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The C*-Algebras of a Free Boson Field

I. Discussion of the Basic Facts

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Abstract. We give a systematic description of several C^*-algebras associated with a free Boson field. In this first part the structure of the one-particle space enters only through its symplectic form σ and a directed absorbing set of finite-dimensional subspaces on which σ is non-degenerate. The Banach *-algebras $\mathscr{L}_1(\mathfrak{E}, \sigma)$ and $\mathscr{M}_1(\mathfrak{E}, \sigma)$ of absolutely continuous resp. bounded measures on a finite-dimensional symplectic space $(\mathfrak{E}, \sigma)$, with their "twisted convolution product" stemming from Weyl's commutation relations, are studied as the analogues of the $\mathscr{L}_1$ resp. $\mathscr{M}_1$ algebras of a locally compact group. The fundamental "vacuum idempotent" Ω determines their (unique) Schrödinger representation, Schrödinger A^*-norm and Schrödinger C^*-completions $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ and $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$. After a study of these one proceeds to a construction as an inductive limit of the algebras $\mathscr{M}_1(\mathfrak{H}, \sigma)$ and $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ for an infinite-dimensional symplectic space $(\mathfrak{H}, \sigma)$. The "Fock representations" (with the corresponding "field operators") are presented as the infinite-dimensional generalization of the Schrödinger representation. The paper ends with a discussion of several possible choices for the "free Boson C^*-algebra".

§ 1. Introduction

The present paper is the first part of a study of different C^*-algebras associated with the free Bose field. Our aim is to investigate on the example of the free relativistic Bose field a number of questions which arise in HAAG's approach to field theory based on "local rings" [1—5] — particularly in the version of this approach based on C^*-algebras [6, 7]. Some of the questions we have in view are the following:

1) Amongst the different, more or less "rich" C^*-algebras which can be associated with the relativistic free Bose field, which one should be chosen as the "quasi-local algebra"?

2) What is the relation between the space-time structure and the algebraic structure ("diamond theorem" . . . etc.)?

3) Does HAAG's conjecture that local factors are of Type I [8] hold for some adequately chosen faithful representation of the quasi-local algebra (it has been shown not to hold for the standard Fock representation by ARAKI [9])? This question leads to formulating the following conjecture: for adequately chosen "commuting" space-time

domains $\mathscr{B}$ and $\mathscr{B}'$ ($\mathscr{B}'$ is the set of points lying space-like to all points of $\mathscr{B}$ and conversely) the whole quasi-local algebra $\mathfrak{A}$ can be written as the "*C-*tensor product*" $\mathfrak{A} = \mathfrak{A}(\mathscr{B}) \otimes \mathfrak{A}(\mathscr{B}')$*. If this conjecture holds true for free fields one would be tempted to take it as an axiom for coupled fields and to attempt a construction of the quasi-local algebra as an infinite C^*-tensor product [10].

Technically this work is an amplification of Von Neumann's fundamental paper on the uniqueness of representation of the commutation relations for finite systems [11]. Our approach, paralleling the theory of the group algebras of a locally compact group is tailored to afford the maximum flexibility for the choice of the quasi-local algebra about which we want to remain open-minded in view of question 1). In that respect our presentation should help to unify the different standpoints of previous related works: Van Hove [12], Cook [13], Friedrichs [14], Gårding and Wightman [15], Haag [16, 17], Segal [18], Coester and Haag [19], Araki [20, 21], Lew [22], Fukutome [23], Gelfand and Vilenkin [24], Bargmann [25], Shale [28], Klauder [29], McKenna and Klauder [30].

In this first part we treat only some of the possible algebras (see discussion of § 7). To motivate their abstract synthetic construction presented in the sections to follow we devote the rest of this introduction to a heuristic analytic comment.

Let us start from the free scalar boson field operator

$$A(f) = \int f(x)\, A(x)\, dx$$

smeared out with a test function f. $A(f)$ operates on Fock space and is equal to

$$A(f) = A\{\psi\} = a^+\{\psi\} + a^-\{\psi\}\,,$$

$a^{\pm}\{\psi\}$ being the $\begin{matrix}\text{creator}\\ \text{annihilator}\end{matrix}$ of a particle with the wave function

$$\psi = \Delta^+ * f\,,$$

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* The concept of C^*-tensor product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ has been developed by T. Turumaru, Tôhoku Math. Journ. 4, 242 (1953), 5, 1 (1953) and A. Wulfsohn, Bull. Sci. Math., 87, 13 (1963). It can be stated most simply as follows: choose two respective faithful representations π_1 and π_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and take the smallest norm closed algebra of operators on the tensor product Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$ which contains all finite sums $\sum_i \pi_1(A_i) \otimes \pi_2(B_i)$ with $A_i \in \mathfrak{A}_1$, $B_i \in \mathfrak{A}_2$. This construction is independent of the choice of the (faithful) representations π_1 and π_2 and has accordingly a purely algebraic character. The a priori possibility that Haag's conjecture, although incorrect in the original "W^*-form", could be true in "C^*-form" as stated here exists according to A. Guichardet (oral communication).

the convolution product of the test function f and the singular function Δ^{+}*. ψ is a square-integrable positive-energy solution of the Klein-Gordon equation and the set of all such ψ is a dense linear manifold in the free boson one-particle Hilbert space $\mathfrak{H}$. Owing to the commutation relations between creators and annihilators:

$$[a^+\{\psi_1\}, a^+\{\psi_2\}] = [a^-\{\psi_1\}, a^-\{\psi_2\}] = 0$$
$$[a^-\{\psi_1\}, a^+\{\psi_2\}] = (\psi_1 \mid \psi_2) ,$$

where $(\,|\,)$ denotes the scalar product in $\mathfrak{H}$, one has for the field operator the commutation relations

$$[A\{\psi_1\}, A\{\psi_2\}] = (\psi_1|\psi_2) - (\psi_2|\psi_1) = 2i\sigma(\psi_1, \psi_2) \tag{1}$$

(we denote by $s(\psi_1, \psi_2)$ and $i\sigma(\psi_1, \psi_2)$ respectively the real and purely imaginary part of the complex scalar product $(\psi_1|\psi_2)$ in H).

$A\{\psi\}$ being an unbounded self adjoint operator [13] we will get an everywhere defined bounded (in fact unitary) operator by taking

$$U\{\psi\} = e^{iA\{\psi\}} . \tag{2}$$

By formal manipulation of (1) and (2) (where we replace the exponential by its Mac Laurin expansion) we find as a substitute for (1) Weyl's commutation relations [31]

$$U\{\psi_1\}\, U\{\psi_2\} = e^{-2i\sigma(\psi_1,\psi_2)}\, U\{\psi_2\}\, U\{\psi_1\} ,$$

or equivalently the multiplication law

$$U\{\psi_1 + \psi_2\} = e^{i\sigma(\psi_1,\psi_2)}\, U\{\psi_1\}\, U\{\psi_2\} , \tag{3}$$

which resembles the addition law of an Abelian group (the difference consisting in the factor $e^{i\sigma(\psi_1,\psi_2)}$ which destroys commutativity – we will accordingly call it *twisted addition*).

This analogy will help us to construct the algebras we are aiming at as the objects analoguous to the "group algebras" of group theory [32–35]. Let us first consider the case of a subsystem with n degrees of freedom, i.e. take the operators $U\{\psi\}$ given by (2) with ψ confined to an n-dimensional complex subspace $\mathfrak{E}$ of the Hilbert space $\mathfrak{H}$. In its natural topology $\mathfrak{E}$ is locally compact and the analogue of the group algebra (of a locally compact group) is then obtained as follows: to each function f on $\mathfrak{E}$ integrable with respect to the Lebesgue measure $d\psi$ of $\mathfrak{E}$ we associate the operator**

$$U(f) = \int_{\mathfrak{E}} f(\psi)\, U\{\psi\}\, d\psi . \tag{4}$$

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* See for instance D. KASTLER: Introduction à l'Electrodynamique quantique Dunod Paris (1961) Chap. V.

** This integral is to be understood in the weak sense i.e.

$$(\Phi_1|\, U(f)\,|\Phi_2) = \int_{\mathfrak{E}} f(\psi)\, (\,\Phi_1|\, U\{\psi\}\,|\Phi_2)\, d\psi$$

where Φ_1 and Φ_2 are arbitrary vectors in Fock space. The integral exists because $(\Phi_1|\, U\{\psi\}\,|\Phi_2)$ is a bounded continuous function of ψ.

The multiplication law (3) for the $U\{\psi\}$ then entails the following multiplication law for the operators $U(f)$:

$$U(f_1)\,U(f_2) = U(f_1 \times f_2)\,, \tag{5}$$

where $f_1 \times f_2$ is the (Lebesgue integrable) function on E defined by

$$\begin{aligned}(f_1 \times f_2)(\psi) &= \int e^{-i\sigma(\xi,\psi)}\, f_1(\xi)\, f_2(\psi - \xi)\, d\xi \\ &= \int e^{i\sigma(\xi,\psi)}\, f_1(\psi - \xi)\, f_2(\xi)\, d\xi\,.\end{aligned} \tag{6}$$

The product $\times$ defined by (6) is a bilinear associative composition law for integrable functions analoguous to the convolution familiar in group theory (the difference consisting in the exponential modulating factor under the integral – we shall accordingly call it *twisted convolution*). (3) implies also that

$$U(f)^* = U(f)^{-1} = U(f^*)\,, \tag{7}$$

the integrable function f^* (called adjoint of f) being defined as

$$f^*(\psi) = \overline{f(-\psi)}\,. \tag{8}$$

Under the product (6) and the adjoint operation (8) the set $\mathscr{L}_1(\mathfrak{E}, \sigma)$* of Lebesgue integrable functions on $\mathfrak{E}$ (equipped with the $\mathscr{L}_1$ norm $\|f\|_1$ of functions) is a non-commutative Banach *-algebra which parallels the group algebra of a locally compact group for $\mathfrak{E}$ equipped with the "twisted addition" (3). Furthermore $f \to U(f)$ is a faithful (continuous)** *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$. Setting

$$\|f\| = \|U(f)\| = \text{norm of the operator } U(f)\,,$$

one defines on $\mathscr{L}_1(\mathfrak{E}, \sigma)$ a norm with respect to which the completion $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ is a C^*-algebra which we might call the *$\mathscr{L}_1 - C^*$-algebra of the n-dimensional subsystem defined by the n-dimensional subspace $\mathfrak{E}$.*

In order to define a C^*-algebra corresponding to the infinite dimensional system of free bosons we have to take a kind of an inductive limit of the C^*-algebras of all n-dimensional subsystems. To be able to do this it is necessary that given two mutually included finite dimensional subspaces $\mathfrak{E} \subset \mathfrak{F}$ there be a corresponding inclusion of their C^*-algebras. This is not realized with the $\mathscr{L}_1 - C^*$-algebras of $\mathfrak{E}$ and $\mathfrak{F}$ (there is no inclusion of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ in $\mathscr{L}_1(\mathfrak{F}, \sigma)$, the elements of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ being *measures* on $\mathfrak{F}$ instead of integrable functions). This urges us to take instead of the $\mathscr{L}_1 - C^*$-algebras the *$\mathscr{M}_1 - C^*$-algebras* defined as follows: given an n-dimensional subspace $\mathfrak{E} \subset \mathfrak{H}$ we extend the definition (4) to *bounded*

* We write $\mathscr{L}_1(\mathfrak{E}, \sigma)$ ($\mathscr{M}_1(\mathfrak{E}, \sigma)$) instead of $\mathscr{L}_1(\mathfrak{E})$ ($\mathscr{M}_1(\mathfrak{E})$) as a reminder that the multiplication law (twisted convolution $\times$) depends on the symplectic form σ.

** For C^*-algebras we use the terminology of the Appendix I of [6].

measures μ on $\mathfrak{E}$*

$$U(\mu) = \int U\{\psi\}\, d\mu(\psi)\,. \tag{9}$$

One has now

$$U(\mu_1)\, U(\mu_2) = U(\mu_1 \times \mu_2)$$

$$U(\mu)^* = U(\mu^*)\,,$$

where $\times$ and $*$ are now a "twisted convolution" and an adjoint operation defined on bounded measures. Under these operations the set $\mathcal{M}_1(\mathfrak{E},\sigma)$** of bounded measures on $\mathfrak{E}$ is again a Banach $*$-algebra faithfully represented by $\mu \to U(\mu)$. Setting again

$$\|\mu\| = \|U(\mu)\|$$

and completing with respect to this norm we now get C^*-algebras $\overline{\mathcal{M}_1(\mathfrak{E},\sigma)}$ (called the $\mathcal{M}_1 - C^*$*-algebras* of the corresponding spaces) such that to an inclusion $\mathfrak{E} \subset \mathfrak{F}$ of finite dimensional spaces there corresponds a natural inclusion of their $\mathcal{M}_1 - C^*$-algebras. We are now in a position to define the $\mathcal{M}_1 - C^*$-algebra $\overline{\mathcal{M}_1(\mathfrak{H},\sigma)}$ of the infinite dimensional $\mathfrak{H}$ as an inductive limit.

We close here this motivating introduction and proceed afresh to construct $\overline{\mathcal{M}_1(\mathfrak{H},\sigma)}$ rigorously, forgetting about the origin of the problem for the sake of generality. We shall recover a posteriori the field operator $A\{\psi\}$ on Fock space.

We raise a number of lemmas to the dignity of theorems for the convenience of monotonic numbering. Before starting let us say that another technique for treating the relations (3) in a group – theoretic spirit would consist in building a group extension of the additive group of $\mathfrak{E}$ by the one-dimensional torus so as to relate (3) with representations of this group extension [36]. This aspect will be described in a forthcoming paper by LOUPIAS and MIRACLE which will in addition contain a more detailed description of the "regular representation" of theorem 5 as well as a discussion of the relations between our "twisted convolution" and the BARGMANN [25] and WIGNER-MOYAL [26, 27] formalisms.

* We recall that the *bounded measures* on the locally compact space $\mathfrak{E}$ are the (regular) complex measures μ on $\mathfrak{E}$ for which $|\mu|(\mathfrak{E}) < \infty$. The set $\mathcal{M}_1(\mathfrak{E})$ of bounded measures on $\mathfrak{E}$ can be considered as the topological conjugate space of the Banach space $\mathscr{C}_0(\mathfrak{E})$ of complex continuous functions on E vanishing at ∞. $\mathscr{C}_0(\mathfrak{E})$ and $\mathcal{M}_1(\mathfrak{E})$ are Banach spaces under the norms

$$\|f\|_\infty = \operatorname*{Sup}_{\psi\in\mathfrak{E}} |f(\psi)| \qquad f \in \mathscr{C}_0(\mathfrak{E})$$

$$\|\mu\|_1 = \operatorname*{Sup}_{\substack{f\in\mathscr{C}_0(\mathfrak{E})\\ \|f\|_\infty = 1}} |\mu(f)| = \int_{\mathfrak{E}} d|\mu|\,(\psi) \qquad \mu \in \mathcal{M}_1(\mathfrak{E})$$

The integral (9) is to be understood in the weak sense (cf. footnote ** on p. 16).

** See footnote * on p. 17.

§ 2. Twisted convolution of measures on a finite-dimensional symplectic space. The algebra $\mathscr{M}_1(\mathfrak{E}, \sigma)$ and its subalgebras

A *symplectic form* $\sigma(\psi_1, \psi_2)$ on a real vector space $\mathfrak{E}$ is a regular*, antisymmetric**, real bilinear form on $\mathfrak{E}$. In this section we shall be concerned with a *finite-dimensional symplectic space* $(\mathfrak{E}, \sigma)$, that is, a finite-dimensional real vector space $\mathfrak{E}$ equipped with a symplectic form σ. $\mathfrak{E}$ is then of even (real) dimension $m = 2n$ and there exist in $\mathfrak{E}$ *symplectic bases* of vectors e_i, f_i, $i = 1, 2, \ldots, n$, that is, reference systems such that***

$$\begin{aligned}\sigma(e_j, e_k) &= \sigma(f_j, f_k) = 0\\ \sigma(e_j, f_k) &= -\sigma(f_k, e_j) = \delta_{jk}\,.\end{aligned} \qquad j, k = 1, 2, \ldots, n \tag{10}$$

The coordinates (ξ^i, η^i) of a vector $\psi \in \mathfrak{E}$ in a symplectic base $\left(\psi = \sum_{i=1}^{n} \cdot \cdot (\xi^i e_i + \eta^i f_i)\right)$ are called *symplectic coordinates*. The measure $dm_\sigma = \prod_{i=1}^{n} d\xi^i\, d\eta^i$ is the same for all systems of symplectic coordinates†. We call it the *symplectic measure* of $(\mathfrak{E}, \sigma)$.

Theorem 0. *Let μ be a bounded measure†† on $(\mathfrak{E}, \sigma)$ and let $f \in \mathscr{C}_0(\mathfrak{E})$. The function $\mu \times f$ defined by*

$$(\mu \times f)(\psi) = \int e^{-i\sigma(\xi, \psi)} f(\psi - \xi)\, d\mu(\xi) \tag{11}$$

is again an element of $\mathscr{C}_0(\mathfrak{E})$. One has

$$\|\mu \times f\|_\infty \leqq \|\mu\|_1 \|f\|_\infty \tag{12}$$

and

$$\mu(f) = (\mu \times \check{f})(0) \tag{13}$$

$\check{f}$ being the function of $\mathscr{C}_0(\mathfrak{E})$ defined by $\check{f}(\psi) = f(-\psi)$.

