semi-norms of the form j|| $|||_T$, where

$$|||\Phi|||_T = |||T\Phi||| ,$$

for operators T in the algebra generated by all $a(\varphi^*)$ and $a^*(\varphi)$ and all normalized bi- quantizations K of self-ad joint operators $k \in R$.

By exactly the same line of reasoning as was applied in Section 2 one proves

(3.31) Theorem. $\mathfrak{S}$ *is a maximal space of type* $\mathfrak{S}$.

There is only one detail in the proof of this which is not obvious, and that is the fact that the operators a and a^*: $\mathscr{S} \frown L(\mathfrak{S}, \mathfrak{S})$ are continous. The proof of this is postponed until later (lemma (3.42)).

First, we prove that $\mathfrak{S}$ is metrizable by exhibiting a sequence of semi-norms which determine the topology of 0.

(3.32) Theorem. *The topology of the maximal space $\mathfrak{S}$ is determined by the sequence of semi-norms* $|||\ \ |||_r$ *given by*

$$|||\Phi|||_0 = |||\Phi|||$$

$$|||\Phi|||_r^2 = \langle\langle \Phi, H^r \Phi \rangle\rangle \ \textit{for}\ r > 0 ,$$

where H is the normalized bi-quantization of the operator h — bb^* in $\mathscr{S}$. For $r > 0$, these semi-norms are increasing, and they are not norms.

Proof: Let $\mathscr{T}'$be the topology on $\tilde{\tilde{\mathfrak{S}}}'$ determined by the semi-norms $|||\ \ |||_r$, $r = 0, 1, 2, \ldots$ We shall prove that $\mathscr{T}' = \mathscr{T}$, and just as in the proof of the theorem (2.16) we note that it is sufficient to prove that all operators $a(\varphi^*)$ and $a^*(\varphi)$ as well as all normalized bi-quantizations K of self-adjoint operators $k \in R$ are continuous in the topology $\mathcal{T}'$.

We first note that if

$$\Phi = \{\varphi^{(0)}, \ldots, \varphi^{(N)}, 0, \ldots\} \in \mathfrak{S}',$$

then we have for $r > 0$

$$\langle\langle \Phi, H^r \Phi \rangle\rangle = \sum_{n=l}^{N} \|\varphi^{(n)}\|_r^2. \tag{3.33}$$

It then follows from the theorem (2.16) that for $r > 1$

$$\begin{aligned} |||\Phi|||_r^2 &= \sum_{n=1}^{N} \|\varphi^{(n)}\|_r^2 \\ &\geqq \sum_{n=1}^{N} n \|\varphi^{(n)}\|_{r-1}^2 \\ &\geqq |||\Phi|||_{r-1}^2. \end{aligned}$$

We divide the remaining part of the proof of (3.32) into three separate lemmas.

(3.34) Lemma. *The operator $a(\varphi^*)$ on $\tilde{\mathfrak{S}}'$ is continuous in the topology $\mathcal{T}'$ for every $\varphi \in \mathcal{S}$.*

Proof: Assume that $\varphi \in \mathcal{S}$ and that

$$\Phi = \{\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(N)}, 0, \ldots\} \in \tilde{\mathfrak{S}}'.$$

Then, we get from (3.21)

$$a(\varphi^*)\,\Phi = \{\langle \varphi, \varphi^{(1)} \rangle, \sqrt{2}\, \langle \varphi, \varphi^{(2)} \rangle_{(1)}, \ldots, \sqrt{N}\, \langle \varphi, \varphi^{(N)} \rangle_{(1)}, 0, \ldots\},$$

whence by (3.23) and (2.18) (for r even)

$$\begin{aligned} |||a(\varphi^*)\Phi|||_r^2 &= \sum_{n=1}^{N} n \| \langle \varphi, \varphi^{(n)} \rangle_{(1)} \|_r^2 \\ &\leqq \sum_{n=1}^{N} n \|\varphi\|^2 \|\varphi^{(n)}\|_r^2 \\ &\leqq \|\varphi\|^2 |||\Phi|||_{r+1}^2. \end{aligned} \tag{3.35}$$

(3,36) Lemma. *The operator a^* (99) on $\tilde{\mathfrak{S}}'$ is continuous in the topology $\mathcal{T}'$ for every $\varphi \in \mathcal{S}$.*

Proof: First, note that

$$|||a^*(\varphi)\Phi|||^2 = |||a(\varphi^*)\Phi|||^2 + \|\varphi\|^2 |||\Phi|||^2 \tag{3.37}$$

by (3.2) and (3.3).

Next, by (3.9) we get

$$\begin{aligned} H^s a^*(\varphi) &= H^{s-1} a^*(h\varphi) + H^{s-1} a^*(\varphi) H \\ &= \sum_{i=0}^{s} \binom{s}{i} a^*(h^i \varphi) H^{s-i}. \end{aligned} \tag{3.38}$$

The formulas (3.37), (3.38), and (3.35) give

$$(3.39)\qquad \begin{aligned} |||a^*(\varphi)\Phi|||_{2s} &= |||H^s a^*(\varphi)\Phi||| \\ &\leqq \sum_{i=0}^{s} \binom{s}{i} |||a^*(h^i\varphi)H^{s-i}\Phi||| \\ &\leqq \sum_{i=0}^{s} \binom{s}{i} \|\varphi\|_{2i}(|||\Phi|||_{2(s-i)} + |||\Phi|||_{2(s-i)+1}). \end{aligned}$$

(3.40) Lemma. *The normalized bi-quantization K of any operator $k \in L(\mathscr{S}, \mathscr{S})$ is continuous on $\tilde{\mathfrak{S}}'$ in the topology $\mathscr{T}'$.*

Proof: An immediate consequence of (3.33) and (3.26).

This completes the proof of theorem (3.32).

(3.41) Theorem. *The maximal space $\mathfrak{S}$ can be identified with the space of all sequences*

$$\Phi = \{\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(n)}, \ldots\}$$

for which all semi-norms $|||\ \ |||_r$ defined by

$$|||\Phi|||_r^2 = \sum_{n=0}^{\infty} \|\varphi^{(n)}\|_r^2$$

are finite, the topology being determined by these semi-norms.

