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Karakostas Fixed Point Theorem and the Existence of Solutions for Impulsive Semilinear Evolution Equations with Delays and Nonlocal Conditions

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Abstract

We prove the existence and uniqueness of the solutions for the following impulsive semilinear evolution equations with delays and nonlocal conditions:

$$\begin{cases} z' = -Az + F(t, z_t), & z \in Z, \quad t \in (0,\tau], t \neq t_k, \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k = 1, 2, 3, \dots, p. \end{cases}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < r < \tau$, Z is a Banach space Z, z_t defined as a function from $[-r,0]$ to Z^α by $z_t(s) = z(t+s), -r \leq s \leq 0$, $g : C([-r,0];Z_q^\alpha) \to C([-r,0];Z^\alpha)$ and $J_k : Z^\alpha \to Z^\alpha$, $F : [0,\tau] \times C(-r,0;Z^\alpha) \to Z$. In the above problem, $A : D(A) \subset Z \to Z$ is a sectorial operator in Z with $-A$ being the generator of a strongly continuous compact semigroup $\{T(t)\}_{t\geq 0}$, and $Z^\alpha = D(A^\alpha)$. The novelty of this work lies in the fact that the evolution equation studied here can contain non-linear terms that involve spatial derivatives and the system is subjected to the influence of impulses, delays and nonlocal conditions, which generalizes many works on the existence of solutions for semilinear evolution equations in Banch spaces. Our framework includes several important partial differential equations such as the Burgers equation and the Benjamin-Bona-Mohany equation with impulses, delays and nonlocal conditions..

AMS Subject Classification: 34K30, 34k35, 35R10; secondary: 93B05, 93C10.

Keywords: Sectorial operator, Fractional power spaces, Semigroups, Semilinear evolution equations, Impulses, Delays,nonlocal conditions, Karakostas fixed point theorem

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1 Introduction

In the context of semilinear evolution equations in functions spaces, difficulties arise when the nonlinear term consists of a composition operator, normally called Nemytskii's operator, which almost never maps a functions space into itself unless the generator function is affine. This work was motivated primarily by the Burgers Equation and the Benjamin-Bona-Mahony (BBM) equation with impulses, delays and nonlocal conditions, which involve a nonlinear term with spatial derivatives, this greatly complicates the problem when one tries to study the approximate controllability of this equation on a fixed interval $[0,\tau]$ because for each control we need to have a corresponding solution defined on the same fixed interval of time. To address this problem we must use the fact that the Laplacian Operator generates an analytic semigroup, which is compact; and the use of fractional powered spaces to formulate the problem as an abstract evolution equations in a suitable Hilbert space. The fundamental problem is that the composition operator associated to the nonlinear term is well defined only from an adequate fractional power spaces to the $L^2(\Omega)$ space. We have spent a lot of time looking for good results that can be applied to the Burgers Equations and the Benjamin-Bona-Equation with impulses, delays and nonlocal conditions, but we did not find any. In fact, the examples others authors present do not involve nonlinear terms with spatial derivatives. Therefore, the novelty of this work lies in the fact that we allow nonlinear terms involving spatial derivative, the use of fractional power space and the Karakostas Fixed Point Theorem [7]. Moreover, our technique can be applied to others equations like the Navier Stokes Equation.
In this regards we study the existence and uniqueness of the solutions for the following semilinear evolution equation with impulses, delay and nonlocal conditions

$$\begin{cases} z' = -Az + F(t, z_t), & z \in Z, t \in (0,\tau], t \neq t_k, \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(z(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \tag{1.1}$$

where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < r < \tau$ and Z is a Banach space. Given the delay $r > 0$, we denote by z_t the function from $[-r, 0]$ to Z^α defined by $z_t(s) = z(t+s), -r \leq s \leq 0$. Using the notation $Z^\alpha := D(A^\alpha)$, $J_k : Z^\alpha \to Z^\alpha$, $g : C([-r,0]; Z_q^\alpha) \to C([-r,0]; Z^\alpha)$ and $F : [0,\tau] \times C(-r, 0; Z^\alpha) \to Z$ are smooth functions, and $A : D(A) \subset Z \to Z$ is a sectorial operator in Z, and $-A$ generates a strongly continuous compact semigroup $\{T(t)\}_{t \geq 0} \subset Z$. This work is an extension of the previous result obtained in [9] for the existence of solutions of system (2.8) without nonlocal conditions.