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* A bilinear form $\varphi(\psi_1, \psi)_2$ on $\mathfrak{H}$ is *regular* if $\varphi(\psi_1, \psi_2) = 0$ for arbitrary $\psi_2 \in \mathfrak{H}$ implies $\psi_1 = 0$.

** A bilinear form $\varphi(\psi_1, \psi_2)$ on $\mathfrak{H}$ is *symmetric*, resp. *antisymmetric* if $\varphi(\psi_2, \psi_1) = \varphi(\psi_1, \psi_2)$, resp. $\varphi(\psi_2, \psi_1) = -\varphi(\psi_1, \psi_2)$ for arbitrary $\psi_1, \psi_2 \in \mathfrak{H}$.

*** We can construct a symplectic base on the following way. Choose $e_1 \neq 0 \in \mathfrak{E}$. Since σ is regular there exists $f_1 \neq \lambda e_1 \in \mathfrak{E}$ such that $\sigma(e_1, f_1) = 1$. $\mathfrak{E}$ is the direct sum of the plane (e_1, f_1) and the subspace $\mathfrak{F}$ of elements $v \in \mathfrak{E}$ such that $\sigma(e_1, v) = \sigma(f_1, v) = 0$: an arbitrary $\psi \in \mathfrak{E}$ can namely be written in a unique way as $\psi = u + v$ with $u = \xi^1 e_1 + \eta^1 f_1$ and $v \in F$ (by taking $\xi^1 = \sigma(\psi, f_1)$ and $\eta^1 = \sigma(e_1, \psi)$). Now the restriction of σ to the subspace $\mathfrak{F}$ is bilinear, antisymmetric and regular (for $\psi_1 \in \mathfrak{F}$, $\sigma(\psi_1, v) = 0$ for all $v \in \mathfrak{F}$ implies $\sigma(\psi_1, \psi_2) = 0$ for all $\psi_2 \in \mathfrak{E}$ and thus $\psi_1 = 0$). $(\mathfrak{F}, \sigma)$ is thus a symplectic space of dimension $m - 2$ in which we can choose e_2, f_2 such that $\sigma(e_2, f_2) \neq 0 \ldots$ etc.; after $\frac{m}{2} = n$ steps our construction will be completed.

† One passes from a system of symplectic coordinates to another one by means of a matrix of determinant ± 1 (proof as for orthogonal transformations).

†† See footnote * on p. 18.

Proof: For fixed ψ, $e^{-i\sigma(\xi,\psi)} f(\psi - \xi)$ is an element of $\mathscr{C}_0(\mathfrak{E})$ as a function of ξ, whence the existence of the integral (11). We want to show the continuity of (11) with respect to ψ around $\psi_0 \in \mathfrak{E}$: let us first choose a compact set K such that $|f(\psi_0 - \xi)| < \varepsilon/6 \,\|\mu\|_1$ for $\xi \notin K$. There exists a neighborhood V_1 of 0 such that for $h \in V_1$, $\xi \notin K$, $|f(\psi_0 + h - \xi) \leqq \leqq 2\varepsilon/6 \,\|\mu\|_1$ (we can take V_1 such that $h \in V_1$ implies $|f(\psi + h) - f(\psi)| \leqq \leqq \varepsilon/6 \,\|\mu\|_1$ for all $\psi \in E$, which is possible since $f \in \mathscr{C}_0(\mathfrak{E})$ is uniformly continuous*. For $h \in V_1$ we then have

$$|(\mu \times f)(\psi_0 + h) - (\mu \times f)(\psi_0)| \leqq$$

$$\leqq \frac{\varepsilon}{2} + \int_K |e^{-i\sigma(\xi,\psi_0+h)} f(\psi_0 + h - \xi) - e^{-i\sigma(\xi,\psi)} f(\psi_0 - \xi)| \, d\mu(\xi) \leqq$$

$$\leqq \frac{\varepsilon}{2} + \int_K |e^{-i\sigma(\xi,h)} - 1| \, |f(\psi_0 + h - \xi)| \, d\mu(\xi) +$$

$$+ \int_K |f(\psi_0 + h - \xi) - f(\psi_0 - \xi)| \, d\mu(\xi) \leqq$$

$$\leqq \frac{\varepsilon}{2} + \left\{ \|\mu\|_1 \|f\|_\infty \operatorname*{Sup}_{\xi \in K} |e^{-i\sigma(\xi,h)} - 1| + \operatorname*{Sup}_{\xi \in K} |f(\psi_0 + h - \xi) - f(\psi_0 - \xi)| \right\},$$

which can be made arbitrarily small because of the uniform continuity in ξ of $\sigma(\xi, h)$ and $f(\psi - \xi)$ on the compact set K.

We have now to show that $(\mu \times f)(\psi)$ tends to 0 for ψ tending to ∞. Since we have the majorization

$$|(\mu \times f)(\psi)| \leqq \int |f(\psi - \xi) \, d|\mu|(\xi) \tag{11a}$$

and since $|f|$ is a function of $\mathscr{C}_0(\mathfrak{E})$ and $|\mu|$ a bounded measure on $\mathfrak{E}$ the desired result reduces to the analoguous one for the usual convolution**.

Theorem 1. *Let μ and ν be bounded measures on $(\mathfrak{E}, \sigma)$. There exists one and only one bounded measure $\nu \times \mu$ on $\mathfrak{E}$ (called twisted convolution of ν and μ) such that*

$$\nu \times (\mu \times f) = (\nu \times \mu) \times f \tag{14}$$

for every $f \in \mathscr{C}_0(\mathfrak{E})$. One has

$$\{\nu \times \mu\}(f) = \int d\nu(\eta) \int d\mu(\xi) \, e^{-i\sigma(\eta,\xi)} f(\xi + \eta) \tag{15}$$

and

$$\|\nu \times \mu\|_1 \leqq \|\nu\|_1 \|\mu\|_1 \,. \tag{16}$$

Proof: One has according to (11)

$$\{\nu \times (\mu \times f)\}(\psi) = \int d\nu(\eta) \int d\mu(\xi) \, e^{-i\sigma(\eta,\xi)} e^{i\sigma(\psi,\eta+\xi)} f(\psi - \xi - \eta) \,. \tag{17}$$

The last assertion of Theorem 0 then shows that if $\nu \times \mu$ exists with the required properties it satisfies (15), whence the uniqueness. Now for

* Because of Prop. V, § 27, n° 3 of Ref. [32], p. 369 and the fact that each $f \in \mathscr{C}_0(\mathfrak{E})$ is a uniform limit of continuous functions with compact support.

** See for instance Ref. [34], p. 264, Lemma (19.5).

bounded measures ν, μ the integral exists as the value of $\nu \times (\mu \times \check{f})$ for $\psi = 0$ and furthermore (12) shows that

$$|\{\nu \times \mu\}(f)| = |\{\nu \times \mu \times \check{f}\}(0)| \leqq \|\nu \times \mu \times \check{f}\|_\infty \leqq \|\nu_1\| \, \|\mu_1\| \, \|f\|_\infty ,$$

whence (16). On the other hand (15) and (11) imply that $[(\nu \times \mu) \times f](\psi)$ equals the right side of (17), whence (14).

Theorem 2. *Given a bounded measure μ on $(\mathfrak{E}, \sigma)$ there exists one and only one bounded measure μ^* such that*

$$\mu^*(f) = \overline{\mu(f^*)} \tag{18}$$

where $f^* = \check{\bar{f}}$. One has

$$\|\mu^*\|_1 = \|\mu\|_1 \tag{19}$$

and

$$(\nu \times \mu)^* = \mu^* \times \nu^* . \tag{20}$$

The proof is immediate and left to the reader.

Theorem 3. *Equipped with the norm $\| \, \|_1$, the product $\times$ (twisted convolution) and the adjoint operation $*$, the set of bounded measures on $(\mathfrak{E}, \sigma)$ is a Banach $*$-algebra. We denote it by $\mathscr{M}_1(\mathfrak{E}, \sigma)$ to remind that the algebraic product depends on σ.*

The product $\times$ is evidently bilinear. It is associative as a consequence of (14). The $*$ operation being evidently antilinear (19) and (20) then imply that $\mathscr{M}_1(\mathfrak{E}, \sigma)$ is a normed $*$-algebra. As the topological dual space of the Banach space $\mathscr{C}_0(\mathfrak{E})$ it is in addition complete.

The Dirac measures δ_ψ, $\psi \in \mathfrak{E}$, defined by

$$\delta_\psi(f) = f(\psi) \qquad f \in \mathscr{C}_0(\mathfrak{E}) \tag{21}$$

are elements of $\mathscr{M}_1(\mathfrak{E}, \sigma)$. One has according to (15) and (18)

$$\delta_{\psi_1} \times \delta_{\psi_2} = e^{-i\sigma(\psi_1, \psi_2)} \, \delta_{\psi_1 + \psi_2} \tag{22}$$

or more generally, for $\psi_1, \psi_2, \psi \in \mathfrak{E}$,

$$\{\delta_{\psi_1} \times f\} \, (\psi_2) = e^{-i\sigma(\psi_1, \psi_2)} \, f(\psi_2 - \psi_1) \tag{22a}$$

$$\{f \times \delta_{\psi_1}\} \, (\psi_2) = e^{i\sigma(\psi_1, \psi_2)} \, f(\psi_2 - \psi_1) \tag{22b}$$

$$\delta_\psi^* = \delta_\psi^{-1} = \delta_{-\psi} . \tag{23}$$

(22) is identical with the "twisted addition" (3). The algebra $\mathscr{M}_1(\mathfrak{E}, \sigma)$ is accordingly not commutative. (22a) shows that $\mathscr{M}_1(\mathfrak{E}, \sigma)$ has an identity element, namely δ_0.

Theorem 4. *Let f, g be functions on $(\mathfrak{E}, \sigma)$ (defined up to a set of measure zero) integrable with respect to the symplectic measure dm_σ. The measures $f \, dm_\sigma$, $g \, dm_\sigma$ are elements of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ and one has*

$$\|f\|_1 = \int |f(\psi)| \, dm_\sigma(\psi) = \|f \, dm_\sigma\|_1 , \tag{24}$$

$$(f \, dm_\sigma) \times (g \, dm_\sigma) = (f \times g) \, dm_\sigma , \tag{25}$$

$$(f \, dm_\sigma)^* = f^* \, dm_\sigma , \tag{26}$$

where $f \times g$ and f^* are again integrable functions given by

$$(f \times g)(\psi) = \int e^{-i\sigma(\xi,\psi)} f(\xi)\, g(\psi - \xi)\, dm_\sigma(\xi) \\ = \int e^{i\sigma(\xi,\psi)} f(\psi - \xi)\, g(\xi)\, dm_\sigma(\xi)\,, \tag{27}$$

$$f^*(\psi) = \overline{f(-\psi)}\,. \tag{28}$$

*Equipped with the norm (24), the multiplication (27) and the adjoint operation (28) the set $\mathscr{L}_1(\mathfrak{E}, \sigma)$ of functions on $\mathfrak{E}$ integrable with respect to dm_σ is a Banach *-algebra which can be considered as a closed subalgebra of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ via the (injective) correspondance $f \in \mathscr{L}_1(\mathfrak{E}, \sigma) \to f\, dm_\sigma \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ (in this sense we shall write f instead of $f\, dm_\sigma$). Furthermore $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is a two -sided ideal of $\mathscr{M}_1(\mathfrak{E}, \sigma)$: for $f \in \mathscr{L}_1(\mathfrak{E}, \sigma)$ and $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ one has $\mu \times f$, $f \times \mu \in \mathscr{L}_1(\mathfrak{E}, \sigma)$ with*

$$(\mu \times f)(\psi) = \int e^{-i\sigma(\xi,\psi)} f(\psi - \xi)\, d\mu(\xi)\,, \tag{29}$$

$$(f \times \mu)(\psi) = \int e^{i\sigma(\xi,\psi)} f(\psi - \xi)\, d\mu(\xi)\,. \tag{30}$$

For $f \in \mathscr{C}_0(\mathfrak{E})$ (29) coincides with (11).*

Proof: (24) is classical. (25), (27) can be inferred from (15) and Fubini's theorem. (26), (28) is obvious. $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is known to be complete in the norm (24). The existence almost everywhere of the integrals (27), (29), (30) results from Fubini's theorem. (29) and (30) are implied by (15) and Fubini's theorem.

Theorem 5. *Let f be a function of $\mathscr{C}_0(\mathfrak{E})$ of integrable square with respect to m_σ and set as usual*

$$\|f\|_2 = \{\int |f(\psi)|^2\, dm_\sigma(\psi)\}^{\frac{1}{2}}\,.$$

μ being an arbitrary element of $\mathscr{M}_1(\mathfrak{E}, \sigma)$, $\mu \times f$ is again such a function and we have

$$\|\mu \times f\|_2 \leqq \|\mu\|_1\, \|f\|_2\,; \tag{31}$$

the mapping

$$f \to \pi_2(\mu)\, f = \mu \times f \tag{32}$$

*can therefore be extended to a bounded operator on the Hilbert space $\mathscr{L}_2(\mathfrak{E})$ and one gets in that way a continuous faithful *-representation of the Banach-*-algebra $\mathscr{M}_1(\mathfrak{E}, \sigma)$. π_2 will be called the regular representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ (or of its subalgebras).*

Proof: $\mu \times f \in \mathscr{C}_0(\mathfrak{E})$ by Theorem 0. (31) is reduced to the analoguous result for the ordinary convolution** using the majorization (11a) as in the proof of Theorem 0. The fact that $\pi_2(\mu \times \nu) = \pi_2(\mu)\, \pi_2(\nu)$ is obvious from the definition (32). The fact that $\pi_2(\mu^*) = \pi_2(\mu)^*$ is obtained from (18) and the change of variable $\psi \to \psi + \xi$ in the integral

$$(g \,|\, \pi_2(\mu) \,|\, f) = \int dm_\sigma(\psi)\, \overline{g(\psi)} \int e^{-i\sigma(\xi,\psi)} f(\psi - \xi)\, d\mu(\xi)$$

---

* $\mathscr{C}_0(\mathfrak{E}) \cap \mathscr{L}_1(\mathfrak{E})$ is accordingly a two-sided ideal in $\mathscr{M}_1(\mathfrak{E}, \sigma)$.

** See for instance Ref. [32], p. 383, Prop. V of § 28, n° 2.

(we again use here Fubini's theorem). Equation (31) implies that

$$\|\pi_2(\mu)\| \leqq \|\mu\|_1\,, \qquad \mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)\,. \tag{31a}$$

Finally the representation π_2 is faithful because $\mathscr{C}_0(\mathfrak{E})$ is dense in $\mathscr{L}_2(\mathfrak{E})$ and $\mu \neq 0$ implies the existence of $f \in \mathscr{C}_0(\mathfrak{E})$ such that $\mu(\check{f}) = (\mu \times f)(0) \neq 0$.

Corollary. *For $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ $\mu^* \times \mu = 0$ implies $\mu = 0$.*

Proof: The representation π_2 injects $\mathscr{M}_1(\mathfrak{E}, \sigma)$ into a C^*-algebra where the property is true.