Proof: Identical with the proof of the theorem (2.19).

For elements $\Phi \in \mathfrak{S}$, the sequences $\{\varphi^{(n)}\}$ are rapidly decreasing with respect to n in virtue of the inequality (2.18).

(3.42) Lemma. *The operators a and a^* are continuous from $\mathscr{S}$ into $L(\mathfrak{S}, \mathfrak{S})$.*

Proof: As $\mathscr{S}$ is metrizable, it is sufficient to prove that when φ runs through a bounded set in $\mathscr{S}$ and Φ through a bounded set in $\mathfrak{S}$, then $a(\varphi^*)\Phi$ and $a^*(\varphi)\Phi$ run through bounded sets in β (lemma (A. 7) and lemma (A. 10)). This is an immediate consequence of the estimates (3.35) and (3.39), which in view of the theorem (3.41) are valid not only in $\tilde{\mathfrak{S}}'$, but also in $\mathfrak{S}$.

4. The dual spaces $\tilde{6}^*$ and $\mathfrak{S}^*$

The dual spaces $\mathscr{S}^{n*}$ and $\mathscr{S}^{n*}_{\pm}$

As a preparation for the discussion of the dual spaces of the extreme spaces of type $\mathfrak{S}$, we first make a few remarks concerning $\mathscr{S}^{n*} = L(\mathscr{S}^n, C)$, which, as we know, may be identified with SCHWARTZ' space of tempered distributions.

The space $\mathscr{S}^n$ may be identified with the space $\mathfrak{s}^n$ of all rapidly decreasing multiple sequences $c = \{c_\nu\}_{\nu \in N^n}$. Hence, the dual space $\mathscr{S}^{n*}$ may be identified with the space $\mathfrak{s}^{n*}$ of all multiple sequences $T = \{t_\nu\}_{\nu \in N^n}$, which are tempered in the index. By this we mean that for each element T there exists a polynomial p such that

$$|t_\nu| \leqq p(\nu), \quad \text{for all } \nu \in N^n.$$

Thus, there is a natural embedding of $\tilde{\mathfrak{s}}^n$, and of $\mathfrak{s}^n$, into $\mathfrak{s}^{n*}$, and hence of $\tilde{\mathcal{S}}^n$ and $\mathcal{S}^n$ into ff**. Let T be an element of $\mathcal{S}^{n*}$. It is easily seen that the sequence of elements $T_k \in \tilde{\mathcal{S}}^n, k = 1, 2, \ldots$, obtained by truncating T in the obvious way, converges to T uniformly on any bounded set in $\mathcal{S}^n$. Hence, under the mentioned natural embedding, $\tilde{\mathcal{S}}^n$, and a fortiori also $\mathcal{S}^n$, is dense in $\mathcal{S}^{n*}$.

The operators b_i, b_i^*, $i = 1, \ldots, n$, sym, and the conjugation can be defined on $\mathcal{S}^{n*}$as the continuous extensions of these operators in $\tilde{\mathcal{S}}^n$or $\mathcal{S}^n$.With this definition, b_i, b_i^*, and sym in $\mathcal{S}^{n*}$ are the duals (cf. Appendix A) of the operators b_i^*, b_i, and sym respectively in $\mathcal{S}^n$.Also, for the dual $\mathcal{S}_+^{n*}$ of ff\ $=$ sym $\mathcal{S}^n$, we have

$$\mathcal{S}_+^{n*} = \operatorname{sym} \mathcal{S}^{n*}.$$

The space $\tilde{\mathfrak{S}}^*$

It follows from the definition of $\tilde{\mathfrak{S}}$ that the dual space $\tilde{\mathfrak{S}}^*$ may be identified with the space of all sequences

$$(4.1) \qquad T = \{T^{(0)}, T^{(1)}, \ldots, T^{(n)}, \ldots\}, \qquad T^{(n)} \in \mathcal{S}_+^{n*},$$

with the topology of coordinate-wise convergence. By this identification the formula

$$\langle\langle T, \Phi \rangle\rangle = \sum_{n=0}^{\infty}{}' \langle T^{(n)}, \varphi^{(n)} \rangle$$

holds for all elements $\Phi = \{\varphi^{(n)}\}$of $\tilde{\mathfrak{S}}$.

For $T^{(n)} \in \mathcal{S}_+^{n*}$we define $\Psi^n[T^{(n)}]$ as that element of $\tilde{\mathfrak{S}}^*$ which is defined by

$$\langle\langle \Psi_n[T^{(n)}], \Phi \rangle\rangle = \langle T_n, \varphi^{(n)} \rangle .$$

Then, for any $T = \{T^{(n)}\} \in \tilde{\mathfrak{S}}^*$, we have

$$(4.2) \qquad \boldsymbol{T} = \sum_{n=0}^{\infty} \Psi_n[T^{(n)}]$$

in $\tilde{\mathfrak{S}}^*$.

It is now easily seen that $\mathfrak{S}$ is dense in $\tilde{\mathfrak{S}}^*$.

TOe *space* $\mathfrak{S}^*$

Since $\tilde{\mathfrak{S}}$ C $\mathfrak{S}$ algebraically and topologically, and since $\mathfrak{S}$ is dense in 0, we have $\mathfrak{S}^*$ C $\tilde{\mathfrak{S}}^*$ Thus, also elements of $\mathfrak{S}^*$ may be written in the forms (4.1) and (4.2). However, obviously not all sequences (4.1) belong to $\mathfrak{S}^*$.

(4.3) Theorem. *The space $\mathfrak{S}^*$ consists of those sequences $T = \{T^{(n)}\}$ of symmetric tempered distributions for which the series*

$$(4.4) \qquad \sum_{n=0}^{\infty} \langle T^{(n)}, \varphi^{(n)} \rangle$$

is convergent for all elements $\Phi = \{\varphi^{(n)}\}$ *of* 0, *and then* $\langle\langle \boldsymbol{T}, \Phi \rangle\rangle$ *is equal to the sum of this series. Moreover, the series* (4.2) *is convergent in* 0* *with the sum* T.