There are many practical examples of impulsive systems with delays, e.g., chemical reactor systems, financial systems with two state variables; namely, the amount of money in a market and the savings rate of a central bank, and the growth of population diffusing in its habitat modeled by a reaction-diffusion equation. One may easily visualize situations in these examples where abrupt changes such as disasters, meltdowns and instantaneous shocks may occur. Real life problems are modeled by impulsive differential equations, cf. e.g., Lakshmikantham [8] and Samoilenko and Perestyuk [12].
The existence and the asymptotic behavior of a functional differential equations without impulses have been studied by S. M. Rankin III in [11] using fractional power spaces.

The existence of solutions for impulsive abstract partial differential equations with state dependent delay has been studied by E. Hernandez, M. Pierri and G. Goncalves [6] without using fractional power spaces since the nonlinear term does not involve spatial derivative. Likewise, the existence of solutions for semilinear differential evolution equations with impulses and delay has been studied by N. Abada, M. Benchohra and H. Hammouche in [1] and by N. Abada and M. Benchohra in [1] without using fractional power spaces. The existence and stability for partial functional differential equations has been studied by C.C. Travis and G.F. Webb in [14] . On the other hand, the existence and the asymptotic behavior of a functional differential equations without impulses have been studied by S. M. Rankin III in [11] using fractional power spaces. Approximate controllability of semilinear partial neutral functional differential systems has been studied by Xianlong Fu and Kaidong Mei in [3] using also fractional power spaces. In the latter work, since the nonlinear terms involve spatial derivative and hence, spaces of fractional exponents are used. More recently, in [9], the fractional power spaces and the Karakosta's fixed point Theorem is used to prove the existence of solutions for semilinear evolution equations, but without nonlocal conditions. Here, we have it all, impulses, delays and nonlocal conditions.

Our results will be applied to the following impulsive semilinear Burgers equation with delays and nonlocal conditions.

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} & = \nu z_{xx}(t,x) - z(t-r,x)z_x(t-r,x) + f(t,z(t-r)), \\ z(t,0) & = z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) & +h(z(\tau_1+s,x),\dots,z(\tau_q+s,x)) = \phi(s,x), \quad x \in [0,1], \\ z(t_k^+,x) & = z(t_k^-,x) + J_k(z(t_k,x)), \quad x \in \Omega, \quad k = 1,2,3,\dots,p, \end{cases} \tag{1.2}$$

where $\phi \in C([-r,0];H_0^1) = C([-r,0];Z^{1/2})$, with $Z = L_2[0,1]$, $Z^{1/2} = D((-\Delta)^{1/2})$ and the functions f, J_k, h are globally Lipschitz.
The following Burgers Equations with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} & = \nu z_{xx}(t,x) - z(t,x)z_x(t-r,x), \\ z(t,0) & = z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) & = \phi(s,x), \quad s \in [-r,0], \quad x \in [0,1], \end{cases} \tag{1.3}$$

has been studied by Weijiu Liu [10], Yanbin Tang and Ming Wang [14] and Yanbin Tang [13] where the existence and uniqueness of global solutions has been proved.
The Benjamin-Bona-Mahony (BBM) equation with impulses, delay and nonlocal conditions

$$\begin{cases} z_t - az_{xxt} & = bz_{xx} - z(t-r,x)z_x(t-r,x) + f(t,z(t-r,x)), \\ z(t,0) & = z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) & +h(z(\tau_1+s,x),\dots,z(\tau_q+s,x)) = \phi(s,x), \quad x \in [0,1], \\ z(t_k^+,x) & = z(t_k^-,x) + J_k(t_k,z(t_k,x)), \quad x \in \Omega, \end{cases} \tag{1.4}$$

where $a \geq 0$ and $b > 0$ are constants, $\phi \in C([-r,0];H_0^1)$, is also analyzed as an applycation of our results