Definition: *$(\mathfrak{E}, \sigma)$ being a real vector space equipped with a symplectic form σ, the real vectorial subspace $\mathfrak{F} \subset \mathfrak{E}$ is called regular if the restriction of the bilinear form σ to $\mathfrak{F}$ is regular in $\mathfrak{F}$*.*

Theorem 6. *Let $\mathfrak{F}$ be a regular subspace of $(\mathfrak{E}, \sigma)$. To each $\mu \in \mathscr{M}_1(\mathfrak{F}, \sigma)$ we associate the measure $\tilde{\mu}$ on $\mathfrak{E}$ defined by*

$$\tilde{\mu}(f) = \mu(f|\mathfrak{F}) = \int_{\mathfrak{F}} f(\xi)\, d\mu(\xi) \tag{33}$$

where f runs through $\mathscr{C}_0(\mathfrak{E})$ and $f|\mathfrak{F}$ denotes its restriction to $\mathfrak{F} \subset \mathfrak{E}$. $\tilde{\mu}$ is an element of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ which we call the natural extension of μ. One has

$$\widetilde{\alpha_1 \mu_1 + \alpha_2 \mu_2} = \alpha_1 \tilde{\mu}_1 + \alpha_2 \tilde{\mu}_2\,, \tag{34}$$

$$\|\tilde{\mu}\|_1 = \|\mu\|_1\,, \tag{35}$$

$$\tilde{\nu} \times \tilde{\mu} = \widetilde{\nu \times \mu}\,, \tag{36a}$$

$$\tilde{\mu}^* = \widetilde{\mu^*}\,, \tag{36b}$$

so that $\mathscr{M}_1(\mathfrak{F}, \sigma)$ can be considered as a closed sub--algebra of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ via the correspondance $\mu \in \mathscr{M}_1(\mathfrak{F}, \sigma) \to \tilde{\mu} \in \mathscr{M}_1(\mathfrak{E}, \sigma)$**. If $\mathfrak{F}'$ is another regular subspace of $(\mathfrak{E}, \sigma)$ perpendicular to $\mathfrak{F}$ with respect to the symplectic form σ the subalgebras $\mathscr{M}_1(\mathfrak{F}, \sigma)$ and $\mathscr{M}_1(\mathfrak{F}', \sigma)$ of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ commute with each other.*

Proof: For $f \in \mathscr{C}_0(\mathfrak{E})$ one has $f|\mathfrak{F} \in \mathscr{C}_0(\mathfrak{F})$ with $\|f|\mathfrak{F}\|_\infty \leqq \|f\|_\infty$. Conversely to each $g \in \mathscr{C}_0(\mathfrak{F})$ there exists a $f \in \mathscr{C}_0(\mathfrak{E})$ such that $f|\mathfrak{F} = g$ with $\|f\|_\infty = \|g\|_\infty$. This justifies (33) and proves (34). In order to show (35) we write for $f \in \mathscr{C}_0(\mathfrak{E})$ using (33), (13), (14) and the fact that the operation $f \to \check{f}$ commutes with the restriction to $\mathfrak{F}$:

$$\widetilde{\nu \times \mu}(f) = \{\nu \times \mu\}(f|\mathfrak{F}) = \{\nu \times \mu \times (\check{f}|\mathfrak{F})\}(0) = \nu[(\mu \times (\check{f}|\mathfrak{F}))^{\check{}}]$$

$$(\tilde{\nu} \times \tilde{\mu})(f) = \{\tilde{\nu} \times \tilde{\mu} \times \check{f}\}(0) = \tilde{\nu}[(\tilde{\mu} \times \check{f})^{\check{}}] = \nu[((\tilde{\mu} \times \check{f})|\mathfrak{F})^{\check{}}]$$

(35) is thus reduced to

$$\mu \times (g|\mathfrak{F}) = (\tilde{\mu} \times g)|\mathfrak{F} \qquad g \in \mathscr{C}_0(\mathfrak{E}) \tag{33a}$$

* See footnote * on p. 19.

** We shall accordingly write $\tilde{\mu}$ instead of μ whenever this does not cause confusion.

which results immediately from (33) and (11): for $\psi \in \mathfrak{F}$ one has

$$\{\mu \times (g|\mathfrak{F})\}(\psi) = \int_{\mathfrak{F}} e^{-i\sigma(\xi,\psi)} g(\psi - \xi)\, d\mu(\xi)$$

whilst

$$\{\tilde{\mu} \times g\}(\psi) = \int_{\mathfrak{E}} e^{-i\sigma(\xi,\psi)} g(\psi - \xi)\, d\tilde{\mu}(\xi)$$

$$= \int_{\mathfrak{F}} e^{-i\sigma(\xi,\psi)} g(\psi - \xi)\, d\mu(\xi)\,.$$

The last assertion of the theorem results from (33) and (15).

Theorem 7. *Let $(\mathfrak{E}, \sigma)$ be a finite dimensional symplectic space. The formula*

$$U(f) = \int_{\mathfrak{E}} f(\psi)\, U\{\psi\}\, dm_\sigma(\psi) \tag{4a}$$

*establishes a one-to-one correspondance between the essential continuous *-representations $f \in \mathscr{L}_1(\mathfrak{E}, \sigma) \to U(f)$ of the algebra $\mathscr{L}_1(\mathfrak{E}, \sigma)$ and the weakly continuous unitary representations $\psi \in \mathfrak{E} \to U\{\psi\}$ of the canonical commutation relations. This correspondance carries over irreducibility and cyclicity of vectors**.

Proof: Let us start from a weakly continuous unitary representation of the canonical commutation relations on the Hilbert space $\mathscr{H}$, i.e. a weakly continuous mapping $\psi \to U\{\psi\}$ of $\mathfrak{E}$ into the unitary operators on $\mathscr{H}$ obeying the relation (3) for all $\psi_1, \psi_2 \in \mathfrak{E}$. Given a $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ we define the operator $U(\mu)$ on $\mathscr{H}$ by

$$(\Phi|U(\mu)|\Phi') = \int_{\mathfrak{E}} (\Phi|U\{\psi\}|\Phi')\, d\mu(\psi) \tag{9a}$$

for all $\Phi, \Phi' \in \mathscr{H}$. Since $(\Phi|U\{\psi\}|\Phi')$ is a continuous function of ψ bounded by $\|\Phi\| \cdot \|\Phi'\|$ the integral (9a) exists and is bounded in modulus by $\|\mu_1\| \cdot \|\Phi\| \cdot \|\Phi'\|$. (4a) thus defines a linear operator on $\mathscr{H}$ depending linearly on μ and of bound not larger than $\|\mu\|_1$. The fact that $\mu \to U(\mu)$ is a *-representation of the algebra $\mathscr{M}_1(\mathfrak{E}, \sigma)$ is shown by the relations

$$(\Phi|U(\nu)\, U(\mu)|\Phi') = \sum_i (\Phi|U(\nu)|\Phi_i)(\Phi_i|U(\mu)|\Phi') \qquad \left(\sum_i |\,\Phi_i\right)(\Phi_i| = I)$$

$$= \sum_i \iint (\Phi|U(\xi)|\Phi_i)(\Phi_i|U(\eta)|\Phi')\, d\nu(\xi)\, d\mu(\eta)$$

$$= \iint (\Phi|U(\xi)\, U(\eta)|\Phi')\, d\nu(\xi)\, d\mu(\eta) = (\Phi|U(\nu \times \mu)|\Phi')$$

* Theorems 7 and 7a are the analogues of well known theorems in the theory of locally compact groups, see Ref. [32], §§ 29 and 30. A representation π of a *-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ is called *essential* if $\pi(\mathfrak{A})\, \mathscr{H}$ is dense in $\mathscr{H}$ (i.e. if it does not admit the null-representation as a subrepresentation). *Cyclicity* of the vector Ψ_0 for the representation $\psi \to U\{\psi\}$ on $\mathscr{H}$ means that the $U\{\psi\}\, \Psi_0$, $\psi \in \mathfrak{E}$, form a total set in $\mathscr{H}$ (i.e. generate linearly a dense set in $\mathscr{H}$). Theorems 7, 7a and 8 are not indispensable for the comprehension of the sequel.

(where use was made of Fubini's theorem, Lebesgue's dominated convergence theorem for exchange of summation and integration, formulae (3) and (15)) and

$$\begin{aligned}\overline{(\Phi\,|\,U(\mu)^*\,|\,\Phi')} &= (\Phi'\,|\,U(\mu)\,|\,\Phi) = \int (\Phi'\,|\,U\{\psi\}\,|\,\Phi)\,d\mu(\psi)\\ &= \int \overline{(\Phi\,|\,U\{-\psi\}\,|\,\Phi')}\,d\mu(\psi) = \overline{(\Phi\,|\,U(\mu^*)\,|\,\Phi')}\end{aligned}$$

cf (18). We have thus associated to the mapping $\psi \to U\{\psi\}$ a continuous $*$-representation of $\mathscr{M}_1(\mathfrak{E},\sigma)$ whose restriction to the subalgebra $\mathscr{L}_1(\mathfrak{E},\sigma)$ defines a continuous $*$-representation $f \to U(f)$ of $\mathscr{L}_1(\mathfrak{E},\sigma)$ (notice that $U(\delta_\psi) = U\{\psi\}$). The fact that $f \to U(f)$ is essential is obvious: a $\Phi \in \mathscr{H}$ such that $(\Phi\,|\,U(f)\,|\,\Phi')$ vanishes for all $f \in \mathscr{L}_1(\mathfrak{E},\sigma)$ and $\Phi' \in \mathscr{H}$ is such that $(\Phi\,|\,U\{\psi\}\,|\,\Phi') = 0$ for all Φ', whence $\Phi = 0$.

Conversely let $f \to U(f)$ be an essential $*$-representation of $\mathscr{L}_1(\mathfrak{E},\sigma)$ i.e. a $*$-representation such that the set $\mathscr{H}_0$ of finite sums $\sum_{i=1}^{n} U(f_i)\,\Phi_i$, $f_i \in \mathscr{L}_1(\mathfrak{E},\sigma)$, $\Phi_i \in \mathscr{H}$, is dense in $\mathscr{H}$. We define the action of $U\{\psi\}$ on $\mathscr{H}_0$ by

$$\text{(a)}\qquad U\{\psi\}\sum_{i=1}^{n} U(f_i)\,\Phi_i = \sum_{i=1}^{n} U(\delta_\psi \times f_i)\,\Phi_i\,.$$

Since

$$\begin{aligned}\Big\|\sum_{i=1}^{n} U(\delta_\psi \times f_i)\,\Phi_i\Big\|^2 &= \sum_{n,i=1}^{n} (\Phi_i\,|\,U(f_i^* \times \delta_{-\psi} \times \delta_\psi \times f_j)\,|\,\Phi_j)\\ &= \Big\|\sum_{i=1}^{n} U(f_i)\,\Phi_i\Big\|^2,\end{aligned}$$

this definition is coherent ((38) applied to the null vector gives again the null vector) and $U\{\psi\}$ is linear isometric on $\mathscr{H}_0$. It can accordingly be extended to a unitary operator on $\mathscr{H}$ and the relation (3) (which need only be verified for $U\{\psi\}$ acting on $\mathscr{H}_0$) is a consequence of (a) and (22). Let us now show that $U\{\psi\}$ is weakly continuous in ψ. It suffices to prove the continuity of $(\Phi_1\,|\,U\{\psi\}\,|\,\Phi_2)$ in ψ for Φ_1 and Φ_2 running through the total set of the $U(g)\Phi$ where $\Phi \in \mathscr{H}$ and g runs through the set $\mathscr{K}(\mathfrak{E})$ of continuous functions on $\mathfrak{E}$ with compact support (the totality of the set of $U(g)\Phi$ stems from the inequality $\|U(g)\Phi\| < \|g\|_1\|\Phi\|$ and the density of $\mathscr{K}(\mathfrak{E})$ in $\mathscr{L}_1(\mathfrak{E},\sigma)$). We have now for $f, g \in \mathscr{K}(\mathfrak{E})$:

$$\begin{aligned}&(U(f)\Phi\,|\,U\{\psi\} - U\{\psi_0\}\,|\,U(g)\Phi') = \big(\Phi\,|\,U(f \times (\delta_\psi - \delta_{\psi_0}) \times g)\,|\,\Phi'\big) \leqq\\ &\leqq \|\Phi\|\cdot\|\Phi'\|\cdot\|f^* \times (\delta_\psi - \delta_{\psi_0}) \times g\|_1 \leqq \|\Phi\|\cdot\|\Phi'\|\cdot\|f\|_1\cdot\|(\delta_\psi - \delta_{\psi_0}) \times g\|_1\end{aligned}$$

the function

$$\{(\delta_\psi - \delta_{\psi_0}) \times g\}(u) = e^{-i\sigma(\psi,u)} f(u-\psi) - e^{-i\sigma(\psi_0,u)} f(u-\psi_0)$$

converging to zero within a fixed compact set.

Now we have shown how to assign to every unitary representation of the canonical commutation relations an essential continuous *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ and conversely. To see that the correspondence is one-to-one we need only verify the relations

$$\int \{\delta_\psi \times g\}(u)\ (\Phi|U(u)|\Phi')\ dm_\sigma(u) = \int g(u)\ (\Phi|U\{\psi\}\ U(u)|\Phi')\ dm_\sigma(u)$$

and

$$\int (\Phi|U(\delta_\psi \times f)|\Phi')\, g(\psi)\, dm_\psi = (\Phi|U(g \times f)|\Phi')$$

which is straightforward with some amount of vectorial integration.

The relations (4a) and (38) show that it is equivalent, for a $\Phi \in \mathscr{H}$, to be perpendicular to all $U\{\psi\}\Psi_0$, $\psi \in \mathfrak{E}$, or to all $U(f)\Psi_0$, $f \in \mathscr{L}_1(\mathfrak{E}, \sigma)$, ($\Psi_0$ being a fixed element of $\mathscr{H}$). Ψ_0 is thus cyclic with respect to $\psi \to U\{\psi\}$ if and only if it is cyclic with respect to $f \to U(f)$. The statement about irreducibility results from the fact that irreducibility means cyclicity for every vector.

We notice that the construction of $U(f)$ from $U\{\psi\}$ required only the weak measurability of $U\{\psi\}$. By recovering $U\{\psi\}$ from $U(f)$ one then sees that weak measurability of $U(\psi)$ implies weak continuity. We notice also that the process $U(f) \to U\{\psi\} \to U(\mu)$ defines a "canonical extension" of any essential continuous *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ to an essential continuous *-representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$.

Definition: *A complex function $\psi \in (\mathfrak{E}, \sigma) \to \varphi\{\psi\}$ on the finite dimensional symplectic space $(\mathfrak{E}, \sigma)$ is called of positive type if*

$$\sum_{j,k=1}^{n} C_j \bar{C}_k\, e^{i\sigma(\psi_j, \psi_k)}\ \varphi\{\psi_k - \psi_j\} \geqq 0 \tag{37}$$

for every choice of elements $\psi_j \in (\mathfrak{E}, \sigma)$ and complex constants $C_j, j = 1, 2, \ldots n$.

Theorem 7a. *The formula*

$$\varphi\{\psi\} = (\Phi_0|U\{\psi\}|\Phi_0) \tag{38}$$

establishes a one-to-one correspondance between continuous functions of positive type on $(\mathfrak{E}, \sigma)$ and unitary cyclic weakly continuous representations of the canonical commutation relations with cyclic vector Φ_0. The formula

$$\varphi(f) = \int_{\mathfrak{E}} f(\psi)\ \varphi\{\psi\}\ dm_\sigma(\psi) \tag{38a}$$

establishes a one-to-ome correspondance between the continuous functions of positive type $\psi \in (\mathfrak{E}, \sigma) \to \varphi\{\psi\}$ on $(\mathfrak{E}, \sigma)$ and the positive forms $f \in \mathscr{L}_1(\mathfrak{E}, \sigma) \to \varphi(f)$ on $\mathscr{L}_1(\mathfrak{E}, \sigma)$. The mapping $f \to U(f)$ defined by (4a) *is the *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ associated with the positive form $f \to \varphi(f)$:* i.e. $\varphi(f) = (\Phi_0|U(f)|\Phi_0)$.

Proof: The fact that $\varphi\{\psi\}$ defined in (38) satisfies (37) is shown by calculating $\|\sum_{i=1}^{n} C_i U\{\psi_i\}\|^2$ using (3). Conversely starting from a $\varphi\{\psi\}$

satisfying (37), the formula

$$(f|g) = \sum_{\xi, \eta \in G} \overline{f(\xi)}\, g(\eta)\, e^{i\sigma(\xi,\eta)}\, \varphi\{\eta - \xi\} \qquad f, g \in \mathscr{F}$$

defines a semi-definite sesquilinear form on the vector space $\mathscr{F}$ of complex functions on $\mathfrak{E}$ vanishing on all but a finite number of points. Passing to the quotient $\mathscr{H}_0 = \mathscr{F}/\mathscr{N}$ of $\mathscr{F}$ modulo the null space $\mathscr{N}$ (consisting of all $f \in \mathscr{F}$ such that $(f|f) = 0$ or equivalently $(f|g) = 0$ for all $g \in \mathscr{F}$) our sesquilinear form becomes a strictly positive Hermitean product. Now the definition

$$[U\{\psi\} f](\xi) = e^{-i\sigma(\psi,\xi)} f(\xi - \psi)\,, \qquad f \in \mathscr{F}$$

determines coherently a linear isometric operator $\psi \to U\{\psi\}$ on $\mathscr{H}_0$ as is shown by the relation $(U\{\psi\} f | U\{\psi\} g) = (f|g)$, easily obtained by shifting summation variables. $U\{\psi\}$ extends to a unitary operator on the completion $\mathscr{H}$ of $\mathscr{H}_0$. The relation (3) follows from the fact that the definition of $U\{\psi\}$ formally coincides with (22a). The element $\Phi_0 = f_0 + \mathscr{N}$ of $\mathscr{H}_0$, where f_0 is the characteristic function of the set $\{0\}$, is easily seen to be a cyclic vector for $\psi \to U\{\psi\}$. Finally the relation

$$(f|U\{\psi\} g) = \sum_{\xi, \eta \in G} \overline{f(\xi)}\, g(\eta)\, e^{i\sigma(\xi,\eta) - i\sigma(\psi, \eta+\xi)}\, \varphi\{\eta - \xi + \psi\}$$

shows both that $U\{\psi\}$ is weakly continuous in ψ, and by making $f = g = f_0$, that it satisfies (3). (3) establishes a one-to-one correspondance between $\varphi\{\psi\}$ and $U\{\psi\}$ because our construction applied to $\varphi\{\psi\}$ given by (3) is merely a reinterpretation of the cyclic component of Φ_0 (via $\sum_{i=1}^{n} C_i\, U\{\psi_i\}\, \Phi_0 \to f + \mathscr{N}$ with f vanishing everywhere but in the points ψ_i where it takes the values C_i).