Proof: Since 0 is a complete metrizable space, the principle of uniform boundedness holds. Hence, if the series (4.4) is convergent for all $\Phi \in 0$, the partial sums are equicontinuous, and from ARZELA's theorem it follows that (4.4) converges uniformly on compact sets.

It is easily seen (cf. the lemma (B. 2)) that bounded sets in 0 are relatively compact, and hence (4.2) is convergent in 0*.

Since 0 is a complete metrizable space (an F-space), and bounded sets are relatively compact, & is a so-called Montel space (M-space). Such spaces are known to be reflexive. Thus, we have

(4.5) Theorem. *The space* 0 *is reflexive, i.e. the second dual space* $\mathfrak{S}^{**}$ *may be identified algebraically and topologically with* 0.

We remark that

$$0 \ C \ \beta \ C \ \mathfrak{S}^* \ C \ \tilde{\mathfrak{S}}^*$$

algebraically and topologically, where each of the spaces is dense in each of the spaces it may be embedded in.

For the benefit of the reader we give a direct proof of the theorem (4.5) in Appendix B.

Extensions of the mappings a^* *and* a

Since & is complete and 0 is dense in $\mathfrak{S}^*$, every mapping A of 0 into 0 which is continuous with respect to the topology induced on 0 by the topology of 0* has a unique continuous extension from 0* into 0*.

The definition of the semi-norms in 0* shows immediately that if A has an adjoint A^*, which is continuous from 0 into 0, then A is continuous on 0 with respect to the topology of $\mathfrak{S}^*$.

It follows that we have

(4.6) Theorem. *The mappings* $a^*(\varphi)$ *and* $a(\varphi^*)$, $\varphi \in \mathcal{S}$, *have unique continuous extensions from* 0* *into* $\mathfrak{S}^*$.

Let us note

(4.7) Lemma. *// S is a reflexive locally convex space, then the mapping*

$$A \curvearrowright A^*$$

of $L(S, S)$ *into* $L(S^*, S^*)$ *is a conjugate linear isomorphism and a topological homeomorphism.*

Proof: This follows from the definition of the dual operator and the topologies of the various spaces.

We shall prove below (lemma (4.9)) that the mapping a^* from $\mathcal{S}$ into $L(\mathfrak{S}^*, \mathfrak{S}^*)$ is continuous when $\mathcal{S}$ is given the topology of $\mathcal{S}^*$. Since

$$a(\varphi^*) = (a^*(\varphi))^*$$

for all $\varphi \in \mathscr{S}$, it follows from the lemma (4.7) and the reflexivity of $\mathfrak{S}$ that the mapping a from $\mathscr{S}$ into $L(\mathfrak{S}, \Theta)$ is continuous when $\mathscr{S}$ is given the topology of $\mathscr{S}^*$. Since $L(\mathfrak{S}, 6)$ is complete (theorem (A. 1)), $L(\mathfrak{S}^*, \mathfrak{S}^*)$ is also complete by the lemma (4.7). Thus we get:

(4.8) Theorem. *The mappings a and a^* from ff> into $L(\mathfrak{S}, \mathfrak{S})$ and $L(\mathfrak{S}^*, \mathfrak{S}^*)$ respectively have unique continuous extensions to $\mathscr{S}^*$. These extensions are also denoted a and a^*, and they are given in the Fock representations of $\mathfrak{S}$ and $\mathfrak{S}^*$ by formulas completely analogous to* (3.21) *and* (3.20). *The operators $a^*(T)$ in $\mathfrak{S}^*$ and $a(T^*)$ in $\mathfrak{S}$ are dual for every $T \in \mathscr{S}^*$.*

(4.9) Lemma. *The mapping a^* from $\mathscr{S}$ into $L(\mathfrak{S}^*, \mathfrak{S}^*)$ is continuous when $\mathscr{S}$ is given the topology of $\mathscr{S}^*$.*

Proof: Let p be one of the topology-determining semi-norms in $L(\mathfrak{S}^*, \mathfrak{S}^*)$, that is, p is defined in terms of a bounded set B^* in $\mathfrak{S}^*$ and a bounded set B in $\mathfrak{S}$. For an element $\varphi \in \mathscr{S}$ we have

$$p(a^*(\varphi)) = \sup_{T \in B^*} \sup_{\Omega \in B} |\langle\!\langle a^*(\varphi)\, \boldsymbol{T}, \Omega\rangle\!\rangle| \,.$$

If

$$\varphi = \sum_{\nu=0}^{\infty} c_\nu \psi_\nu$$

is the Fourier development of φ with respect to the Hermite elements, then we have

$$\begin{aligned} \langle\!\langle a^*(\varphi)\, \boldsymbol{T}, \Omega\rangle\!\rangle &= \langle\!\langle \boldsymbol{T}, a(\varphi^*)\,\Omega\rangle\!\rangle \\ &= \sum_{\nu=0}^{\infty} (-1)^\nu \bar{c}_\nu \langle\!\langle \boldsymbol{T}, a(\psi_\nu^*)\,\Omega\rangle\!\rangle \qquad (4.10) \\ &= \sum_{\nu=0}^{\infty} \tilde{c}_\nu \xi_\nu \end{aligned}$$

in view of the continuity of a as a mapping of $\mathscr{S}$ into $L(\mathfrak{S}, \mathfrak{S})$.

We shall prove below that there exist constants $M(r) = M(r; B, B^*)$, $r = 0, 1, \ldots$, such that

$$\sup_{\nu} (\nu + 1)^r |\xi_\nu| \leq M(r) \qquad (4.11)$$

for aU $T \in B^*$ and all $\Omega \in B$. Consequently,

$$\xi = \xi(\boldsymbol{T}, \Omega) = \sum_{\nu=0}^{\infty} \xi_\nu \psi_\nu$$

is an element of $\mathscr{S}$ for all $T \in B^*$ and all $\Omega \in B$, and these ξ's run through a certain bounded subset A of $\mathscr{S}$.

It then follows from (4.10) that

$$p(a^*(\varphi)) = \sup_{\xi \in A} \mathrm{I}\langle \varphi, \xi\rangle| \,,$$

and this proves the theorem.