2 Preliminaries

Throughout this paper, the operator $A : D(A) \subset Z \to Z$ is sectorial and $-A$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $\{T(t)\}_{t\geq 0} \subset Z$, with $0 \in \rho(A)$. Therefore, fractional power operators A^{α}, $0 < \alpha \leq 1$, are well defined. And since A^{α} is a closed operator, its domain $D(A^{\alpha})$ is a Banach space endowed with the graph norm

$$||z||_{\alpha} = ||A^{\alpha}z||, \quad z \in D(A^{\alpha}).$$

This Banach space is denoted by $Z^{\alpha} = D(A^{\alpha})$ and is dense in Z. Moreover, for $0 < \beta < \alpha \leq 1$ the embedding $Z^{\alpha} \hookrightarrow Z^{\beta}$ is compact whenever the resolvent operator of A is compact.
For this semigroup the following properties will be used:
There are constants, $\eta > 0$, $M \geq 1$, $M_{\alpha} \geq 0$ and $C_{1-\alpha}$ such that

$$\begin{aligned}
||T(t)|| &\leq M \quad , \ t \geq 0, && (2.1)\\
||A^{\alpha}T(t)|| &\leq \frac{M_{\alpha}}{t^{\alpha}}e^{-\eta t}, \ \ t > 0, && (2.2)\\
A^{\alpha}T(t)z &= T(t)A^{\alpha}z, \quad \forall z \in Z^{\alpha}, && (2.3)\\
||(T(t) - I)z|| &\leq \frac{C_{1-\alpha}}{\alpha}t^{\alpha}||A^{\alpha}z||, \ \ t > 0, \quad \forall z \in Z^{\alpha}. && (2.4)
\end{aligned}$$

For more properties of sectorial operators and strongly continuous semi-group is good see the book by D. Henry [5] and the book by Jerome A. Goldstein [4].
The functions $J_k \in C(Z^{\alpha}; Z^{\alpha})$ and the function $F : [0,\infty) \times \mathcal{D}_{\alpha} \to Z$ is a smooth function where the set D_{α} denotes the space

$$\mathcal{D}_{\alpha} = \{\phi : [-r, 0] \to Z^{\alpha} : \phi \quad \text{is continuous}\}$$

endowed with the norm

$$||\phi||_{d} = \sup_{-r\leq s\leq 0} ||\phi(s)||_{\alpha}.$$

One natural space to work evolution equations with delay and impulses is the following Banach space: With the notation $J := [-r, \tau]$, and $J' = [-r, \tau] \backslash \{t_1, t_2, \ldots, t_p\}$, define $PC_{\alpha} =$

$$PC(J; Z^{\alpha}) := \{z : J \to Z^{\alpha} : z \in C(J'; Z^{\alpha}) : \forall\, k, z(t_k^{+}), z(t_k^{-}) \text{ exist, and } z(t_k) = z(t_k^{-})\}$$

endowed with the norm

$$||z|| = \sup_{t\in[-r,\tau]} ||z(t)||_{\alpha}.$$

Also, we shall consider the following Banch spaces:

$$Z_q^{\alpha} = Z^{\alpha} \times Z^{\alpha} \times \cdots Z^{\alpha} = \prod_{k=1}^{q} Z^{\alpha},$$

endowed with the norm

$$||y||_q^{\alpha} = \sum_{i=1}^{q} ||y_i||_{\alpha}, \quad y = (y_1, y_2, \ldots, y_q)^T \in Z_q^{\alpha},$$

and the norm in the space $C([-r,0];Z_q^\alpha)$ is given by

$$\|y\|_q = \sup_{t\in[-r,\tau]} \|y(t)\|_q^\alpha = \sup_{t\in[-r,\tau]} \left(\sum_{i=1}^{q} \|y_i(t)\|_\alpha \right), \quad \forall y \in C([-r,0];Z_q^\alpha).$$

For a function $y \in PC([-r,\tau];Z^\alpha)$ and $i = 1,2,\ldots,p$, we define the function $\tilde{y}_i \in C([t_i,t_{i+1}];Z^\alpha)$ such that

$$\tilde{y}_i(t) = \begin{cases} y(t), & \text{for} \quad t \in (t_i,t_{i+1}], \\ y(t_i^+), & \text{for} \quad t = t_i. \end{cases} \tag{2.5}$$

For $W \subset PC([-r,\tau];Z^\alpha)$ and $i = 1,2,\ldots,p$, we define $\tilde{W}_i = \{\tilde{y}_i : y \in W\}$, and following the Arzela-Ascoli classical Theorem one gets a characterization of compactness in $PC([-r,\tau];Z^\alpha)$.