Let us now take $\psi \to \varphi\{\psi\}$ and $\psi \to U\{\psi\}$ related by (3) and let $f \to U(f)$ be the cyclic *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ defined by (4a). One has by (4a)

$$\varphi(f) = (\Phi_0 | U(f) | \Phi_0) = \int_{\mathfrak{E}} f(\psi)\, \varphi\{\psi\}\, dm_\sigma(\psi)$$

whence, by Theorem 7, formula (38a) and the second part of our theorem. The continuous positive form φ extends canonically to a (continuous) positive form on $\mathscr{M}_1(\mathfrak{E}, \sigma)$ by using (9a):

$$\varphi(\mu) = (\Phi_0 | U(\mu) | \Phi_0) = \int_{\mathfrak{E}} \varphi\{\psi\}\, d\mu(\psi) = \mu(\varphi)\,, \tag{38b}$$

this extension being such that $\varphi\{\psi\} = \varphi(\delta_\psi) = \delta_\psi(\varphi)$. The possibility of this extension shows that the function $\varphi\{\psi\}$ is bounded:

$$\|\varphi\|_\infty = \|\varphi\| = \varphi(\delta_0) = \varphi\{0\}\,,$$

which could have been obtained directly from (37) for $m = 2$.

Theorem 8. *Each non vanishing continuous *-representation ϱ of the algebra $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is such that $\varrho(f) \neq 0$ for each $f \in \mathscr{L}_1(\mathfrak{E}, \sigma)$ with nowhere vanishing Fourier transform $\tilde{f}$.*

Proof: If $\varrho(f) = 0$ one has $\varrho(\delta_u \times f \times \delta_u) = 0$ for all $u \in \mathfrak{E}$. Since, by (22a, b)

$$\{\delta_u \times f \times \delta_u\}(\psi) = f(\psi - 2u),$$

the theorem follows if $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is the smallest closed subspace generated by all translates of f. This is so because this subspace is an ideal and f is contained in no proper ideal since its Fourier transform vanishes nowhere (Ref. [32], p. 426, Folgerung 1 of § 31, n^0 8. The meaning of the word ideal is here the usual one in commutative harmonic analysis.)

§ 3. The Schrödinger representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ and the associated C^*-Algebra $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$

We know from Theorem 5 that the algebra $\mathscr{M}_1(\mathfrak{E}, \sigma)$ of bounded measures on a finite dimensional symplectic space $(\mathfrak{E}, \sigma)$ can be faithfully represented by operators on a Hilbert space. In this section we shall see that $\mathscr{M}_1(\mathfrak{E}, \sigma)$ has a unique faithful *irreducible* (continuous) *-representation (up to unitary equivalence). This is essentially Von Neumann's uniqueness theorem for the representation of the canonical commutation relations for systems of n degrees of freedom (see Ref. [11]) which we have to complement in some respects for our further study of the infinite-dimensional case. We begin with the

Definition: *Let $\mathfrak{E}$ be a real vector space (of arbitrary dimension) on which a symplectic form σ is defined (we recall that σ is a real-bilinear regular antisymmetric form — this forces the dimension of $\mathfrak{E}$ to be even if it is finite). We say that $\mathfrak{E}$ is equipped with a σ-allowed prehilbertian structure if*

1) $\mathfrak{E}$ is a complex vector space for which the respective addition of vectors and multiplication by complex numbers of the form $\alpha + i0$ coincides with the addition and multiplication by reals for the initial real vector space structure,

2) as a complex vector space $\mathfrak{E}$ has a hermitian positive definite scalar product h whose purely imaginary part is equal to $i\sigma$.

Comment on this definition: from the point of view of the real vector space structure multiplication in $\mathfrak{E}$ times the imaginary unit is a real-linear operator J of square -1:

$$J(\alpha_1\psi_1 + \alpha_2\psi_2) = \alpha_1 J\psi_1 + \alpha_2 J\psi_2 \qquad \begin{matrix}\alpha_1, \alpha_2 \text{ reals} \\ \psi_1, \psi_2 \in \mathfrak{E}\end{matrix} \tag{39}$$

$$J^2 = -1 \tag{39a}$$

satisfying in addition

$$\sigma(J\psi_1, J\psi_2) = \sigma(\psi_1, \psi_2) \qquad \psi_1, \psi_2 \in \mathfrak{E}. \tag{40}$$

The real part s of the hermitean scalar product:

$$h(\psi_1, \psi_2) = s(\psi_1, \psi_2) + i\sigma(\psi_1, \psi_2), \qquad s, \sigma \text{ real} \tag{41}$$

is a real-bilinear symmetric positive-definite form on $\mathfrak{E}$ satisfying

$$s(J\psi_1, J\psi_2) = s(\psi_1, \psi_2) \qquad \psi_1, \psi_2 \in \mathfrak{E} \tag{42}$$

and

$$s(\psi_1, \psi_2) = -\sigma(J\psi_1, \psi_2) \qquad \psi_1, \psi_2 \in \mathfrak{E} \tag{43}$$

or equivalently

$$\sigma(\psi_1, \psi_2) = s(J\psi_1, \psi_2) \qquad \psi_1, \psi_2 \in \mathfrak{E} \tag{43a}$$

so that

$$J = -\sigma^{-1} s\,. \tag{44}$$

The situation characterized by the presence on a real vector space of a symplectic form σ and a σ-allowed prehilbertian structure can be reconstructed

— either from a symplectic form σ and a J satisfying (39), (39a) and (40); s is then given by (43)
— or from a real-bilinear symmetric positive-definite (i.e. *Euclidean*) s and a J satisfying (39), (39a) and (42); σ is then given by (43a)
— or from a symplectic σ and a Euclidean s satisfying (44); J is then given by (44).

In the rest of this section we shall be concerned with a finite-dimensional symplectic space $(\mathfrak{E}, \sigma)$. In this case the existence of σ-allowed (pre)hilbertian structures is guaranteed. In fact every symplectic base (e_k, f_k), $k = 1, 2, \ldots, n$ provides one by defining

$$\begin{cases} J e_k = i e_k = f_k \\ J f_k = i f_k = -e_k \end{cases} \qquad k = 1, 2, \ldots, n \tag{45}$$

which implies

$$\begin{cases} s(e_i, e_k) = s(f_i, f_k) = \delta_{ik} \\ s(e_i, f_k) = 0 \end{cases} \qquad i, k = 1, 2, \ldots, n \tag{46}$$

and therefore

$$h(\psi_1, i\psi_2) = -h(i\psi_1, \psi_2) = ih(\psi_1, \psi_2)\,.$$

The e_i, $i = 1, 2, \ldots, n$ then constitute a complex orthonormal base for $\mathfrak{E}$. It is important to realize that there are plenty of σ-allowed prehilbertian structures on $(\mathfrak{E}, \sigma)$, the preceding construction giving in general two different σ-allowed prehilbertian structures if applied to two different symplectic bases.

The main tool for the construction of the Schrödinger representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ will now be the

Theorem 9. *Let* $(\mathfrak{E}, \sigma)$ *be a finite-dimensional symplectic space on which we choose a* σ*-allowed (pre)hilbertian structure. The function*

$\Omega \in \mathscr{C}_0(\mathfrak{E}) \cap \mathscr{L}_1(\mathfrak{E}, \sigma)$ *defined by*

$$\Omega(\psi) = a^{-1} e^{-\frac{1}{2} s(\psi,\psi)} \tag{47}$$

with

$$a = \int e^{-s(\psi,\psi)}\, dm_\sigma(\psi) \tag{48}$$

gives rises for each $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ *to the relation*

$$\Omega \times \mu \times \Omega = \omega(\mu)\Omega\,, \tag{49}$$

where

$$\omega(\mu) = \int e^{-\frac{1}{2} s(\psi,\psi)}\, d\mu(\psi) = \mu(a\Omega)\,. \tag{50}$$

We prove (49) by direct calculation using (27), (29) and Fubini's theorem: we have

$$\begin{aligned}\{\Omega \times \mu \times \Omega\}(\psi) &= a^{-2} \int dm_\sigma(v)\, e^{i\sigma(v,\psi)}\, \Omega(\psi - v) \int d\mu(u)\, e^{-i\sigma(u,v)}\, \Omega(v-u) \\ &= a^{-2} \int d\mu(u) \int dm_\sigma(v)\, e^{i\sigma(v,\psi+u)}\,\cdot \\ &\quad \cdot\, e^{-\frac{1}{2}[s(\psi-v,\psi-v) + s(v-u,v-u)]} \\ &= a^{-1}\,\Omega(\psi) \int d\mu(u)\, e^{\frac{1}{4} s(\psi+u,\psi+u) - \frac{1}{2} s(u,u)} \cdot I\,.\end{aligned}$$

with

$$\begin{aligned} I &= \int dm_\sigma(v)\, e^{i\sigma(v,\psi+u)}\, e^{-s\left(v - \frac{\psi+u}{2},\, v - \frac{\psi+u}{2}\right)} \\ &= \int dm_\sigma(v)\, e^{i\sigma(v,\psi+u)}\, e^{-s(v,v)}\end{aligned}$$

which we evaluate using symplectic coordinates: $u = (u^j, u'^j)$, $v = (v^j, v'^j)$ $\psi = (\psi^j, \psi'^j)$. We get

$$\begin{aligned} I &= \prod_{j=1}^{n} \{\int e^{-[(v^j)^2 + i v^j(\psi'^j + a'^j)]}\, dv^j \cdot \int e^{-[(v'^j)^2 - i v'^j(\psi^j + u^j)]}\, dv'^j \\ &= a\, e^{-\frac{1}{4} s(\psi+u,\, \psi+u)}\,, \end{aligned}$$

q.e.d.

Corollary. *Ω is a self-adjoint idempotent of $\mathscr{M}_1(\mathfrak{E}, \sigma)$.*

Proof: One gets $\Omega^* = \Omega$ from (28) and $\Omega \times \Omega = \Omega$ by setting $\mu = \delta_0$ (see (22a)) in (49).

Corollary. *The function ω on $\mathscr{M}_1(\mathfrak{E}, \sigma)$ defined by* (50) *is a continuous positive linear form on $\mathscr{M}_1(\mathfrak{E}, \sigma)$ such that*

$$|\omega(\mu)| \leqq \|\pi_2(\mu)\| \leqq \|\mu\|_1 \tag{51}$$

*where π_2 is the regular representation defined in Theorem 5**.

Proof: ω is obviously linear. Setting $\mu = \nu^* \times \nu$ in (50) and applying π_2 one gets

$$\pi_2(\mu \times \Omega)^*\, \pi_2(\mu \times \Omega) = \omega(\mu^* \times \mu)\, \pi_2(\Omega)$$

* The positive form ω is associated to the function of positive type $a\Omega$ as described by formula (38b).

so that $\omega(\mu^* \times \mu)$ appears as the proportionality factor of two positive operators: it is therefore non-negative. Since $\mathscr{M}_1(\mathfrak{E}, \sigma)$ has a unit, ω is automatically continuous. $\omega(\mu^*) = \omega(\mu)$ for all $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ because of positivity (or directly from the definition). Applying π_2 to (49) we get

$$|\omega(\mu)| \, \|\pi_2(\Omega)\| = \|\pi_2(\Omega \times \mu \times \Omega)\| \leqq \|\pi_2(\mu)\| \, \|\pi_2(\Omega)\|^2 ,$$

whence (51) since $\|\pi_2(\Omega)\| = 1$ because of the first Corollary to Theorem 9.

Definition: *Let $\mathfrak{A}$ be a sub *-algebra of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ and $\omega | \mathfrak{A}$ the restriction of ω to $\mathfrak{A}$: $\omega | \mathfrak{A}$ is a continuous positive linear form on $\mathfrak{A}$. The *-representation of $\mathfrak{A}$ obtained from $\omega | \mathfrak{A}$ through the Gelfand-Segal construction* [37]* *is called the Schrödinger representation of $\mathfrak{A}$ and denoted $\pi_{\omega, \mathfrak{A}}$.*

We recall that $\pi_{\omega, \mathfrak{A}}$ is obtained by considering in $\mathfrak{A}$ the left ideal

$$\mathfrak{N}_{\omega, \mathfrak{A}} = \{\mu \in \mathfrak{A} \mid \omega(\mu^* \times \mu) = 0\} , \tag{52}$$

then defining $\pi_{\omega, \mathfrak{A}}(\mu)$ acting on $\mathfrak{A}/\mathfrak{N}_{\omega, \mathfrak{A}}$ by

$$\pi_{\omega, \mathfrak{A}}(\mu) \, [\nu + \mathfrak{N}_{\omega, \mathfrak{A}}] = \nu \times \mu + \mathfrak{N}_{\omega, \mathfrak{A}} , \tag{53}$$

and finally extending $\pi_{\omega, \mathfrak{A}}(\mu)$ to the completion of $\mathfrak{A}/\mathfrak{N}_{\omega, \mathfrak{A}}$ with respect to the norm associated with the scalar product

$$(\mu_1 + \mathfrak{N}_{\omega, \mathfrak{A}} \mid \mu_2 + \mathfrak{N}_{\omega, \mathfrak{A}}) = \omega(\mu_1^* \times \mu_2) \tag{54}$$

which is possible if $\pi_{\omega, \mathfrak{A}}(\mu)$ is continuous for that norm. This circumstance usually arises from the fact that the *-algebra under consideration is a Banach *-algebra with approximate unit. We here instead notice that (51) allows to extend ω to the C^*-completion of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ in the π_2 norm. Denoting by $\leqq$ the order relation in that C^*-algebra we have for $\mu, \nu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$

$$\nu^* \times \mu^* \times \mu \times \nu \leqq \|\pi_2(\mu^* \times \mu)\| \, \nu^* \times \nu$$

whence, upon applying ω and using (31) the inequalities

$$\|\pi_{\omega, \mathfrak{A}}(\mu)\|^2 \leqq \|\pi_2(\mu^* \times \mu)\| \leqq \|\mu\|_1^2 . \tag{55}$$

In order to study the Schrödinger representation we shall now describe it in another way supplied by the following theorems.

Theorem 10..** *Let $\mathfrak{A}$ be a *-subalgebra of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ containing Ω. The sub *-algebra $\Omega \times \mathfrak{A} \times \Omega$ of $\mathfrak{A}$ is a field. Specifically the mapping*

$$\mu \in \Omega \times \mathfrak{A} \times \Omega \to \omega(\mu) \in \mathbf{C} \tag{56}$$

*is a *-isomorphism of $\Omega \times \mathfrak{A} \times \Omega$ with the complex number field $\mathbf{C}$.*

Proof: The elements of $\mathfrak{A}$ of the form $\mu = \Omega \times \nu \times \Omega$, $\nu \in \mathfrak{A}$ clearly build a sub *-algebra of $\mathfrak{A}$. They are such that

$$\mu = \omega(\mu) \, \Omega , \qquad \mu \in \Omega \times \mathfrak{A} \times \Omega , \tag{57}$$

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* See also Ref. [32], § 17, n⁰ 3.

** Theorem 10, the minimality of the ideal $I_\Omega \cap \mathfrak{A}$ in Theorem 11, and Theorem 17 are not necessary for the understanding of subsequent constructions.

which shows that the mapping (56) is multiplicative and injective, therefore surjective since $\mathfrak{A} \neq 0$. It is on the other hand evidently linear and we noticed earlier that $\omega(\mu^*) = \omega(\mu)^*$.