In order to prove (4.11) we first note that in view of the boundedness of B^* in $\mathfrak{S}^*$, it is sufficient to prove that the set

$$B_r = \{(\nu+1)^r a(\psi_\nu^*)\Omega \mid \Omega \in B, \nu = 0, 1, \ldots\}$$

is bounded in 0 for every r.

Now, let $\Omega = \{\omega^{(n)}\} \in 6$. As $(\nu+1)\psi_\nu^* = h\psi_\nu^*$, we have in the Fock-representation

$$(\nu+1)^r a(\psi_\nu^*)\Omega = \{\sqrt{n+1}\,\langle\psi_\nu, h_1^r \omega^{(n+1)}\rangle_{(1)}\}$$

with the notation $\langle\psi_\nu, \ \rangle_{(1)}$ of Section 3. Hence, we find by (3.23)

$$\begin{aligned} |||(\nu+1)^r a(\psi_\nu^*)\Omega|||_s^2 &\leq \sum_n n\,\|\psi_\nu\|^2\,\|h_1^r \omega^{(n)}\|_s^2 \\ &\leq \sum_n n\,\|\omega^{(n)}\|_{2r+s}^2 \leq |||\Omega|||_{2r+s+1}^2, \end{aligned}$$

where the inequality (2.18) has been used in the last step.

It follows that B_r is bounded in 6 for every bounded set B.

Since continuity properties of mappings into and from the spaces $\tilde{\mathfrak{S}}$ and $\tilde{0}^*$ can be determined by looking only at the summand-spaces $\mathscr{S}_+^n$ and the factor-spaces $\mathscr{S}_+^{n*}$, the following theorems are easier to prove than the corresponding theorems above. We state them without proofs.

(4.12) Theorem. *The mapping a has a unique continuous extension from $\mathscr{S}$ into $L(\tilde{\mathfrak{S}}^*, \tilde{\mathfrak{S}}^*)$.*

(4.13) Theorem. *The mapping a has a unique continuous extension from $\mathscr{S}^*$ into $L(\tilde{\mathfrak{S}}, \mathfrak{S})$.*

(4.14) Theorem. *The mapping a^* has a unique continuous extension from $\mathscr{S}^*$ into $L(\tilde{\mathfrak{S}}^*, \tilde{\mathfrak{S}}^*)$.*

The mappings $a^{*n\otimes} \otimes a^{m\otimes}$ and their extensions

It is clear that if $\mathfrak{S}^?$ is a space of type $\mathfrak{S}$, then the mapping

$$\varphi_1 \quad \varphi_n\psi_1 \quad \psi_m \rightsquigarrow a^*(\varphi_1) \ . \ a^*(\varphi_n)a(\psi_1^*) \quad a(\psi_m^*)$$

can be extended to a linear mapping $a^{*n\otimes} \otimes a^{m\otimes}$ from $\mathscr{S}^{(n+m)\otimes}$ into $L(\mathfrak{S}^?, \mathfrak{S}^?)$.

By exactly the same reasoning as in the proof of the theorem (3.27) we conclude that this mapping is continuous from $\mathscr{S}^{(n+m)}$® into $L(\mathfrak{S}^?, \mathfrak{S}^?)$. Thus, if $L(\mathfrak{S}^?, \mathfrak{S}^?)$ is complete, and, in particular, if $\mathfrak{S}^? = \mathfrak{S}$ or $\mathfrak{S}^? = \tilde{\mathfrak{S}}$, then $a^{*n\otimes} \otimes a^{m\otimes}$ has a unique continuous extension from $\mathscr{S}^{n+m}$into $L(\mathfrak{S}^?, \mathfrak{S}^?)$.

Observe that now the element $\Psi_n[\varphi^{(n)}]$ of $\tilde{\mathfrak{S}}$ may be written

$$\Psi_n[\varphi^{(n)}] = \frac{1}{\sqrt{n!}} a^{*n\otimes}(\varphi^{(n)})\Psi_0, \quad \varphi^{(n)} \in \mathscr{S}_+^n,$$

and that — corresponding to (3.14) — we have

$$\Phi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} a^{*n\otimes}(\varphi^{(n)})\Psi_0,$$

a relation, which also holds for $\Phi \in \mathfrak{S}$, the series being convergent in $\mathfrak{S}$.

(4.15) Theorem. *The mapping* $a^{*n\otimes} \otimes a^{m\otimes}$ *has a unique continuous extension from* $\mathcal{S}^{(n+m)*}$ *into* $L(\mathfrak{S}, \mathfrak{S}^*)$.

Proof: This may be proved in essentially the same way as the lemma (4.9). However, utilizing results of the Appendix B we here give a somewhat simpler proof.

Let $\Psi = \{\psi_n\}$ and $\Phi = \{\varphi_n\}$ be elements of $\mathfrak{S}$, and let χ and ω be elements of $\mathcal{S}$. When the explicit formulas (3.13) and (3.21) are used, an easy calculation gives

$$\langle\langle \Phi, a^*(\chi)^n a(\omega)^m \Psi \rangle\rangle$$
$$= \sum_{t=0}^{\infty} \frac{\sqrt{(t+n)!\,(t+m)!}}{t!} \langle \langle \chi^n, \varphi_{t+n} \rangle_{(n)}, \langle \omega^{*m}, \psi_{t+m} \rangle_{(m)} \rangle,$$

where $\langle \chi^n, \ \rangle_{(n)}$ denotes the n-fold repetition of $(\chi, \cdot \rangle_{(1)}$. Hence, by the corollary (2.23) and by continuity, we find for all elements $\omega_{nm} \in \mathcal{S}^{n+m}$,

$$\langle\langle \Phi, a^{*n\otimes} \otimes a^{m\otimes}(\omega_{nm})\, \Psi \rangle\rangle$$
$$= \sum_{t=0}^{\infty} \frac{\sqrt{(t \quad n)!\,(t+m)!}}{t!} \int \varphi^*_{t+n}(x,z)\, \omega_{nm}(x,y)\, \psi_{t+m}(y,z)\; dx\, dy\, dz,$$