Lemma 2.1. A set $W \subset PC([-r,\tau];Z^\alpha)$ is relatively compact in $PC([-r,\tau];Z^\alpha)$ if, and only if, each set $\tilde{W}_i$, $\quad i = 1,2,\ldots,p$, with $t_0 = 0$ and $t_{p+1} = \tau$, is relatively compact in $C([t_i,t_{i+1}];Z^\alpha)$.

Theorem 2.1. (G. L. Karakostas [7]) Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{B} : D \to Y$ be a continuous operator such that $\mathcal{B}(D)$ is a relatively compact subset of Y, and

$$\mathcal{T} : D \times \overline{\mathcal{B}(D)} \to D \tag{2.6}$$

a continuous operator such that the family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then the operator equation

$$\mathcal{T}(z,\mathcal{B}(z)) = z, \tag{2.7}$$

admits a solution on D.

Lemma 2.2. ([8],[12] generalized Gronwall-Bellman inequality) Let a nonnegative function $z \in PC([-r,\infty);\mathbb{R})$ satisfy, for $t \geq t_0$, the inequality

$$z(t) \leq C + \int_{t_0}^{t} v(s)z(s)ds + \sum_{t_0<t_k<t} \beta_k u(t_k),$$

where $C \geq 0$, $\beta_k \geq 0$, $v(s) > 0$, and t_k's are the discontinuity points of first type for the function z. Then we have,

$$z(t) \leq C \prod_{t_0<t_k<t} (1+\beta_k) e^{\int_{t_0}^{t} v(s)ds}.$$

In the context of semilinear evolution equations in functions spaces, difficulties arise when the nonlinear term consists of a composition operator, normally called Nemytskii's operator, which almost never maps a functions space into itself unless the generator function is affine.

This work was motivated primarily by the Burgers Equation and the Benjamin-Bona-Mahony (BBM) equation with impulses, delays and nonlocal conditions, which involve a nonlinear term with spatial derivatives, this greatly complicates the problem when one tries to study the approximate controllability of this equation on a fixed interval $[0,\tau]$ because for each control we need to have a corresponding solution defined on the same fixed interval of

time. To address this problem we must use the fact that the Laplacian Operator generates an analytic semigroup, which is compact; and the use of fractional powered spaces to formulate the problem as an abstract evolution equations in a suitable Hilbert space. The fundamental problem is that the composition operator associated to the nonlinear term is well defined only from an adequate fractional power spaces to the $L^2(\Omega)$ space. We have spent a lot of time looking for good results that can be applied to the Burgers Equations and the Benjamin-Bona-Equation with impulses, delays and nonlocal conditions, but we did not find any. In fact, the examples others authors present do not involve nonlinear terms with spatial derivatives. Therefore, the novelty of this work lies in the fact that we allow nonlinear terms involving spatial derivative, the use of fractional power space and the Karakostas Fixed Point Theorem [7]. Moreover, our technique can be applied to others equations like the Navier Stokes Equation.
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where $0 < t_1 < t_2 < t_3 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2 < \cdots < \tau_q < r < \tau$ and Z is a Banach space. Given the delay $r > 0$, we denote by z_t the function from $[-r, 0]$ to Z^α defined by $z_t(s) = z(t+s), -r \leq s \leq 0$. Using the notation $Z^\alpha := D(A^\alpha)$, $J_k : Z^\alpha \to Z^\alpha$, $g : C([-r, 0]; Z_q^\alpha) \to C([-r, 0]; Z^\alpha)$ and $F : [0, \tau] \times C(-r, 0; Z^\alpha) \to Z$ are smooth functions, and $A : D(A) \subset Z \to Z$ is a sectorial operator in Z, and $-A$ generates a strongly continuous compact semigroup $\{T(t)\}_{t \geq 0} \subset Z$. This work is an extension of the previous result obtained in [9] for the existence of solutions of system (2.8) without nonlocal conditions.

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neutral functional differential systems has been studied by Xianlong Fu and Kaidong Mei in [3] using also fractional power spaces. In the latter work, since the nonlinear terms involve spatial derivative and hence, spaces of fractional exponents are used. More recently, in [9], the fractional power spaces and the Karakosta's fixed point Theorem is used to prove the existence of solutions for semilinear evolution equations, but without nonlocal conditions. Here, we have it all, impulses, delays and nonlocal conditions.