Theorem 11. *Let $\mathfrak{A}$ be as in Theorem 10 and consider in $\mathfrak{A}$ the left ideals $\mathfrak{J}_\Omega \cap \mathfrak{A}$ and $\mathfrak{J}'_\Omega \cap \mathfrak{A}$ intersections of $\mathfrak{A}$ with the closed left ideals*

$$\begin{cases} \mathfrak{J}_\Omega = \mathcal{M}_1(\mathfrak{E}, \sigma) \times \Omega = \{\mu \times \Omega \mid \mu \in \mathcal{M}_1(\mathfrak{E}, \sigma)\} \\ \qquad = \{\mu \in \mathcal{M}_1(\mathfrak{E}, \sigma) \mid \mu \times \Omega = \mu\} \\ \mathfrak{J}'_\Omega = \{\mu \in \mathcal{M}_1(\mathfrak{E}, \sigma) \mid \mu \times \Omega = 0\} \end{cases} \tag{58}$$

of $\mathcal{M}_1(\mathfrak{E}, \sigma)$. These ideals are directly defined as

$$\begin{cases} \mathfrak{J}_\Omega \cap \mathfrak{A} = \mathfrak{A} \times \Omega = \{\mu \in \mathfrak{A} \mid \mu \times \Omega = \mu\} \\ \mathfrak{J}'_\Omega \cap \mathfrak{A} = \{\mu \in \mathfrak{A} \mid \mu \times \Omega = 0\}\,. \end{cases} \tag{59}$$

They are complementary

$$\mathfrak{A} = \mathfrak{J}_\Omega \cap \mathfrak{A} \oplus \mathfrak{J}'_\Omega \cap \mathfrak{A} \tag{60}$$

and closed if $\mathfrak{A}$ is closed. $\mathfrak{J}_\Omega \cap \mathfrak{A}$ is a minimal iedal of $\mathfrak{A}$ on which the sesquilinear form

$$(\mu \mid \nu) = \omega(\mu^* \times \nu) \tag{61}$$

defines a prehilbertian structure. $\mathfrak{J}'_\Omega \cap \mathfrak{A}$ is a maximal ideal of $\mathfrak{A}$ coinciding with the null-space $\mathfrak{N}_{\omega,\mathfrak{A}}$ of the form (61). Finally the restriction π of the left-regular representation of $\mathfrak{A}$ to the prehilbertian space $\mathfrak{J}_\Omega \cap \mathfrak{A}$:

$$\pi(\mu)\nu = \mu \times \nu \qquad \mu \in \mathfrak{A},\ \nu \in \mathfrak{J}_\Omega \cap \mathfrak{A} \tag{62}$$

*is an irreducible *-representation consisting of bounded operators whose extension to the completion of $\mathfrak{J}_\Omega \cap \mathfrak{A}$ is unitarily equivalent to the Schrödinger representation $\pi_{\omega,\mathfrak{A}}$.*

Proof: The equalities between sets stated in (58) and (59) are immediate. Continuity of the mappings $\mu \to \mu \times \Omega - \mu$ and $\mu \to \mu \times \Omega$ implies that $\mathfrak{J}_\Omega$ and $\mathfrak{J}'_\Omega$ are closed. Equation (60) amounts to the existence and uniqueness of the decomposition for each $\mu \in \mathfrak{A}$

$$\mu = \mu_1 + \mu_2 \quad \text{with} \quad \mu_1 \in \mathfrak{J}_\Omega \cap \mathfrak{A},\ \mu_2 \in \mathfrak{J}'_\Omega \cap \mathfrak{A}\,. \tag{63}$$

Now right multiplication of (63) by Ω fixes $\mu_1 = \mu \times \Omega$ whence $\mu_2 = \mu - \mu \times \Omega$ which actually fulfills (63). Let us now show the minimality of $\mathfrak{J}_\Omega \cap \mathfrak{A}$: we start with a left ideal $\mathfrak{L} \neq 0$ of $\mathfrak{A}$ contained in $\mathfrak{J}_\Omega \cap \mathfrak{A}$. The square $\mathfrak{L}^2 = \{\Sigma\, \mu_i \times \nu_i \mid \mu_i, \nu_i \in \mathfrak{L}\}$ of the ideal $\mathfrak{L}$ cannot be (0) because taking $\mu \in \mathfrak{L}$, whence $\mu^* \times \mu \in \mathfrak{L}$, this would imply $(\mu^* \times \mu)^2 = 0$, whence $\mu^* \times \mu = 0$ and $\mu = 0$ by the corollary to Theorem 5. Now take $\mu_1 \times \Omega$, $\mu_2 \times \Omega \in \mathfrak{L}$ such that $\mu_1 \times \Omega \times \mu_2 \times \Omega \neq 0$ whence $\Omega \times \mu_2 \times \Omega \neq 0$, there exist by Theorem 10 $\nu \in \Omega \times \mathfrak{A} \times \Omega$ such that $\nu \times \Omega \times \mu_2 \times \Omega = \Omega$ so that

$$\mathfrak{J}_\Omega \cap \mathfrak{A} = \mathfrak{A} \times \Omega \supset \mathfrak{L} \supset \mathfrak{A} \times \mu_2 \times \Omega \supset \mathfrak{A} \times \nu \times \Omega \times \mu_2 \times \Omega \supset \mathfrak{A} \times \Omega$$

whence $\mathfrak{L} = \mathfrak{J}_\Omega \cap \mathfrak{A}$, q.e.d. The maximality of $\mathfrak{J}'_\Omega \cap \mathfrak{A}$ now immediately results from the minimality of $\mathfrak{J}_\Omega \cap \mathfrak{A}$ and (60). To study the sesquilinear form (61) we write according to (49):

$$\Omega \times \mu^* \times \mu \times \Omega = (\mu \times \Omega)^* \times (\mu \times \Omega) = \omega(\mu^* \times \mu)\Omega$$

this implies via Theorem 5 the equivalences: $\omega(\mu^* \times \mu) = 0 \Leftrightarrow \mu \times \Omega = 0 \Leftrightarrow$ $\Leftrightarrow \mu \in \mathfrak{J}'_\Omega \cap \mathfrak{A}$, whence $\mathfrak{J}'_\Omega \cap \mathfrak{A} = \mathfrak{N}_{\omega,\mathfrak{A}}$. Equation (60) then implies the strict positivity of (61) on $\mathfrak{J}_\Omega \cap \mathfrak{A}$.

π defined by (62) is a $*$-representation of $\mathfrak{A}$ on $\mathfrak{J}_\Omega \cap \mathfrak{A}$ (obvious from Theorem 3). One sees easily that it is unitarily equivalent to the Schrödinger representations $\pi_{\omega,\mathfrak{A}}$: because of (60) the class $\mu + \mathfrak{N}_{\omega,\mathfrak{A}} \in$ $\in \mathfrak{A}/\mathfrak{N}_{\omega,\mathfrak{A}}$ contains a unique element of $\mathfrak{J}_\Omega \cap \mathfrak{A}$, namely $\mu \times \Omega$. This establishes a one-to-one mapping between these two spaces which is isometric for the scalar products (54) and (61) and transforms (53) into (62). The extension to the completions is obvious. π is irreducible owing to the minimality of the left ideal $\mathfrak{J}_\Omega \cap \mathfrak{A}$.

Theorem 12. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two sub $*$-algebras of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ such that $\mathfrak{J}_\Omega \subset \mathfrak{A}_1 \subset \mathfrak{A}_2$. The Schrödinger representation $\pi_{\omega,\mathfrak{A}_1}$ of $\mathfrak{A}_1$ is unitarily equivalent to the restriction $\pi_{\omega,\mathfrak{A}_2} | \mathfrak{A}_1$ of the Schrödinger representation of $\mathfrak{A}_2$ to $\mathfrak{A}_1$. This holds in particular if $\mathscr{L}_1(\mathfrak{E}, \sigma) \subset \mathfrak{A}_1 \subset \mathfrak{A}_2$. For $*$-algebras $\mathfrak{A}$ such that $\mathfrak{J}_\Omega \subset \mathfrak{A} \subset \mathscr{M}_1(\mathfrak{E}, \sigma)$ we shall accordingly write π_ω instead of $\pi_{\omega,\mathfrak{A}}$ (or $\pi_{\omega, M_1(\mathfrak{E},\sigma)} | \mathfrak{A})$).*

Proof: Results immediately from Theorem 11 and the fact that $\mathfrak{J}_\Omega \cap \mathfrak{A}_1 = \mathfrak{J}_\Omega \cap \mathfrak{A}_2 = \mathfrak{J}_\Omega$. For $\mathfrak{A} \supset \mathscr{L}_1(\mathfrak{E}, \sigma)$ one has $\mathfrak{A} \supset \mathfrak{J}_\Omega$ because the ideal $\mathscr{L}_1(\mathfrak{E}, \sigma)$ contains Ω and therefore $\mathfrak{J}_\Omega$.

Theorem 13.* *Let us denote by ${}^u\Omega$, $u \in (\mathfrak{E}, \sigma)$, the following element of $\mathfrak{J}_\Omega \subset \mathscr{C}_0(\mathfrak{E})$*

$$^u\Omega = \delta_u \times \Omega \,. \tag{64}$$

As a function on $\mathfrak{E}$ ${}^u\Omega$ is given by

$$^u\Omega(\psi) = a^{-1}\, e^{-i\sigma(u,\psi)}\, e^{-\frac{1}{2}s(\psi-u,\, \psi-u)} \qquad \psi \in \mathfrak{E}\,. \tag{65}$$

The scalar product (61) *in $\mathfrak{J}_\Omega$ of ${}^u\Omega$* and *${}^v\Omega$, $u, v \in \mathfrak{E}$, is equal to*

$$({}^u\Omega | {}^v\Omega) = a\, {}^v\Omega(u)\,. \tag{66}$$

In particular ${}^u\Omega$ is of unit norm. One has furthermore, for any $f \in \mathscr{C}_0(\mathfrak{E})$,

$$f \times \Omega = \int f(v)\, {}^v\Omega\, dm_\sigma(v) \tag{67}$$

$$({}^u\Omega | f \times \Omega) = a\{f \times \Omega\}(u)\,. \tag{68}$$

The set of all ${}^u\Omega$, u running through $\mathfrak{E}$, is a total set in $\mathfrak{J}_\Omega$ (or in the Hilbert space of the Schrödinger representation).

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* The states ${}^u\Omega$ are the same as the "coherent states" introduced by R. GLAUBER, Phys. Rev., **130**, 2529 (1963); **131**, 2766 (1963).

Proof: (65) is immediate inserting (21) and (47) into (27). By (64), (49) and (22), (23), the left side of (66) is seen to be equal to

$$\omega(\delta_{-u} \times \delta_v) = e^{i\sigma(u,v)}\, \delta_{v-u}(a\Omega) = a\, e^{i\sigma(u,v)}\, e^{-\frac{1}{2}(v-u,\, v-u)}.$$

Using (27) one has on the other hand for $f \in \mathscr{C}_0(\mathfrak{E})$

$$\begin{aligned}(f \times \Omega)(\psi) &= a^{-1} \int e^{-i\sigma(v,\psi)}\, e^{-\frac{1}{2}s(\psi-v,\, \psi-v)} f(v)\, dm_\sigma(v) \\ &= \int {}^v\Omega(\psi)\, f(v)\, dm_\sigma(v)\end{aligned} \tag{67a}$$

which is the same as (67) where the integral on the right side is to be understood in the topology of point-wise convergence on $\mathscr{C}_0(\mathfrak{E})$. (68) finally obtains by forming the scalar product of (67) with ${}^u\Omega$ to the left and using (66) and (67). For any $f \in \mathfrak{I}_\Omega$, (by Theorem 0, $f = f \times \Omega \in \mathscr{C}_0(\mathfrak{E})$) (68) now shows that $({}^u\Omega|f) = 0$ for all $u \in \mathfrak{E}$ implies $f = 0$: the set of ${}^u\Omega$, $u \in \mathfrak{E}$, is therefore total in $\mathfrak{I}_\Omega$ (i.e.) the finite linear combinations of the ${}^u\Omega$ are dense in $\mathfrak{I}_\Omega$).

Theorem 14. *The Schrödinger representation π_ω of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ is faithful.*

Proof: Take $\mu \neq 0 \in \mathscr{M}_1(\mathfrak{E}, \sigma)$. To show that $\pi_\omega(\mu) \neq 0$ we shall exhibit $f \in \mathscr{C}_0(\mathfrak{E})$ such that $\pi_\omega(\mu)\, \pi_\omega(f) = \pi_\omega(\mu \times f) \neq 0$. Since $\mu \neq 0$ there exists $f \in \mathscr{C}_0(\mathfrak{E})$ such that $\mu(\check{f}) = (\mu \times f)(0) \neq 0$, whence $\mu \times f = g \neq 0$. Now $g \in \mathscr{C}_0(\mathfrak{E})$ (Theorem 0) is such that $\pi_\omega(g) \neq 0$ because otherwise one would have $g \times {}^u\Omega = 0$ for all u and therefore

$$(g \times {}^u\Omega)(\psi) = e^{-i\sigma(u,\psi)} \int e^{-i\sigma(\xi,\psi+u)}\, \Omega(\psi - u - \xi)\, g(\xi)\, dm_\sigma(\xi) = 0$$

for all ψ and u. Uniqueness of Fourier transform would then imply $\Omega(\psi - u - \xi)\, g(\xi) = 0$ for all ξ, whence $g = 0$ contrary to hypothesis.

Theorem 15. *Let $\mathfrak{A}$ be a sub-*-algebra of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ containing Ω. Each *-representation of $\mathfrak{A}$ on a Hilbert space is the direct sum of a multiple of the Schrödinger representation $\pi_{\omega,\mathfrak{A}}$ and a *-representation ϱ_1 such that $\varrho_1(\Omega) = 0$. Consequently each irreducible *-representation ϱ such that $\varrho(\Omega) \neq 0$ (in particular each faithful irreducible *-representation) is unitarily equivalent to $\pi_{\omega,\mathfrak{A}}$. This theorem is a slight extension of Von Neumann's uniqueness theorem (see Ref.* [11]*).*

Proof: Let ϱ be a *-representation of $\mathfrak{A}$ on the Hilbert space $\mathscr{H}$. Ω being a self-adjoint idempotent $\varrho(\Omega)$ is a self-adjoint projector with range $\mathscr{K}$ and null space $\mathscr{K}^\perp$. Let $\{f_\alpha, \alpha \in I\}$ be a complete orthonormal system for $\mathscr{K}$. Owing to (49) we have for $\mu, \nu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$

$$(f_\alpha|\varrho(\mu)|f_\beta) = (f_\alpha|\varrho(\Omega \times \mu \times \Omega)|f_\beta) = \omega(\mu)\,(f_\alpha|\varrho(\Omega)|f_\beta) = \delta_{\alpha\beta}\,\omega(\mu)$$

and

$$(\varrho(\mu)\, f_\alpha|\varrho(\nu)\, f_\beta) = \delta_{\alpha\beta}\, \omega(\mu^* \times \nu).$$

Thus the cyclic components of the f_α, $\alpha \in I$, for the representation ϱ are mutually orthogonal, ϱ reducing on each of them to a subrepresentation with expectation $\omega(\mu)$, i.e. unitarily equivalent to $\pi_{\omega,\mathfrak{A}}$. Moreover since $\sum\limits_{\alpha \in I}^{\oplus} \varrho(\mathfrak{A}) f_\alpha \supseteqq \mathscr{K}$ the remaining component ϱ_1 of ϱ acts in $\mathscr{K}^\perp$ and is thus such that $\varrho_1(\Omega) = 0$. In the special event that ϱ is irreducible and such that $\varrho(\Omega) \neq 0$ one has $\varrho_1 = 0$ and $\mathscr{K}$ is one-dimensional.

We notice that as a result of Theorem 8, ϱ_1 annuls not only Ω but the whole of $\mathscr{L}_1(\mathfrak{E}, \sigma)$. In particular we have the

Theorem 15a. *Each *-representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is a direct sum of a multiple of the Schrödinger representation and the null representation. Each irreducible representation of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ is unitarily equivalent to the Schrödinger representation.*

Corollary. *The regular representation π_2 of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ is a multiple of the Schrödinger representation π_ω. Accordingly $\|\pi_2(\mu)\| = \|\pi_\omega(\mu)\|$ for all $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$.*

Proof: We have to show that the subspace $\mathscr{K}^\perp$ in the proof of Theorem 15 reduces to the null vector for $\varrho = \pi_2$. For any $f \in \mathscr{K}^\perp \subset \mathscr{L}_2(\mathfrak{E}, dm_\sigma)$ one has $\pi_2(g) f = g \times f = 0$ for all $g \in \mathscr{L}_1(\mathfrak{E}, \sigma)$. In particular for each $g \in \mathscr{L}_1(\mathfrak{E}, \sigma) \cap \mathscr{C}_0(\mathfrak{E})$, $g \times f \in \mathscr{C}_0(\mathfrak{E})$ by Theorem 0 and

$$\{g \times f\}(0) = \int g(\xi) f(-\xi)\, dm_\sigma(\xi) = 0$$

which implies that $f = 0$ since $\mathscr{L}_1(\mathfrak{E}, \sigma) \cap \mathscr{C}_0(\mathfrak{E})$ is dense in $\mathscr{L}_2(\mathfrak{E}, dm_\sigma)$.

Theorem 16. *Let $\mathfrak{A}$ be as in Theorem 15. The Schrödinger representation $\pi_{\omega, \mathfrak{A}}$ of $\mathfrak{A}$ is (topologically) irreducible.*

Proof: Let $\hat{\mathfrak{A}}$ be the completion of $\mathfrak{A}$ in the π_2-norm (Theorem 5). $\hat{\mathfrak{A}}$ is a C^*-algebra and possesses therefore an irreducible representation ϱ such that $\varrho(\Omega) \neq 0$*. The restriction of ϱ to $\mathfrak{A}$ is also irreducible since $\mathfrak{A}$ is dense in $\hat{\mathfrak{A}}$ in the norm topology (and a fortiori in the weak topology of operators on the representation space of ϱ). According to the preceding theorem ϱ is then unitarily equivalent to $\pi_{\omega, \mathfrak{A}}$ which is thus irreducible. This result could also have been inferred from the Theorem 19 below.