where $x \in E^n$, $y \in E^m$, $z \in E^t$ and dx, dy, dz denote the respective volume elements, and where, for convenience, SCHWARTZ' representation has been used. It is easily verified that h^r, $r = 0, 1, \ldots$, has a unique continuous inverse h^{-r} in $L(\mathcal{S}^n, \mathcal{S}^n)$, $n = 0, 1, \ldots$ If further the following triangular version of Cauchy-Schwartz inequality

$$|\int \psi_1(x_1, x_2)\, \psi_2(x_2, x_3)\, \psi_3(x_3, x_1)\, dx_1 dx_2 dx_{\overline{3}} \leqq \|\psi_1\| \ \|\psi_2\| \ \|\psi_3\|$$

is taken into account, one finds (with small obvious and convenient changes of notation)

$$|\langle\langle \Phi, a^{*n\otimes} \otimes a^{m\otimes}(\omega_{nm})\, \Psi \rangle\rangle|$$
$$\leqq \sum_{t=0}^{\infty} \frac{\sqrt{(t+n)!\,(t+m)!}}{t!} \sum_{s=0}^{r} \binom{r}{s} \|(h_x + h_y)^{-r}\, \omega_{nm}\| \ \|h_x^{r-s}\, \varphi_{t+n}\| \ \|h_y^s\, \psi_{t+m}\|$$
$$\leqq K_r \|\omega_{nm}\|_{-2r} \sum_{t=0}^{\infty} \sqrt{(t+n)^n}\, \|\varphi_{t+n}\|_{2r} \sqrt{(t+m)^m}\, \|\psi_{t+m}\|_{2r},$$

this inequality being valid for any natural number r. Here K_r is a positive constant, and $\| \ \|_{-2r}$ denotes the norm given by $\|\omega\|_{-2r} = \|h^{-r}\omega\|$. Finally, we make use of the inequality (2.18), and obtain

(4.16) $$|\langle\langle \Phi, a^{*n\otimes}\ 9\ a^{m\otimes}(\omega_{nm})\Psi \rangle\rangle|$$
$$\leqq K_r \|\omega_{nm}\|_{-2r} |||\Phi|||_{2r+n} |||\Psi|||_{2r+m},$$

$r = 0, 1, \ldots$ Let $\mathscr{H}^{n+m}_{-2r}$ denote the Hilbert-space obtained by completion of $\mathscr{S}^{n+m}$ in the norm $\| \;\|_{-2r}$ The inequality (4.16) proves the existence of a unique continuous extension from $\mathscr{H}^{n+m}_{-2r}$ into $L(\mathfrak{S}, \mathfrak{S}^*)$ (lemmas (A. 7), (A. 10), and (A. 11)) for every $r = 0, 1, \ldots$ Since $\mathscr{S}^{(n+m)*}$ is the inductive limit of the Hilbert-spaces $\mathscr{H}^{n+m}_{-2r}$, $r = 0, 1, \ldots$ (cf. Appendix B), the theorem follows.

The proof of the following theorem is simpler (cf. the comments before the theorems (4.12)—(4.14)).

(4.17) Theorem. *The mapping* $a^{*n\otimes} \otimes a^{m\otimes}$ *has a unique continuous extension from* $\mathscr{S}^{(n+m)*}$ *into* $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$. *For any family of elements* $T_{n,m} \in \mathscr{S}^{(n+m)*}$, $n, m = 0, 1, \ldots$, *the series*

$$\sum_{n,m} a^{*n\otimes} \otimes a^{m\otimes}(T_{n,m})$$

is unconditionally convergent in $L(\tilde{\mathfrak{S}}, \tilde{\mathfrak{S}}^*)$.

Appendix A

Locally convex spaces

In this appendix a selection of notions and results from the theory of locally convex topological vector spaces is given. For proofs we refer to the many excellent textbooks, some of which are found in the references [1, 8, 12, and 14].

Let S be a vector space over the complex field C. A real function $\| \;\|$ on S is called a *semi-norm* iff

$$\|x\| \geqq 0 \qquad \text{for all } x \in S\,,$$

$$\|tx\| = |t|\,\|x\| \qquad \text{for all } t \in C \text{ and all } x \in S\,,$$

$$\|x + y\| \leqq \|x\| + \|y\| \qquad \text{for all } x, y \in S\,.$$

A semi-norm is called a *norm* iff

$$\|x\| = 0 \Rightarrow x = 0\,.$$

Let P be a family of semi-norms on S. By *the topology determined by P* we understand the translation-invariant topology on S characterized in the following way: a subset U of S is a neighbourhood of 0 iff there exist semi-norms $\| \;\|_r$, $r = 1, \ldots, k$, in P and a positive real number ε such that

$$U \supseteq \left\{x \mid \sum_{r=1}^{k} \|x\|_r < \varepsilon\right\}.$$

It is easily proved that every semi-norm $\| \;\|$ in P is continuous with respect to the topology determined by P.

By a *locally convex space* (more correctly, a locally convex topological vector space) one understands a vector space equipped with a topology determined by some family of semi-norms.

(A. 1) Lemma. *If 8 is a locally convex space whose topology is determined by a family P of semi-norms, then a semi-norm \\ ‖ is continuous on 8 iff there exist semi-norms \\ ‖$_r$, r— 1, . . . , k, in P and a positive real number K such that*

$$\mathrm{M1} \leqq K\sum_{r=1}^{k} \|x\|_r \quad \textit{for all} \quad x \in S\,.$$

(A. 2) Lemma. *Two families P and P′ of semi-norms determine the same topology on S iff each semi-norm in one of the families is continuous with respect to the topology determined by the other family and conversely. In particular, the topology of a locally convex space S is determined by the family of all continuous semi-norms on S.*

Bounded sets

A subset B of a locally convex space *8* is called *bounded* iff it is *absorbed* by every neighbourhood of 0, i. e. iff for every neighbourhood U of 0 there exists a real number K such that

$$B \subseteqq K U\,.$$

(A. 3) Lemma. *// B is bounded, then every continuous semi-norm on S is bounded on B. Conversely, if every semi-norm in some topology-determining family of semi-norms is bounded on B, then B is bounded*

(A. 4) Lemma. *// B is a bounded subset of a locally convex space S, then the closure $\overline{B}$ of B and the convex hull* $\mathrm{conv}(B)$ *of B are bounded.*

The space $L(S_1, S_2)$ of continuous linear transformations

(A. 5) Lemma. *// S_1 and S_2 are locally convex spaces and T is a linear transformation from S_1 into S_2, then T is continuous with respect to the topologies of S_1 and S_2 iff the semi-norm p_2T defined by*

$$(p_2T)(x) = p_2(Tx) \qquad \textit{for all } x \in S_1,$$

is continuous on S_1 for every continuous semi-norm p_2 on S_2, or, equivalently, if this holds for every semi-norm p_2 in some topology-determining family P_2 of semi-norms on S_2.