Our results will be applied to the following impulsive semilinear Burgers equation with delays and nonlocal conditions.

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= \nu z_{xx}(t,x) - z(t-r,x)z_x(t-r,x) + f(t,z(t-r)), \\ z(t,0) &= z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) &+h(z(\tau_1+s,x),\dots,z(\tau_q+s,x)) = \phi(s,x), \quad x \in [0,1], \\ z(t_k^+,x) &= z(t_k^-,x) + J_k(z(t_k,x)), \quad x \in \Omega, \quad k = 1,2,3,\dots,p, \end{cases} \tag{2.9}$$

where $\phi \in C([-r,0];H_0^1) = C([-r,0];Z^{1/2})$, with $Z = L_2[0,1]$, $Z^{1/2} = D((-\Delta)^{1/2})$ and the functions f, J_k, h are globally Lipschitz.
The following Burgers Equations with delay

$$\begin{cases} \dfrac{\partial z(t,x)}{\partial t} &= \nu z_{xx}(t,x) - z(t,x)z_x(t-r,x), \\ z(t,0) &= z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) &= \phi(s,x), \quad s \in [-r,0], \quad x \in [0,1], \end{cases} \tag{2.10}$$

has been studied by Weijiu Liu [10], Yanbin Tang and Ming Wang [14] and Yanbin Tang [13] where the existence and uniqueness of global solutions has been proved.
The Benjamin-Bona-Mahony (BBM) equation with impulses, delay and nonlocal conditions

$$\begin{cases} z_t - az_{xxt} &= bz_{xx} - z(t-r,x)z_x(t-r,x) + f(t,z(t-r,x)), \\ z(t,0) &= z(t,1) = 0, \quad t \in [0,\tau] \\ z(s,x) &+h(z(\tau_1+s,x),\dots,z(\tau_q+s,x)) = \phi(s,x), \quad x \in [0,1], \\ z(t_k^+,x) &= z(t_k^-,x) + J_k(t_k,z(t_k,x)), \quad x \in \Omega, \end{cases} \tag{2.11}$$

where $a \geq 0$ and $b > 0$ are constants, $\phi \in C([-r,0];H_0^1)$, is also analyzed as an applycation of our results

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$$\|z\|_\alpha = \|A^\alpha z\|, \quad z \in D(A^\alpha).$$

This Banach space is denoted by $Z^\alpha = D(A^\alpha)$ and is dense in Z. Moreover, for $0 < \beta < \alpha \leq 1$ the embedding $Z^\alpha \hookrightarrow Z^\beta$ is compact whenever the resolvent operator of A is compact.

For this semigroup the following properties will be used:
There are constants, $\eta > 0$, $M \geq 1$, $M_\alpha \geq 0$ and $C_{1-\alpha}$ such that

$$\begin{aligned} \|T(t)\| &\leq M \quad , \ t \geq 0, && (3.1)\\ \|A^\alpha T(t)\| &\leq \frac{M_\alpha}{t^\alpha} e^{-\eta t}, \ \ t > 0, && (3.2)\\ A^\alpha T(t) z &= T(t) A^\alpha z, \quad \forall z \in Z^\alpha, && (3.3)\\ \|(T(t) - I) z\| &\leq \frac{C_{1-\alpha}}{\alpha} t^\alpha \|A^\alpha z\|, \ \ t > 0, \quad \forall z \in Z^\alpha. && (3.4) \end{aligned}$$

For more properties of sectorial operators and strongly continuous semi-group is good see the book by D. Henry [5] and the book by Jerome A. Goldstein [4].
The functions $J_k \in C(Z^\alpha; Z^\alpha)$ and the function $F : [0, \infty) \times \mathcal{D}_\alpha \to Z$ is a smooth function where the set D_α denotes the space

$$\mathcal{D}_\alpha = \{\phi : [-r, 0] \to Z^\alpha : \phi \quad \text{is continuous}\}$$

endowed with the norm

$$\|\phi\|_d = \sup_{-r \leq s \leq 0} \|\phi(s)\|_\alpha.$$

One natural space to work evolution equations with delay and impulses is the following Banach space: With the notation $J := [-r, \tau]$, and $J' = [-r, \tau] \backslash \{t_1, t_2, \ldots, t_p\}$, define $PC_\alpha =$