The C^*-algebra $\overline{\mathscr{M}_1}(\mathfrak{E}, \sigma)$. Let $(\mathfrak{E}, \sigma)$ be a finite-dimensional symplectic space on which one has defined two different σ-allowed Hilbertian structures $s_1 + i\sigma$ and $s_2 + i\sigma$. Let Ω_1 and Ω_2 be defined as in (47) in terms of s_1 and s_2 respectively, and let ω_1 and ω_2 be the corresponding positive forms on $\mathscr{M}_1(\mathfrak{E}, \sigma)$. Theorem 15 shows us that for a sub-*-algebra $\mathfrak{A}$ of $\mathscr{M}_1(\mathfrak{E}, \sigma)$ containing Ω_1 and Ω_2, in particular for $\mathscr{M}_1(\mathfrak{E}, \sigma)$ itself, ω_1 and ω_2 define the same (faithful) *-representation $\pi_{\omega_1, \mathfrak{A}} = \pi_{\omega_2, \mathfrak{A}}$ up to unitary equivalence. The following definition pertains therefore only to the symplectic structure of $(\mathfrak{E}, \sigma)$:

Definition. *Let $(\mathfrak{E}, \sigma)$ be a finite-dimensional symplectic space. For $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ we denote by $\|\mu\|$ the norm of the corresponding operator in*

* See for instance Ref. [5], p. 324, Theorem 4 of § 24, n° 2; p. 314, Prop. II of § 23, n° 3; and p. 320, Prop. I of § 24, n° 1.

the Schrödinger representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$:

$$\|\mu\| = \|\pi_\omega(\mu)\| = \sup_{\nu \in \mathscr{M}_1(\mathfrak{E},\sigma)} \frac{\omega(\nu^* \times \mu^* \times \mu \times \nu)^{\frac{1}{2}}}{\omega(\nu^* \times \nu)^{\frac{1}{2}}} \tag{69}$$

$\|\mu\|$ *is called the Schrödinger norm of* μ*. *For any sub-*-algebra* $\mathfrak{A}$ *of* $\mathscr{M}_1(\mathfrak{E}, \sigma)$, *we denote by* $\overline{\mathfrak{A}}$ *the completion of* $\mathfrak{A}$ *for the Schrödinger norm* (69). $\overline{\mathfrak{A}}$ *is a* C^**-algebra called the Schrödinger* C^**-completion of* $\mathfrak{A}$. *In particular the respective* C^**-completions of* $\mathscr{M}_1(\mathfrak{E}, \sigma)$ *and* $\mathscr{L}_1(\mathfrak{E}, \sigma)$ *are denoted by* $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ *and* $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$.

It is obvious from this definition that one has $\overline{\mathfrak{A}} \subset \overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$, $\overline{\mathfrak{A}}$ being the sub-C^*-algebra of $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ generated by $\mathfrak{A}$. Setting $\nu = \delta_0$ in (69) and using Schwarz's inequality for ω and (55) one gets the set of inequalities

$$|\omega(\mu)| \leqq \omega(\mu^* \times \mu)^{\frac{1}{2}} \leqq \|\mu\| \leqq \|\mu\|_1 . \tag{69a}$$

The positive form ω on $\mathscr{M}_1(\mathfrak{E}, \sigma)$ therefore extends to a positive form on $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ which we continue to call ω, the representations $\pi_{\omega, \mathscr{M}_1(\mathfrak{E}, \sigma)}$ extending correspondingly to a faithful irreducible representation of $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ which we continue to call the Schrödinger representation and to denote by $\pi_{\omega, \overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}} = \pi_\omega$.

If, as we shall assume in what follows, $\mathfrak{A}$ contains $\mathfrak{J}_\Omega$, $\|\mu\|$ is equal to the operator-norm $\|\pi_{\omega, \mathfrak{A}}(\mu)\|$ i.e. is obtained by taking the Sup in (69) for ν running through $\mathfrak{A}$ (Theorem 12). Theorem 16 then shows that $\overline{\mathfrak{A}}$ is (via $\pi_\omega = \pi_{\omega, \overline{\mathfrak{A}}}$) isomorphic to a topologically (and, by Kadison's Theorem [38], strictly) irreducible operator algebra i.e. $\overline{\mathfrak{A}}$ is *primitive*. The following theorem shows that by completing $\mathfrak{A}$ in the C^*-norm we obtain a new object.

Theorem 17. *Let* $\mathfrak{A}$ *be a sub-*-algebra of* $\mathscr{M}_1(\mathfrak{E}, \sigma)$ *containing* $\mathfrak{J}_\Omega$. $\mathfrak{A}$ *is strictly smaller than its Schrödinger* C^**-completion* $\overline{\mathfrak{A}}$.

Proof: We set

$$\Omega'_\lambda(\psi) = e^{-\frac{1}{2\lambda^2} s(\psi, \psi)} \qquad 0 < \lambda < \infty$$

and easily obtain using Theorem 8 and (24):

$$\|\Omega'_\lambda\|_1^2 = \lambda \sqrt{1 + 1/\lambda^2}\, \|\Omega'^*_\lambda \times \Omega'_\lambda\|_1 .$$

Theorem (4.10.6) of RICKART** then shows that $\mathfrak{J}_\Omega \cap \mathfrak{A}$ is strictly smaller than its Hilbert space completion: because of Kadison's theorem $\mathfrak{A}$ is then strictly smaller than $\overline{\mathfrak{A}}$.

The inequality (69a) allows one to extend (49) to $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$. Reasoning as in the proofs of Theorems 10, 11 and 15 one then gets corresponding results for the C^*-algebra $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$:

* The fact that μ is a norm and not a pseudonorm results from Theorem 14.
** See Ref. [33], p. 263, Theorem (4.10.6).

Theorem 18. *One has for all* $\mu \in \overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$

$$\Omega \times \mu \times \Omega = \omega(\mu)\, \Omega \tag{49'}$$

where ω *denotes the extension of the form* (50) *to* $\mathscr{M}_1(\mathfrak{E}, \sigma)$. *The sub-algebra* $\Omega \times \overline{\mathfrak{A}} \times \Omega$ *of* $\overline{\mathfrak{A}}$ *is a field. The respective Schrödinger* C^**-completions* $\overline{\mathfrak{A} \times \Omega} = \overline{\mathfrak{J}_\Omega \cap \mathfrak{A}} = \overline{\mathfrak{A}} \times \Omega = \overline{\mathfrak{J}}_\Omega \cap \overline{\mathfrak{A}}$ *and* $\overline{\mathfrak{J}'_\Omega \cap \mathfrak{A}} = \overline{\mathfrak{J}'_\Omega} \cap \overline{\mathfrak{A}}$ *of* $\mathfrak{J}_\Omega \cap \mathfrak{A}$ *and* $\mathfrak{J}'_\Omega \cap \mathfrak{A}$ *are complementary left ideals of* $\overline{\mathfrak{A}}$, *the first being a minimal, the second a maximal left ideal.* $\overline{\mathfrak{J}'_\Omega} \cap \overline{\mathfrak{A}}$ *is the null space of the extension of* ω *to* $\overline{\mathfrak{A}}$. $\overline{\mathfrak{J}}_\Omega \cap \overline{\mathfrak{A}}$ *is (by Kadison's theorem) a complete Hilbert space under the scalar product* $(\mu \mid \nu) = \omega(\mu^* \times \nu)$ $(\mathfrak{J}_\Omega \cap \mathfrak{A}$ *has accordingly the same completion in the norms* $\|\mu\|$ *and* $\omega(\mu^* \times \mu)^{\frac{1}{2}})$*. *The extension of the Schrödinger representation* π_ω *to* $\overline{\mathfrak{A}}$ *coincides with the restriction of the left regular representation of* $\overline{\mathfrak{A}}$ *to the minimal left ideal* $\overline{\mathfrak{J}}_\Omega \cap \overline{\mathfrak{A}}$ *(this providing an alternative proof of Theorem* 16*). Every *-representation* ϱ *of* $\overline{\mathfrak{A}}$ *on a Hilbert space is the direct sum of a multiple of the Schrödinger representation and a representation* ϱ_1 *such that* $\varrho_1(\Omega) = 0$. *In particular every irreducible representation* ϱ *of* $\overline{\mathfrak{A}}$ *such that* $\varrho(\Omega) \neq 0$ *is unitarily equivalent to the Schrödinger representation* π_ω. *Every *-representation of* $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ *is the sum of a multiple of the Schrödinger representation and the null representation. Every irreducible representation of* $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ *is unitarily equivalent to the Schrödinger representation. All *-representations of* $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ *are quasi-equivalent.*

The last statement is an immediate consequence of the definition of quasi-equivalence of representation, for which we refer to Appendix I of Ref. [6].

Theorem 19. *The Schrödinger representation maps* $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ *isomorphically onto the compact (= completely continuous) operators of the representation space.*

This theorem is an immediate consequence of A. Rosenberg's result that a concrete C^*-algebra on a separable Hilbert space having only one irreducible representation consists of the compact operators [39].

Theorem 20. *Let* $(\mathfrak{E}, \sigma)$ *be a* $2n$*-dimensional symplectic space,* $\mathfrak{E}_1$ *a* $2p$*-dimensional regular subspace of* $(\mathfrak{E}, \sigma)$, $\mathfrak{E}_2$ *the orthogonal complement of* $\mathfrak{E}_1$ *(with respect to* σ*),* π_ω, π_{ω_1}, π_{ω_2} *the respective Schrödinger representations of* $\mathscr{M}_1(\mathfrak{E}, \sigma)$, $\mathscr{M}_1(\mathfrak{E}_1, \sigma)$, $\mathscr{M}_1(\mathfrak{E}_2, \sigma)$ *acting on the respective spaces* $\mathscr{H}$, $\mathscr{H}_1$ *and* $\mathscr{H}_2$. *One has*

$$\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \tag{70}$$

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* This can be directly inferred from the fact that $\omega(\mu^* \times \mu)^{\frac{1}{2}} = \|\mu\| = \|\mu^* \times \mu\|^{\frac{1}{2}}$ for $\mu \in \mathfrak{J}_\Omega$ (take $\mu^* \times \mu$ for μ in (49)).

and, for $\mu_1 \in \mathcal{M}_1(\mathfrak{E}_1, \sigma)$, $\mu_2 \in \mathcal{M}_2(\mathfrak{E}_2, \sigma)$

$$\pi_\omega(\mu_1) = \pi_{\omega_1}(\mu_1) \otimes \mathbf{1}_{\mathcal{H}_2} \tag{71}$$

$$\pi_\omega(\mu_2) = \mathbf{1}_{\mathcal{H}_1} \otimes \pi_{\omega_2}(\mu_2) \tag{72}$$

$$\pi_\omega(\mu_1 \times \mu_2) = \pi_{\omega_1}(\mu_1) \otimes \pi_{\omega_2}(\mu_2) \tag{73}$$

therefore

$$\|\pi_\omega(\mu_1)\| = \|\pi_{\omega_1}(\mu_1)\| \tag{74}$$

i.e. μ_1 *has the same Schrödinger norm as an element of* $\mathcal{M}_1(\mathfrak{E}_1, \sigma)$ *and as an element of* $\mathcal{M}_1(\mathfrak{E}, \sigma)$. *Consequently one has the inclusion**

$$\overline{\mathcal{M}_1(\mathfrak{E}_1, \sigma)} \subset \overline{\mathcal{M}_1(\mathfrak{E}, \sigma)}\,. \tag{75}$$

Proof: The definition $\mathfrak{E}_2 = \{\psi \in \mathfrak{E} \,|\, \sigma(\psi, \theta) = 0$ for all $\theta \in \mathfrak{E}_1\}$ implies that $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$: since σ is regular one has namely $\dim \mathfrak{E} = \dim \mathfrak{E}_1 + \dim \mathfrak{E}_2$ and on the other hand $\mathfrak{E}_1 \cap \mathfrak{E}_2$ reduces to the null vector owing to the assumed regularity of $\mathfrak{E}_1$. Moreover if $v \in \mathfrak{E}_2$ is such that $\sigma(v, \eta) = 0$ for all $\eta \in \mathfrak{E}_2$ one has $\sigma(v, \psi) = 0$ for all $\psi \in \mathfrak{E}$, and therefore $\mathfrak{E}_2$ is also regular. Taking symplectic bases (e_i, f_i) and (e_j, f_j) respectively in $\mathfrak{E}_1$ and $\mathfrak{E}_2$, $i = 1, 2, \ldots, p$, $j = p + 1, \ldots, n$, we get a symplectic base in $(\mathfrak{E}, \sigma)$ which, upon the construction (45), (46) provides a σ-allowed prehilbertian structure $h = s + i\sigma$ on $(\mathfrak{E}, \sigma)$ for which $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are orthogonal complements. For $\psi = \psi_1 + \psi_2 \in \mathfrak{E}$, $\psi_1 \in \mathfrak{E}_1$, $\psi_2 \in \mathfrak{E}_2$ we set

$$\Omega(\psi) = a^{-1}\, e^{-\frac{1}{2}}\, s(\psi, \psi) = \Omega_1(\psi_1)\, \Omega_2(\psi_2)$$

where

$$\Omega_i(\psi) = a_i^{-1}\, e^{-\frac{1}{2} s(\psi_i, \psi_i)}\,, i = 1, 2$$

$$a_i = \int_{E_i} e^{-s(\psi, \psi)}\, dm_\sigma(\psi)\,,\; a = a_1 a_2$$

and denote by ω, ω_1, ω_2 the corresponding positive forms respectively on $\mathcal{M} = \mathcal{M}_1(\mathfrak{E}, \sigma)$, $\mathcal{M}_1 = \mathcal{M}_1(\mathfrak{E}_1, \sigma)$, $\mathcal{M}_2 = \mathcal{M}_1(\mathfrak{E}_2, \sigma)$. For $\nu_1 \in \mathcal{M}_1$, $\nu_2 \in \mathcal{M}_2$ one has according to (33) and (50):

$$\omega(\nu_1 \times \nu_2) = \omega_1(\nu_1)\, \omega_2(\nu_2)\,. \tag{76}$$

To prove (70) we define the tensor product of $\hat{\mu}_1 = \mu_1 + \mathfrak{N}_{\omega_1, \mathcal{M}_1} \in \mathcal{M}_1/\mathfrak{N}_{\omega_1, \mathcal{M}_1}$ and $\hat{\mu}_2 = \mu_2 + \mathfrak{N}_{\omega_2, \mathcal{M}_2} \in \mathcal{M}_2/\mathfrak{N}_{\omega_2, \mathcal{M}_2}$ as

$$\hat{\mu}_1 \otimes \hat{\mu}_2 = \mu_1 \times \mu_2 + \mathfrak{N}_{\omega, \mathcal{M}} \tag{77}$$

This definition is coherent because $\varrho_1 \in \mathfrak{N}_{\omega_1, \mathcal{M}_1}$, $\varrho_2 \in \mathfrak{N}_{\omega_2, \mathcal{M}_2}$ imply $\varrho_1 \times \varrho_2$, $\varrho_1 \times \mu_2$, $\varrho_2 \times \mu_1 \in \mathfrak{N}_{\omega, \mathcal{M}}$ owing to (76). Furthermore for

$\hat{\mu}_1^i, \hat{\nu}_1^j \in \mathcal{M}_1/\mathfrak{N}_{\omega_1, \mathcal{M}_1}$, $\hat{\mu}_2^i, \hat{\nu}_2^j \in \mathcal{M}_2/\mathfrak{N}_{\omega_2, \mathcal{M}_2}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$

* As a result of theorem 19, $\overline{\mathcal{L}_1(\mathfrak{E}, \sigma)}$ is the C^*-tensor product of $\overline{\mathcal{L}_1(\mathfrak{E}_1, \sigma)}$ and $\overline{\mathcal{L}_1(\mathfrak{E}_2, \sigma)}$. We leave open the question of wether an analoguous result holds for the $\mathcal{M}_1$-C^*-algebras.

one has by (54) and (76):

$$\left(\sum_{i=1}^{k} \hat{\mu}_1^i \otimes \hat{\mu}_2^i \,\middle|\, \sum_{j=1}^{l} \hat{\nu}_1^j \otimes \hat{\nu}_2^j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} (\hat{\mu}_1^i | \hat{\nu}_1^j)_1 \cdot (\hat{\mu}_2^i, \hat{\nu}_2^j)_2 \tag{78}$$

where $(\,|\,)$, $(\,|\,)_1$, $(\,|\,)_2$ denote the scalar products (54) respectively on $\mathscr{M}/\mathfrak{N}_{\omega,\mathscr{M}}$, $\mathscr{M}_1/\mathfrak{N}_{\omega_1,\mathscr{M}_1}$, $\mathscr{M}_2/\mathfrak{N}_{\omega_2,\mathscr{M}_2}$. Since the two last spaces are respectively dense in $\mathscr{H}_1$ and $\mathscr{H}_2$, (77) and (78) show that $\mathscr{H}_1 \otimes \mathscr{H}_2 \subset \mathscr{H}$ and (70) will be proven if we show that this inclusion is an equality. This now follows from the last assertion of Theorem 13: we start from the total systems $\{{}^{u_1}\hat{\Omega}_1, u_1 \in \mathfrak{E}_1\}$, $\{{}^{u_2}\hat{\Omega}, u_2 \in \mathfrak{E}_2\}$ respectively in $\mathscr{H}_1$ and $\mathscr{H}_2$ and show that ${}^{u_1}\hat{\Omega}_1 \otimes {}^{u_2}\hat{\Omega}_2$ is total in $\mathscr{H}$: since $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are orthogonal $\mathscr{H}_1$ and $\mathscr{H}_2$ commute (Theorem 6). One has therefore

$$\begin{aligned} {}^{u_1}\Omega_1 \times {}^{u_2}\Omega_2 &= \delta_{u_1} \times \Omega_1 \times \delta_{u_2} \times \Omega_2 \\ &= \delta_{u_1} \times \delta_{u_2} \times \Omega_1 \times \Omega_2 \approx \delta_{u_1+u_2}\Omega = {}^{u_1+u_2}\Omega\,, \end{aligned}$$

the ${}^{u}\Omega = {}^{u_1+u_2}\Omega$ being a total system in $\mathscr{H}$.