(A. 6) Corollary. *If T is a continuous linear mapping of S_1 into S_2, then the image $T(B_1)$ of a bounded set B_1 in S_1 is bounded in S_2.*

It follows that the vector space $L(S_1, S_2)$ of all continuous linear mappings from S_1 into S_2 can be given the topology of *uniform convergence on bounded sets*, i.e. the topology determined by the semi-norms $\|\ \|_{B_1,p_2}$ defined by

$$\|T\|_{B_1,p_2} = \sup_{x \in B_1} p_2(Tx)$$

for all bounded sets B_1 in S_1 and all continuous semi-norms p_2 on S_2 (or, equivalently, all semi-norms p_2 in some topology-determining family of semi-norms on S_2). In the present work we always assume that $L(S_1, S_2)$ is given this topology.

In particular, the *dual space* $8^* = L(8, C)$ of a locally convex space 8 is given the so-called *strong* topology determined by the semi-norms

$$\|f\|_B = \sup_{x \in B} |\langle f, x \rangle|$$

for all bounded subsets B of 8. Here / $\in S^*$, and $\langle f, x \rangle$ denotes the value of f at $x \in S$.

Note that we define the scalar multiplication in 8^* by

$$\langle tf, x \rangle = t^* \langle f, x \rangle ,$$

where t^* denotes the complex conjugate of t. It follows that the natural embedding of a space with a scalar product into its dual space is linear instead of conjugate linear.

(A. 7) Lemma. *Let S_1 and S_2 be locally convex spaces. Then a subset B of $L(S_1, S_2)$ is bounded iff $B(B_1)$ is a bounded subset of S_2 for every bounded subset B_1 of S_1.*

Metrizable spaces

A topological space is called *metrizable* iff the topology can be defined by a metric.

(A. 8) Lemma. *If the topology of a locally convex space 8 is determined by a sequence of semi-norms, then there exists a translation invariant metric on 8 giving the topology (more correctly, the topology may be determined by a pseudo-metric, since two distinct points may have distance 0 — this possibility, however, will be disregarded in the sequel), and conversely.*

It is clear that one can always assume the topology of a metrizable locally convex space to be given by an increasing sequence of semi-norms.

(A. 9) Lemma. *A semi-norm p on a metrizable locally convex space is continuous iff it is bounded on bounded sets.*

Proof: Let $\| \ \|_r$ be an increasing topology-determining sequence of semi-norms on 8, and assume that p is not continuous. Then, by the lemma (A. 1), for each positive integer r there exists an element $x_r \in S$ such that

$$\|x_r\|_r = 1, \quad p(x_r) > r .$$

By the lemma (A. 3), the sequence $\{x_r\}$ is bounded, and on the other hand p is not bounded on this sequence. The other half of the statement is a consequence of the lemma (A. 3).

(A. 10) Lemma. *A linear transformation T from a metrizable locally convex space S_1 into a locally convex space S_2 is continuous iff it maps bounded sets in S_1 into bounded sets in S_2.*

Proof: Apply (A. 6), (A. 5), and (A. 9).

(A. 11) Theorem. *If* S_1 *is a metrizable locally convex space and* S_2 *a complete locally convex space, then* $L(S_1, S_2)$ *is complete.*

Proof: Let $\{T_\lambda\}, \lambda \in \Lambda$, be a Cauchy net in $L(S_1, S_2)$, i.e. assume that for every bounded set B_1 in S_1 and every continuous semi-norm p_2 on S_2

$$\|T_{\lambda'} - T_{\lambda''}\|_{B_1, p_2} = \sup_{x \in B_1} p_2(T_{\lambda'}x - T_{\lambda''}x) \to 0$$

as $\lambda', \lambda'' \to \infty$ in A

Then, in particular, since every point $x \in S_1$ is a bounded set, and S_2 is complete, the net $\{T_\lambda x\}$ in S_2 converges to a limit Tx for every $x \in S_1$. It is easily seen that T is a linear mapping from S_1 into S_2, and that it maps bounded sets into bounded sets. Hence, by the lemma (A. 10), T is continuous. Finally, it is easily proved that $T_\lambda \to T$ in $L(S_1, S_2)$, and the theorem is proved.

(A. 12) Corollary. *If 8 is a metrizable locally convex space, then* S^* *is complete.*

Direct sums

Let N be any index set, and assume that S_n is a locally convex space for every $n \in N$. By the *direct sum*

$$S = \sum_{n \in N} S_n$$

we understand the set of all functions x on N with the properties

$$x_n \in S_n \text{ for all } n \in N,$$

$$x_n — 0 \quad \text{for all but a finite number of indices } n \in N.$$

It is clear that 8 is a vector space with the vector operations defined pointwise on N, that S contains a canonical copy of S_n (namely the set of functions x for which $x_m — 0$ for all $m \neq n$), which will also be denoted S_n, and that these subspaces S_n generate the whole space 8.

Now, let P_n be a topology-determining family of semi-norms on S_n. For every combination of semi-norms $\| \ \|_n \in P_n$, $n \in N$, and for every real valued positive function $a(n)$ we define a semi-norm || j| on 8 by

$$\|x\| = \sum_{n \in N} a(n) \|x_n\|_n$$

By the *direct sum topology* we understand the topology on 8 determined by these semi-norms.

(A. 13) Lemma. *A semi-norm \\ || on 8 in continuous iff its restriction to each of the spaces* S_n *is continuous.*

(A. 14) Lemma. *A linear transformation* T *from 8 into a locally convex space* S' *is continuous iff its restriction to each of the spaces* S_n *is continuous.*

(A. 15) Lemma. *A convex subset* U *of 8 is a neighbourhood of* 0 *in 8 iff its intersection with each of the spaces* S_n *is a neighbourhood of* 0 *in* S_n.

(A. 16) Lemma. *A subset B of S is bounded iff there exists a finite number of indices $n_1, n_2, \ldots, n_k$ in N and bounded subsets B_{n_i} in each of the spaces S_{n_i}, $i = 1, 2, \ldots, k$, such that*

$$B \subseteq \sum_{i=1}^{k} B_{n_i}.$$

(A. 17) Lemma. *A sequence $\{x^{(i)}\}$ in S is convergent iff there exists a finite subset N' of N such that*

$$x_n^{(i)} = 0 \text{ for all } n \notin N' \text{ and all } i\,,$$

$$\{x_n^{(i)}\} \text{ is convergent in } S_n \text{ for all } n \in N'.$$

(A. 18) Lemma. *S is complete iff each of the spaces S_n is complete.*

It is easily seen (applying in particular the lemma (A. 16)) that

(A. 19) Lemma. *The dual space S^* of a direct sum*

$$S = \sum_{n \in N} S_n$$

can be identified algebraically with the product space

$$\mathop{\times}_{n \in N} S_n^*$$

consisting of all functions f on N with $f(n) \in S^*$ for all $n \in N$.

The strong topology of S^* is the *product topology*, i. e. it is determined by the family of all semi-norms of the form

$$\|f\|_{B_n} = \sup_{x \in B_n} |\langle f, x\rangle|\,,$$

where $B_n, n \in N$, runs through all bounded sets in the canonical copy of S_n in S.

Scalar products

By a *scalar product* in a complex vector space S we shall understand a positive definite Hermitean sesquilinear form $\langle\ ,\ \rangle$ on S. We shall take it to be linear in the second variable, conjugate linear in the first.

We shall say that a scalar product $\langle\ ,\ \rangle$ on a locally convex space S is *continuous* iff it is continuous in both arguments simultaneously, or, equivalently, iff the norm $\|\ \|$ defined by

$$\|x\|^2 = \langle x, x\rangle$$

is continuous.

Two linear operators T and T^* on S are called *adjoint* with respect to the scalar product $\langle\ ,\ \rangle$ iff

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \quad \text{for all } x, y \in S\,.$$

Dual operators

If S is a locally convex space and $T \in L(S, S)$, then we define the *dual operator* $T^* \in L(S^*, S^*)$ by the formula

$$\langle T^* f, x\rangle = \langle f, Tx\rangle$$

for all $f \in S^*$ and all $x \in S$.

In the literature T^* is usually called the adjoint of T, but we use the name dual of T in order that no confusion should arise between this concept and the concept of an adjoint with respect to a scalar product.

Appendix B

A class of sequence spaces

As we have seen, $\mathscr{S}^n_+$ may be represented as a space of multiple sequences. Hence, $\mathfrak{S}$ may be represented as a space of sequences too, and it follows from the theorems (3.41) and (2.19) that this space as well as all spaces $\mathscr{S}^n$ and $\mathscr{S}^n_+$ belongs to the class of spaces S characterized as follows:

Let $\rho = \{\varrho_n\}$ be a non-decreasing sequence of positive real numbers with $\varrho_n \to \infty$ for $n \to \infty$. Denote by S $(= S(\varrho))$ the locally convex space of all complex-valued sequences $x = \{x_n\}$ satisfying

$$\|x\|_r^2 = \sum_{n=1}^{\infty} \varrho_n^r \, |x_n|^2 < \infty\,, \qquad r = 0, 1, 2, \ldots,$$

with the topology determined by the norms $\| \ \|_r$, $r = 0, 1, 2, \ldots$ One proves easily

(B. 1) Lemma. *S is a complete metrizable locally convex space.*

Denote by H_r, r real (not necessarily positive), the Hilbert-space of all complex valued sequences x, for which

$$\|x\|_r^2 = \sum_{n=1}^{\infty} \varrho_n^r \, |x_n|^2.$$

The triple of spaces $S \subset H_0 \subset S^*$ is essentially a Gelfand space-triple (cf. GELFAND and VILENKIN [8]), except for the fact that S need not be nuclear (S is nuclear if and only if there exists a positive real number r such that $\Sigma \varrho_n^{-r} < \infty$ It follows that each of the spaces $\mathscr{S}^n$ and $\mathscr{S}^n_+$ is nuclear, as is also well-known, while the space $\mathfrak{S}$ is *not* nuclear). Furthermore, each space S is a perfect countably Hilbertian space in the terminology of GELFAND and VILENKIN [8]. The results below are all contained in the book by GELFAND and VILENKIN, and they are collected here simply for the purpose of convenient reference.

Obviously, if $r' < r''$, then $H_{r'} \supset H_{r''}$ algebraically and topologically, and we have

$$S = \bigcap_r H_r,$$

or, better, algebraically and topologically, S is the *projective limit* of the spaces H_r, $r \to \infty$.

(B. 2) Lemma. *Every bounded set B in S is relatively compact.*

Proof: Since S is complete and metrizable, it is sufficient to prove that every bounded sequence has a Cauchy subsequence. Since every bounded sequence is coordinate-wise bounded, it has a subsequence for which all coordinate-sequences are convergent. Now, assume that $\{x^{(i)}\}$ is a bounded sequence and that $\{x_n^{(i)}\}$ is convergent for each n. Since

$$\sum_{n>N} \varrho_n^r \, |x_n|^2 \leqq \varrho_N^{-1} \, \|x\|_{r+1}^2,$$

for all $x \in S$ and $\varrho_N \to \infty$ for $N \to \infty$, $\{x^{(i)}\}$ is a Cauchy-sequence with respect to each of the norms $\| \ \|_r$.

It is well known that

(B. 3) Lemma. *Every continuous linear functional $f \in H_r^*$ can be represented in the form*

$$\langle f, x \rangle = \sum_{n=1}^{\infty} \bar{f}_n x_n$$

for some sequence $f = \{f_n\} \in H_{-r}$.