Proof of (73), (71) and (72). We verify (73) on the total set of elements $\hat{\nu}_1 \otimes \hat{\nu}_2 \in \mathscr{H}$, $\nu_1 \in \mathscr{M}_1/\mathfrak{N}_{\omega_1,\mathscr{M}_1}$, $\nu_2 \in \mathscr{M}_2/\mathfrak{N}_{\omega_2,\mathscr{M}_2}$:

$$\begin{aligned} \pi_\omega(\mu_1 \times \mu_2)\, \hat{\nu}_1 \otimes \hat{\nu}_2 &= \mu_1 \times \mu_2 \times \nu_1 \times \nu_2 + \mathfrak{N}_{\omega,\mathscr{M}} \\ &= \mu_1 \times \nu_1 \times \mu_2 \times \nu_2 + \mathfrak{N}_{\omega,\mathscr{M}} = \pi_{\omega_1}(\mu_1)\, \hat{\nu}_1 \otimes \pi_{\omega_2}(\mu_2)\, \hat{\nu}_2 \end{aligned}$$

(71) and (72) are obtained by specifying respectively $\mu_1 = \delta_0$, $\mu_2 = \delta_0$ in (73).

We close this section with the remark that $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ *is a separable C^*-algebra whilst* $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ *is non-separable*. Separability results for $\overline{\mathscr{L}_1(\mathfrak{E}, \sigma)}$ from Theorem 19. The non-separable character of $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ will result from the existence of an irreducible representation π on a non-separable Hilbert space (see Ref. [35] 2.3.3). We proceed to the construction of π. Let e_i, f_i, $i = 1, 2, \ldots, n$, be a symplectic base in $\mathfrak{E}$ with $\psi = \sum_{k=1}^{n} (\xi^k e_k + \eta^k f_k)$, $\psi \in \mathfrak{E}$, and let $\mathfrak{F}$ be the real subspace of $\mathfrak{E}$ spanned by the vectors e_i, $i = 1, 2, \ldots, n$. Consider the commutative subalgebra $\mathfrak{M}$ of $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ consisting of all (canonical extensions of the) bounded measures on $\mathfrak{F}$. An element $\mu \in \mathfrak{M}$ can be considered as a measure $\mu(\xi^k)$ on the space of the coordinates ξ^k and we define as follows its Fourier transform $\hat{\mu}$:

$$\hat{\mu}(\lambda_k) = \int \exp\left[i \sum_{k=1}^{n} \lambda_k \xi^k\right] d\mu(\xi^k)\,.$$

For fixed λ_k $\mu \in \mathfrak{M} \to \hat{\mu}(\lambda_k)$ is a character of $\mathfrak{M}$ because the twisted convolution reduces on $\mathfrak{M}$ to the ordinary convolution. We thus get a one-dimensional representation π_0 of $\mathfrak{M}$ on a complex Hilbert space spanned by the vector u by setting

$$\pi_0(\mu)\, u = \hat{\mu}(\lambda_k)\, u\,, \qquad \mu \in \mathfrak{M}$$

One can now construct an irreducible representation π of $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$ on a Hilbert space $\mathscr{H}$ containing the vector u such that π restricted to $\mathfrak{M}$ acts on u like π_0 (see Ref. [35], 2.10.2). Taking

$$\xi = \sum_{k=1}^{n} \xi^k e_k\,, \qquad \eta = \sum_{k=1}^{n} \eta^k f_k$$

one has then using (22)

$$\begin{aligned} \pi(\delta_\xi)\, \pi(\delta_\eta)\, u &= \exp\Big[-2i \sum_{k=1}^{n} \xi^k \eta^k\Big]\, \pi(\delta_\eta)\, \pi(\delta_\xi)\, u \\ &= \exp\Big[i \sum_{k=1}^{n} (\lambda_k - 2\eta^k)\, \xi^k\Big]\, \pi(\delta_\eta)\, u \end{aligned}$$

which shows that for $\eta \neq \eta'$ $\pi(\delta_\eta)u$ and $\pi(\delta_{\eta'})u$ are eigenvectors of the $\pi(\delta_\xi)$ belonging to different eigenvalues and hence mutually orthogonal. $\mathscr{H}$ contains thus a non-denumerable orthonormal system; q.e.d.

§ 4. The algebras $\mathscr{M}_1(\mathfrak{H}, \sigma)$ and $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ for an infinite-dimensional symplectic space

We consider in this section an *infinite-dimensional symplectic space* $(\mathfrak{H}, \sigma)$, that is, an infinite-dimensional real vector space $\mathfrak{H}$ equipped with a symplectic form and such that in addition there exists a filtrating and absorbing system of regular finite-dimensional subspaces of $\mathfrak{H}$ (we recall that a subspace $\mathfrak{E} \subset \mathfrak{H}$ is *regular* if the restriction of σ to $\mathfrak{E}$ is regular. $\mathscr{S}$ is *filtrating* in the sense that to each pair $\mathfrak{E}_1, \mathfrak{E}_2 \in \mathscr{S}$ there exists $\mathfrak{E}_3 \in \mathscr{S}$ such that $\mathfrak{E}_3 \supset \mathfrak{E}_1 \cup \mathfrak{E}_2$. $\mathscr{S}$ is *absorbing* in that $\bigcup_{\mathfrak{E} \in \mathfrak{S}} \mathfrak{E} = \mathfrak{H}$. These two definitions imply that to each finite-dimensional vectorial subspace $\mathfrak{F} \subset \mathfrak{H}$ there exists an $\mathfrak{E} \in \mathscr{S}$ such that $\mathfrak{F} \subset \mathfrak{E}$).

The infinite-dimensional symplectic spaces which we will have to consider are all provided by some complex prehilbertian space $\mathfrak{H}$ with a complex scalar product $h = s + i\sigma$, $\mathscr{S}$ consisting of all finite-dimensional complex subspaces of $\mathfrak{H}$. In this case one has a "σ-allowed prehilbertian structure" on $(\mathfrak{H}, \sigma)$ as was defined in the beginning of Section 3. Since however other σ-allowed prehilbertian structures on the same $(\mathfrak{H}, \sigma)$ can be of interest (see § 6), it is useful to consider the symplectic space $(\mathfrak{H}, \sigma)$, as above, independently of the way in which it is given in terms of a prehilbertian space.

Our task is to construct the analogues for $(\mathfrak{H}, \sigma)$ of the algebras discussed in §§ 2 and 3 for a finite-dimensional symplectic space. Roughly speaking these are the "union" of the algebras $\mathscr{M}_1(\mathfrak{E}, \sigma)$ (resp. $\overline{\mathscr{M}_1(\mathfrak{E}, \sigma)}$)

corresponding to the subspaces $\mathfrak{E} \in \mathscr{S}$ of $\mathfrak{H}$. For a rigorous construction we start from the set $\mathfrak{M}$ of all couples $\{\mu, \mathfrak{E}\}$ consisting of an $\mathfrak{E} \in \mathscr{S}$ and a $\mu \in \mathscr{M}_1(\mathfrak{E}, \sigma)$ and define on $\mathfrak{M}$ an equivalence relation in the following way: given $\mathfrak{E}_\alpha \subset \mathfrak{E}_\beta \in \mathscr{S}$ let us first denote by $\varphi_{\alpha\beta}$ the homomorphic injection of $\mathscr{M}_1(\mathfrak{E}_\alpha, \sigma)$ into $\mathscr{M}_2(\mathfrak{E}_\beta, \sigma)$ provided by the "natural extension" of measures (see Theorem 6). One has, according to (33)

$$\varphi_{\alpha\alpha} = \text{identity operation on} \quad \mathscr{M}_1(\mathfrak{E}_\alpha, \sigma) \tag{79}$$

$$\varphi_{\gamma\beta} \cdot \varphi_{\beta\alpha} = \varphi_{\gamma\alpha} \quad \text{for} \quad \mathfrak{E}_\alpha \subset \mathfrak{E}_\beta \subset \mathfrak{E}_\gamma \in \mathscr{S} . \tag{80}$$

The *equivalence* $\{\mu_\alpha, \mathfrak{E}_\alpha\} \sim \{\mu_\beta, \mathfrak{E}_\beta\}$ between elements of $\mathfrak{M}$ is now defined as the existence of (at least one) $\mathfrak{E}_\gamma \in \mathscr{S}$ such that $\mathfrak{E}_\gamma \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$ and

$$\varphi_{\gamma\alpha}\mu_\alpha = \varphi_{\gamma\beta}\mu_\beta . \tag{81}$$

One can equivalently require (81) for *all* $\mathfrak{E}_\gamma$ such that $\mathfrak{E}_\gamma \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$. Suppose indeed that (81) holds for $\mathfrak{E}_\gamma$ and that there is a $\mathfrak{E}_\delta \in \mathscr{S}$ such that $\mathfrak{E}_\delta \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$ but $\varphi_{\delta\alpha}\mu_\alpha \neq \varphi_{\delta\beta}\mu_\beta$. Taking $\mathfrak{E}_\varepsilon \in \mathscr{S}$ such that $\mathfrak{E}_\varepsilon \supset \mathfrak{E}_\gamma \cup \mathfrak{E}_\delta$ one would conclude from (81) and the fact that $\varphi_{\varepsilon\delta}$ is injective that

$$\varphi_{\varepsilon\gamma}\varphi_{\gamma\alpha}\mu_\alpha = \varphi_{\varepsilon\gamma}\varphi_{\gamma\beta}\mu_\beta \Rightarrow \varphi_{\varepsilon\alpha}\mu_\alpha = \varphi_{\varepsilon\beta}\mu_\beta .$$

$$\varphi_{\varepsilon\delta}\varphi_{\delta\alpha}\mu_\alpha \neq \varphi_{\varepsilon\delta}\varphi_{\delta\beta}\mu_\beta \Rightarrow \varphi_{\varepsilon\alpha}\mu_\alpha \neq \varphi_{\varepsilon\beta}\mu_\beta .$$

The relation $\sim$ defined by (81) is now easily seen to be an equivalence relation: reflexivity is obvious from (79), reciprocity from the symmetry of (81) in α and β, transitivity from the fact that the equivalences $\{\mu_\alpha, \mathfrak{E}_\alpha\} \sim \{\mu_\beta, \mathfrak{E}_\beta\}$ and $\{\mu_\beta, \mathfrak{E}_\beta\} \sim \{\mu_\gamma, \mathfrak{E}_\gamma\}$ imply for $\mathfrak{E}_\delta \in \mathscr{S}$ such that $\mathfrak{E}_\delta \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$ that

$$\varphi_{\delta\alpha}\mu_\alpha = \varphi_{\delta\beta}\mu_\beta = \varphi_{\delta\gamma}\mu_\gamma ,$$

whence $\{\mu_\alpha, \mathfrak{E}_\alpha\} = \{\mu_\gamma, \mathfrak{E}_\gamma\}$.

We consider now the set $\mathfrak{M}_1$ of the classes determined in $\mathfrak{M}$ by the equivalence relation $\sim$. Let us call $\{\mu_\alpha, \mathfrak{E}_\alpha\} \in \mathfrak{M}$ a *representant* of $\mu \in \mathfrak{M}_1$ if $\{\mu_\alpha, \mathfrak{E}_\alpha\} \in \mu$. For $\mu, \nu \in \mathfrak{M}_1$ with representants $\{\mu_\alpha, \mathfrak{E}_\alpha\}, \{\nu_\beta, \mathfrak{E}_\beta\}$ we can choose other representants $\{\varphi_{\gamma\alpha}\mu_\alpha, \mathfrak{E}_\gamma\}, \{\varphi_{\gamma\beta}\nu_\beta, \mathfrak{E}_\gamma\}$ with the same $\mathfrak{E}_\gamma \in \mathscr{S}$, $\mathfrak{E}_\gamma \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$. We then define $a\mu + b\nu$ (for complex scalars a, b), $\mu \times \nu$ and μ^* as the respective equivalence classes of $\{a\,\varphi_{\gamma\alpha}\mu_\alpha + b\,\varphi_{\gamma\beta}\mu_\beta, \mathfrak{E}_\gamma\}$, $\{(\varphi_{\gamma\alpha}\mu_\alpha) \times (\varphi_{\gamma\beta}\mu_\beta), \mathfrak{E}_\gamma\}$ and $\{(\varphi_{\gamma\alpha}\mu_\alpha)^*, \mathfrak{E}_\gamma\}$. We define in addition

$$\|\mu\|_1 = \|\varphi_{\gamma\alpha}\mu_\alpha\|_1 = \mathscr{M}_1 - \text{norm of } \mu \tag{82}$$

$$\|\mu\| = \|\varphi_{\gamma\alpha}\mu_\alpha\| = \text{Schrödinger norm of } \mu . \tag{83}$$

Owing to (34), (35), (36a, b) and (74) these definitions do not depend on the choice of $\mathfrak{E}_\gamma$. Neither do they depend on the choice of the representants $\{\mu_\alpha, \mathfrak{E}_\alpha\}$ and $\{\nu_\beta, \mathfrak{E}_\beta\}$ of μ and ν because by starting from other representants one would get the same $\{\varphi_{\gamma\alpha}\mu_\alpha, \mathfrak{E}_\gamma\}$, $\{\varphi_{\gamma\beta}\nu_\beta, \mathfrak{E}_\gamma\}$ with an appropriate $\mathfrak{E}_\gamma$. We have therefore defined on $\mathfrak{M}_1$ the structure of a normed *-algebra (in either of the two norms (82) and (83)). The Banach *-algebras $\mathscr{M}_1(\mathfrak{H}, \sigma)$ and $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ are now defined as the completions of $\mathfrak{M}_1 = \mathfrak{M}_1(\mathfrak{H}, \sigma)$ in the respective norms (82) and (83). Since $\|\mu^* \times \mu\| = \|\mu\|^2$ for all $\mu \in \mathfrak{M}_1$, $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ is a C^*-algebra.

The preceding construction depends a priori on the system $\mathscr{S}$. However,

Theorem 21. *$\mathfrak{M}_1(\mathfrak{H}, \sigma)$ as constructed above (and therefore $\mathscr{M}_1(\mathfrak{H}, \sigma)$ and $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$) depend uniquely on the symplectic space $(\mathfrak{H}, \sigma)$ and not on the system $\mathscr{S}$ used for their construction.*

Proof: Take a symplectic space $(\mathfrak{H}, \sigma)$ with two distinct filtrating and absorbing systems $\mathscr{S}$ and $\mathscr{S}_1$ of finite-dimensional regular subspaces. $\mathscr{S}' = \mathscr{S} \cup \mathscr{S}_1$ is a filtrating and absorbing system containing $\mathscr{S}$ (it is filtrating because $\mathscr{S}$ absorbs every finite-dimensional vectorial subspace $\mathfrak{F}$ of $\mathfrak{H}$). Let $\mathfrak{M}^{\mathscr{S}}$ and $\mathfrak{M}_1^{\mathscr{S}}$, resp. $\mathfrak{M}^{\mathscr{S}'}$ and $\mathfrak{M}_1^{\mathscr{S}'}$ denote the spaces constructed above respectively with $\mathscr{S}$ and $\mathscr{S}'$. We want to establish a one-to-one homomorphism between $\mathfrak{M}_1^{\mathscr{S}}$ and $\mathfrak{M}_1^{\mathscr{S}'}$. For this, to each $\mu \in \mathfrak{M}_1^{\mathscr{S}}$ with representant $\{\mu_\alpha, \mathfrak{E}_\alpha\} \in \mathfrak{M}^{\mathscr{S}} \subset \mathfrak{M}^{\mathscr{S}'}$ we assign the class $\mu' \in \mathfrak{M}_1^{\mathscr{S}'}$ of $\{\mu_\alpha, \mathfrak{E}_\alpha\}$ in $\mathfrak{M}^{\mathscr{S}'}$. This procedure is independent of the choice of the representant $\{\mu_\alpha, \mathfrak{E}_\alpha\}$ of μ because if $\{\mu_\alpha, \mathfrak{E}_\alpha\} \sim \{\mu_\beta, \mathfrak{E}_\beta\}$ in $\mathfrak{M}^{\mathscr{S}}$ the same holds obviously in $\mathfrak{M}^{\mathscr{S}'}$. Since the algebraic operations in $\mathfrak{M}^{\mathscr{S}'}$ can be defined using arbitrary representants the mapping $\mu \to \mu'$ is homomorphic. It is on the other hand one-to-one (because of the arbitrariness of $\mathfrak{E}_\gamma$ in the definition of $\sim$) and onto (because $\mathscr{S}$ is absorbing).

Theorem 22. *Let $(\mathfrak{H}, \sigma)$ be an infinite-dimensional symplectic space and $\mathfrak{K}$ a symplectic subspace of $(\mathfrak{H}, \sigma)$ (that is a subspace which is itself a (possibly infinite-dimensional) symplectic space under the induced structure). There is a natural homomorphic injection of $\mathscr{M}_1(\mathfrak{K}, \sigma)$ into $\mathscr{M}_1(\mathfrak{H}, \sigma)$ (respectively of $\overline{\mathscr{M}_1(\mathfrak{K}, \sigma)}$ into $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$). The set of these injections for different subspaces satisfies in addition the rules* (79), (80).