Since every continuous linear functional $f \in S^*$ is continuous with respect to some norm $\| \ \|_r$, and since S is dense in every H_r, we have

(B. 4) Lemma. *Algebraically, $S^* = \bigcup_r H_{-r}$, or, better, S^* is the inductive limit of the spaces H_{-r}, $r \to \infty$.*

(B. 5) Lemma. *A subset B of S^* is bounded iff there exists a number r such that $B \subset H_{-r}$ and B is bounded in H_{-r}.*

Proof: Let B be a subset of S^*, and assume that for every $r = 1, 2, \ldots$ there exists an element $f^{(r)} \in B$ such that

$$\|f^{(r)}\|_{-r}^2 = \sum_{n=1}^{\infty} \varrho_n^{-r} |f_n^{(r)}|^2 > r^2.$$

Determine $N(r)$ such that

$$K_r^2 = \sum_{n=1}^{N(r)} \varrho_n^{-r} |f_n^{(r)}|^2 \geq r^2.$$

Now, define $x^{(r)} = \{x_n^{(r)}\}$ by

$$x_n^{(r)} = \begin{cases} K_r^{-1} \varrho_n^{-r} f_n^{(r)} & \text{for } n \leq N(r) \\ 0 & \text{for } n > N(r). \end{cases}$$

Then, clearly, $x^{(r)} \in S$, and

$$\|x^{(r)}\|_r^2 = 1.$$

For $s \geqq r$ we have

$$\|x^{(s)}\|_r \leqq \|x^{(s)}\|_s = 1\,,$$

so that the sequence $x^{(r)}$, r = 1, 2, . . ., is bounded in S.

On the other hand

$$\langle f^{(r)}, x^{(r)} \rangle = K_r \geqq r\,,$$

so that B is not bounded on the bounded subset $\{x^{(r)}\}$ of S. Hence, B is not bounded in S^* (lemma (A. 7)).

It is easily seen that if B is bounded in H_{-r}, then B is bounded in S^*, and the lemma is proved.

If $\xi = \{\xi_n\}$ is any sequence of complex numbers, we define $P_k \xi$ as the sequence determined by

$$(P_k \xi)_n = \begin{cases} \xi_n & \text{for } n \leqq k \\ 0 & \text{for } n > k\,. \end{cases}$$

(B. 6) Lemma. *For every $f \in S^*$, the sequence $\{P_k f\}$ converges to f in S^*.*

Proof: If $f \in H_{-r}$, then $\|(I - P_k) f\|_{-r} \to 0$, and the result follows from the Cauchy-Schwartz inequality.

(B. 7) Lemma. *A convex subset U^* of S^* is a neighbourhood of* 0 *in S^* iff $U^* \cap H_{-r}$ is a neighbourhood of* 0 *in H_{-r} for every r. Thus, also topologically, S^* is the inductive limit of the spaces H_{-r}, $r \to \infty$.*

Proof: The "only if" part of the lemma follows immediately from the definition of the topology of S^* and the fact that the bounded subsets of S are precisely those which are bounded in every space H_r.

Now, assume that U^* is a convex subset of S^* such that $U^*_{-r} = U^* \cap H_{-r}$ is a neighbourhood of 0 in H_{-r} for every r. It is clear that we may assume $zU^* \subseteq U^*$ for all complex numbers z with $|z| \leqq 1$.

Define

$$\overline{B}_r = \{x \in H_r \mid \sup_{f \in U^*_{-r}} |\langle f, x \rangle| \leqq 1\}$$

and $B_r = \overline{B}_r \cap S$.

It is easily seen that $\overline{B}_r$ is the closure of B_r in H_r, and that

$$\bigcap B_r = \bigcap \overline{B}_r$$

is a bounded subset B of S.

We shall now prove that

$$U^* \supseteq \left\{f \in S^* \mid \sup_{x \in B} |\langle f, x \rangle| \leqq \frac{1}{4}\right\},$$

whence the lemma follows.

Assume that $f \notin U^*$, and choose an integer r_0 such that $f \in H_{-r_0}$. For every integer $r \geqq r_0$, we have $f \in H_{-r}$ and $f \notin U^*_{-r}$, and by the Hahn-Banach theorem there exists an element $x_r \in B_r$, such that

$$\langle f, x_r \rangle \geqq \frac{1}{2}\,.$$

Then, since $s > r \geqq r_0 \Rightarrow x_s \in B_s \subset B_r$, the sequence $\{x_s\}$ is bounded in S, and by the lemma (B. 2), $\{x_s\}$has a subsequence which converges to an element $x \in S$. Since each of the sets $\overline{B}_r$ is closed in the corresponding H_r, it follows that

$$x \in \bigcap_{r \geqq r_0} \overline{B}_r = B ,$$

and since f is continuous, $\langle f, x\rangle \geqq \frac{1}{2}$, so that

$$\sup_{x \in B} |\langle f, x\rangle| \geqq \frac{1}{2} > \frac{1}{4} .$$

(B. 8) Theorem. *S is reflexive, i.e. S^{**} can be identified algebraically and topologically with S.*

Proof: It is clear that the function F_x defined on S^* by

$$\langle F_x, f\rangle = \overline{\langle f, x\rangle}$$

is an element of S^{**} for all $x \in S$, and that the mapping $x \rightsquigarrow F_x$ is one-to-one and linear from S into S^{**}.

Now, let F be any element in S^{**}. Then, by the lemma (B. 7), F is continuous on every H_{-r}, and hence there exists an element $x^{(r)} \in H_r = H^*_{-r}$ such that

$$\langle F, f\rangle = \overline{\langle f, x^{(r)}\rangle} \qquad \text{for all } f \in H_{-r}.$$

Since H_{-r} is contained in H_{-s} and is dense in H_{-s} for r < s, it follows that $x^{(r)} = x^{(s)}$. Thus, all $x^{(r)}$ are equal to an element $x \in \Pi\, H_r = S$, and consequently $F = F_x$ for some $x \in S$.

This shows that S and S^{**} can be identified algebraically. That the topologies coincide follows from the fact (familiar from Hubert space theory) that

$$\|x\|_r = \sup_{\|f\|_{-r} \leqq 1} |\langle f, x\rangle|$$

together with the characterization of bounded sets in S^* given in (*B.* 5).

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(Received March 8, 1965)