$\mathscr{S}$ and $\mathscr{S}_1$ being filtrating absorbing systems for $\mathfrak{H}$ and $\mathfrak{K}$ respectively one sets $\mathscr{S}' = \mathscr{S} \cup \mathscr{S}_1$ and performs with $\mathscr{S}$ and $\mathscr{S}'$ the construction of the preceding theorem, the only difference being that $\mathscr{S}$ is not absorbing for $\mathfrak{H}$, so that the mapping $\mu \to \mu'$ is not onto.

§ 5. The Fock representation of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ associated to a σ-allowed prehilbertian structure and the corresponding field operator

In the last section the algebra $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ has been given an algebraic definition without reference to a special representation. This is made possible by the intrinsic character of the Schrödinger norm (83) due to von Neumann's uniqueness theorem (adapted to the algebra of bounded measures). We now intend to describe the analogue for $\mathscr{M}_1(\mathfrak{H}, \sigma)$ of the Schrödinger representation of $\mathscr{M}_1(\mathfrak{E}, \sigma)$: the analogy with the finite-dimensional case is however only partial: instead of the uniqueness of the Schrödinger representation (following from Theorem 15) we will now have a plurality of "Fock representations" (which depend on the choice of a σ-allowed prehilbertian structure), amongst which the familiar Fock representation of field theory is only one example.

Let $(\mathfrak{H}, \sigma)$ be our infinite-dimensional symplectic space. We start as in § 4 from a σ-allowed prehilbertian structure $h = s + i\sigma$ on $(\mathfrak{H}, \sigma)$. Defining the function Ω'_s on $\mathfrak{H}$ as

$$\Omega'_s(\psi) = e^{-\frac{1}{2}s(\psi,\psi)}, \qquad \psi \in \mathfrak{H} \tag{84}$$

we consider the linear form ω_s on $\mathfrak{M}_1(\mathfrak{H}, \sigma)$ defined as follows: for $\mu \in \mathfrak{M}_1(\mathfrak{H}, \sigma)$ with representant $\{\mu_\alpha, \mathfrak{E}_\alpha\}$, $\mathfrak{E}_\alpha \in \mathscr{S}$, we set

$$\omega_s(\mu) = \mu_\alpha(\Omega'_s | \mathfrak{E}_\alpha) \tag{85}$$

where $\Omega'_s | \mathfrak{E}_\alpha$ is the restriction of Ω'_s to the subspace $\mathfrak{E}_\alpha$.

This definition does not depend on the representant $\{\mu_\alpha, \mathfrak{E}_\alpha\}$: given another representant $\{\mu_\beta, \mathfrak{E}_\beta\}$ we have an $\mathfrak{E}_\gamma \in \mathscr{S}$, $\mathfrak{E}_\gamma \supset \mathfrak{E}_\alpha \cup \mathfrak{E}_\beta$, such that $\varphi_{\gamma\alpha}\mu_\alpha = \varphi_{\gamma\beta}\mu_\beta$, whence

$$\begin{aligned}\mu_\alpha(\Omega'_s | \mathfrak{E}_\alpha) &= \mu_\alpha\{(\Omega'_s | \mathfrak{E}_\gamma) | \mathfrak{E}_\alpha\} = \{\varphi_{\gamma\alpha}\mu_\alpha\}(\Omega'_s | \mathfrak{E}_\gamma) \\ &= \{\varphi_{\gamma\beta}\mu_\beta\}(\Omega'_s | \mathfrak{E}_\gamma) = \mu_\beta(\Omega'_s | \mathfrak{E}_\beta) .\end{aligned}$$

It is obvious from the definition of algebraic operations in $\mathfrak{M}_1(\mathfrak{H}, \sigma)$ that ω_s is a positive linear form on this algebra. One has further from (51), (68), (82), (83)

$$|\omega_s(\mu)| \leqq \|\mu\| \leqq \|\mu\|_1 , \tag{86}$$

therefore ω_s extends to a positive linear form on the Banach *-algebras $\mathscr{M}_1(\mathfrak{H}, \sigma)$ and $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$, which is of unit norm since $\mu(\delta_0) = 1$. The representation π_{ω_s} of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ (and, by restriction of its sub *-algebras) is called the *Fock representation associated with the σ-allowed prehilbertian structure $h = s + i\sigma$*. Unlike the case of a finite number of degrees of freedom (corollary to Theorem 15) π_{ω_s} now essentially depends on s.

If, for $\psi \in \mathfrak{H}$ such that $s(\psi, \psi) = 1$, we define

$$U_s(\psi) = \pi_{\omega_s}(\delta_\psi) , \tag{87}$$

the correspondence

$$\lambda \in R \to U_s(\lambda\psi) \tag{88}$$

is by (22) a unitary representation of the additive real number line. Since

$$\omega_s(\psi) = (\Phi_0 | U_s(\psi) | \Phi_0) = e^{-\frac{1}{2}s(\psi,\psi)} \tag{89}$$

is continuous in ψ this representation is strongly continuous (we denote by Φ_0 the cyclic vector of the Fock representations π_{ω_s} constructed à la Gelfand-Segal from the positive form ω_s). By Stone's theorem there exists therefore a self-adjoint infinitesimal operator

$$A_s(\psi) = -i \lim_{\lambda \to 0} \frac{U(\lambda\psi) - 1}{\lambda} \tag{90}$$

such that

$$U_s(\psi) = e^{i A_s(\psi)} \tag{91}$$

this (unbounded) operator is called *the field operator associated with the Fock representation* π_{ω_s} (or with the σ-allowed structure $h = s + i\sigma$). Let $\overline{\mathfrak{H}}$ be the Hilbert space completion of the prehilbertian space $(\mathfrak{H}, h)$. The field operator (90) is identical with the operator

$$a^+\{\psi\} + a^-\{\psi\}\,, \qquad \psi \in \mathfrak{H} \subset \overline{\mathfrak{H}} \tag{92}$$

defined in the Hilbert space $\mathscr{S}(\overline{\mathfrak{H}}) = \bigoplus_{p=0}^{\infty} S\overline{\mathfrak{H}}^{\otimes p}$. This results from the equality of (89) with

$$(\psi_0 | e^{i[a^+\{\psi\} + a^-\{\psi\}]} | \psi_0) = e^{-\frac{1}{2} s(\psi,\psi)} \tag{93}$$

where ψ_0 is the vacuum of $\mathscr{S}(\overline{\mathfrak{H}})$.

The last formula is easily derived from the fact that the exponential of (92) as applied to the vector ψ_0 is equal to its Mac Laurin expansion (calculation analogous to the derivation of the addition law (3)).

Notice that the Fock representation π_{ω_s} of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ is obviously faithful since $\|\pi_{\omega_s}(\mu)\| = \|\mu\|$ for all μ.

§ 6. The field operator algebra $\mathfrak{A}(\mathfrak{H}, \sigma)$ and its subalgebras $\mathfrak{A}(\mathfrak{E}, \sigma)$

We come now to the description of the "field operator algebra" which is the object of main interest for field theory. $(\mathfrak{H}, \sigma)$ still denoting an (infinite-dimensional) symplectic space let $\mathfrak{F}$ be a finite-dimensional vectorial subspace of $\mathfrak{H}$ and μ a bounded measure on $\mathfrak{F}$. We can give μ a unique meaning as an element of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ by choosing $\mathfrak{E} \in \mathscr{S}$ such that $\mathfrak{E} \supset \mathfrak{F}$, defining $\tilde{\mu} = \varphi_{\mathfrak{E}\mathfrak{F}}\,\mu$ on $\mathfrak{E}$ by (33) and taking the element of $\mathfrak{M}_1(\mathfrak{H}, \sigma) \subset \overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ represented by $(\tilde{\mu}, \mathfrak{E})$. This procedure does not depend on the choice of the subspace $\mathfrak{E}$, because, if we had chosen instead $\mathfrak{E}' \in \mathscr{S}$, there would exist $\mathfrak{E}' \in \mathscr{S}$, $\mathfrak{E}'' \supset \mathfrak{E} \cup \mathfrak{E}'$ such that

$$\varphi_{\mathfrak{E}''\mathfrak{E}'}\,\tilde{\mu}' = \varphi_{\mathfrak{E}''\mathfrak{E}'}\,\varphi_{\mathfrak{E}'\mathfrak{F}}\,\mu = \varphi_{\mathfrak{E}''\mathfrak{F}}\,\mu = \varphi_{\mathfrak{E}''\mathfrak{E}}\,\varphi_{\mathfrak{E}\mathfrak{F}}\,\mu = \varphi_{\mathfrak{E}''\mathfrak{E}}\,\tilde{\mu}$$

or $\{\tilde{\mu}', \mathfrak{E}'\} \sim \{\tilde{\mu}, \mathfrak{E}\}$. In this way we can consider as included in $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ the following decreasing sets: the set $\mathscr{M}_1(\mathfrak{F})$ of bounded measures on $\mathfrak{F}$, the set $\mathscr{L}'_1(\mathfrak{F})$ of bounded measures on $\mathfrak{F}$ absolutely continuous with respect to Lebesgue measure and the set $\mathscr{C}'_0(\mathfrak{F})$ of (bounded) measures with Radon-Nicodym derivatives (with respect to Lebesgue measure) continuous and with compact support.

Definition: Let $\mathfrak{A}(\mathfrak{H}, \sigma)$, resp. $\mathfrak{A}_0(\mathfrak{H}, \sigma)$, be the smallest sub-$C^*$-algebra of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ containing all $\mathscr{L}_1'(\mathfrak{F})$, resp. $\mathscr{C}_0'(\mathfrak{F})$, for all one-dimensional (real) vectorial subspaces of $\mathfrak{H}$. $\mathfrak{A}(\mathfrak{H}, \sigma)$ is called the *field operator algebra of the symplectic space* $(\mathfrak{H}, \sigma)$.

This definition being given for an $(\mathfrak{H}, \sigma)$ of arbitrary dimensionality, it is clear from the inclusion of the $\mathscr{M}_1$-C^*-algebras that $\mathfrak{A}(\mathfrak{H}, \sigma)$ contains as sub-C^*-algebras all the $\mathfrak{A}(\mathfrak{E}, \sigma)$ corresponding to the regular subspaces $\mathfrak{E}$ of $\mathfrak{H}$.

The following theorem gives the connection between $\mathfrak{A}(\mathfrak{H}, \sigma)$ and the field operators of the different Fock representations.

Theorem 23. *Let $h = s + i\sigma$ be an arbitrary σ-allowed prehilbertian structure on $(\mathfrak{H}, \sigma)$ and let π_{ω_s} and $A_s(\psi)$ be the associated Fock representation and field operator. The image of $\mathfrak{A}(\mathfrak{H}, \sigma)$ in π_{ω_s} is the smallest sub-C^*-algebra of $\pi_{\omega_s}(\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)})$ which contains all functions of the different field operators $A_s(\psi)$, $\psi \in \mathfrak{H}$, which are continuous and vanish at infinity.*

This theorem shows that the "algebra generated by the field operator" is independent of the Fock representation (viz σ-allowed prehilbertian structure). This motivates the name "field operator algebra" without further specification.

Proof of the theorem: The specification of a vector ψ in the one dimensional subspace $\mathfrak{F}$ establishes as follows a one-to-one correspondance between Lebesgue-integrable functions and absolutely continuous measures on $\mathfrak{F}$:

$$\begin{cases} f \in \mathscr{L}_1(\mathfrak{F}) \leftrightarrow \mu_f \in \mathscr{L}_1'(\mathfrak{F}) \\ \mu_f(g) = \int\limits_{-\infty}^{+\infty} g(\lambda\psi)\, f(\lambda\psi)\, d\lambda \qquad g \in \mathscr{C}_0(\mathfrak{F}) \,. \end{cases} \tag{94}$$

Let

$$A_s(\psi) = \int\limits_{-\infty}^{+\infty} \xi \, dP(\xi) \tag{95}$$

be the spectral decomposition of the self-adjoint field operator $A_s(\psi)$. Stone's theorem tell us that

$$U_s(\lambda\psi) = \pi_{\omega_s}(\delta_\psi) = e^{i\lambda A_s(\psi)} = \int\limits_{-\infty}^{+\infty} e^{i\lambda\xi}\, dP(\xi) \tag{96}$$

and we know from the theory of the $\mathscr{L}_1$-algebra of the additive group of $\mathfrak{F}$ that for μ_f given by (94)

$$\pi_{\omega_s}(\mu_f) = \int\limits_{-\infty}^{+\infty} U_s(\lambda\psi)\, f(\lambda\psi)\, d\lambda = \int\limits_{-\infty}^{+\infty} \hat{f}(\xi)\, dP(\xi) = \hat{f}(A_s(\psi))\,, \tag{97}$$

where $\hat{f}$ is the Fourier transform of $\lambda \to f(\lambda\psi)$

$$\hat{f}(\xi) = \int\limits_{-\infty}^{+\infty} e^{i\lambda\xi} f(\lambda\psi)\, d\lambda\,; \tag{98}$$

the norm of the operator (97) is given by

$$\|\pi_{\omega_s}(\mu_f)\| = \|\hat{f}\|_\infty = \operatorname*{Sup}_{\xi \in R} |\hat{f}(\xi)| \,. \tag{99}$$

Our theorem then results from the known fact that the set of Fourier transforms (98) of functions of $\mathscr{L}_1(\mathfrak{F})$ is a dense subset of the set $\mathscr{C}_0(\lambda)$ of all continuous functions vanishing at infinity on the real line.

§ 7. Other possible choices of C^*-algebras

The C^*-algebras described above do not exhaust the list of possible choices of a C^*-algebra associated with the Boson field. The following alternatives are to be considered, each of which might possess special virtues:

The algebra $\mathscr{B}(\mathfrak{H}, \sigma)$ defined as the smallest C^*-subalgebra of $\overline{\mathscr{M}_1(\mathfrak{H},\sigma)}$ containing all bounded measures on the different (real) one-dimensional spaces of $\mathfrak{H}$. This algebra could, as well as $\mathfrak{A}(\mathfrak{H}, \sigma)$, claim the name of "field operator algebra" since it is the smallest C^*-algebra of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ containing all bounded continuous functions of the different field operators which are differences of continuous functions of positive type (this is seen by an argument analoguous to the one of the previous section)*.

The algebra $\mathscr{K}(\mathfrak{H}, \sigma)$ is the smallest C^*-subalgebra of $\overline{\mathscr{M}_1(\mathfrak{H}, \sigma)}$ containing all absolutely continuous measures on the different regular subspaces of $\mathfrak{H}$. This algebra is easily seen to be included in all others (by means of the Stone-Weierstrass Theorem). In any concrete realization it consists of tensor products of a compact operator times a unit operator.

SEGAL defines as the "Weyl algebra" the C^*-algebra obtained in the following way. The fact, noted in Theorem 18, that all representations of $\mathscr{L}_1(\mathfrak{E}, \sigma)$ are quasi-equivalent, implies that there is a (unique) one-to-one correspondence between the weak closures (the von Neumann rings generated) in any two different representations. This provides an intrinsic definition of the von Neumann ring $\mathfrak{R}(\mathfrak{E}, \sigma)$. Moreover for $\mathfrak{E}, \mathfrak{F} \in \mathscr{S}$ such that $\mathfrak{E} \subset \mathfrak{F}$ one has $\mathfrak{R}(\mathfrak{E}, \sigma) \subset \mathfrak{R}(\mathfrak{F}, \sigma)$. The Weyl algebra of SEGAL is then defined as the completion in norm of the inductive limit of all $\mathfrak{R}(\mathfrak{E}, \sigma)$, $\mathfrak{E} \in \mathscr{S}$.

The algebra dealt with in Ref. [1] is the C^*-inductive limit of "Haag rings" obtained in the following way. Let $\mathscr{B}$ be an open space-time domain with compact closure; then $\mathfrak{R}(\mathscr{B})$ is the von Neumann algebra generated by the spectral projectors of all field operators $A(f)$ corresponding to test functions f with support in $\mathscr{B}$. In our description $\mathfrak{R}(\mathscr{B})$ could be obtained as the closure in the weak operator topology associated with

* We leave open the question of wether $\mathscr{B}(\mathfrak{H}, \sigma)$ is actually smaller than $\mathscr{M}_1(\mathfrak{H},\sigma)$.

the standard Fock representation of the smallest C^*-subalgebra of $\mathscr{M}_1(\mathfrak{H}, \sigma)$ containing all bounded measures of all rays of $\mathfrak{H}$ of the form $\psi = \Delta^+ * f$, where f is a test function with support in $\mathscr{B}$. One then takes the uniform closure of the inductive limit of all $\mathfrak{R}(\mathscr{B})$. This definition uses both the detailed structure of the one-particle space of free relativistic bosons and the standard Fock representation defined by the Lorentz-invariant vacuum. The question of whether the construction actually depends upon this particular representation is related to the possible "local quasi-equivalence" of all representations.

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