$$PC(J; Z^\alpha) := \{z : J \to Z^\alpha : z \in C(J'; Z^\alpha) : \forall\, k, z(t_k^+), z(t_k^-) \text{ exist, and } z(t_k) = z(t_k^-)\}$$

endowed with the norm

$$\|z\| = \sup_{t \in [-r, \tau]} \|z(t)\|_\alpha.$$

Also, we shall consider the following Banch spaces:

$$Z_q^\alpha = Z^\alpha \times Z^\alpha \times \cdots Z^\alpha = \prod_{k=1}^{q} Z^\alpha,$$

endowed with the norm

$$\|y\|_q^\alpha = \sum_{i=1}^{q} \|y_i\|_\alpha, \quad y = (y_1, y_2, \ldots, y_q)^T \in Z_q^\alpha,$$

and the norm in the space $C([-r, 0]; Z_q^\alpha)$ is given by

$$\|y\|_q = \sup_{t \in [-r, \tau]} \|y(t)\|_q^\alpha = \sup_{t \in [-r, \tau]} \left(\sum_{i=1}^{q} \|y_i(t)\|_\alpha \right), \quad \forall y \in C([-r, 0]; Z_q^\alpha).$$

For a function $y \in PC([-r, \tau]; Z^\alpha)$ and $i = 1, 2, \ldots, p$, we define the function $\tilde{y}_i \in C([t_i, t_{i+1}]; Z^\alpha)$ such that

$$\tilde{y}_i(t) = \begin{cases} y(t), & \text{for} \quad t \in (t_i, t_{i+1}], \\ y(t_i^+), & \quad \text{for} \quad t = t_i. \end{cases} \tag{3.5}$$

For $W \subset PC([-r, \tau]; Z^\alpha)$ and $i = 1, 2, \ldots, p$, we define $\tilde{W}_i = \{\tilde{y}_i : y \in W\}$, and following the Arzela-Ascoli classical Theorem one gets a characterization of compactness in $PC([-r, \tau]; Z^\alpha)$.

Lemma 3.1. A set $W \subset PC([-r,\tau];Z^\alpha)$ is relatively compact in $PC([-r,\tau];Z^\alpha)$ if, and only if, each set $\tilde{W}_i$, $\quad i = 1,2,\ldots,p$, with $t_0 = 0$ and $t_{p+1} = \tau$, is relatively compact in $C([t_i,t_{i+1}];Z^\alpha)$.

Theorem 3.1. (G. L. Karakostas [7]) Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{B} : D \to Y$ be a continuous operator such that $\mathcal{B}(D)$ is a relatively compact subset of Y, and

$$\mathcal{T} : D \times \overline{\mathcal{B}(D)} \to D \tag{3.6}$$

a continuous operator such that the family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then the operator equation

$$\mathcal{T}(z,\mathcal{B}(z)) = z, \tag{3.7}$$

admits a solution on D.

Lemma 3.2. ([8],[12] generalized Gronwall-Bellman inequality) Let a nonnegative function $z \in PC([-r,\infty);\mathbb{R})$ satisfy, for $t \geq t_0$, the inequality

$$z(t) \leq C + \int_{t_0}^{t} v(s)z(s)ds + \sum_{t_0<t_k<t} \beta_k u(t_k),$$

where $C \geq 0$, $\beta_k \geq 0$, $v(s) > 0$, and t_k's are the discontinuity points of first type for the function z. Then we have,

$$z(t) \leq C \prod_{t_0<t_k<t} (1+\beta_k)e^{\int_{t_0}^{t} v(s)ds}.$$

4 Mean Theorems

In this section devoted to prove the main results of this paper, which concerns with the existence and uniqueness of mild solutions for problem (2.8).

Definition 4.1. A function $z \in PC_\alpha$ is said to be a mild solution of problem (2.8) if it satisfies the integral equation

$$\begin{aligned} z(t) &= T(t)\{\phi(0) - (g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(0)\} + \int_0^t T(t-s)F(s,z_s)ds \qquad (4.1)\\ &+ \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), \quad t \in [0,\tau],\\ z(t) &= \phi(t) - (g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(t), \quad t \in [-r,0]. \end{aligned}$$

Let us consider the following hypotheses:
(H1) There exist constants $d_k, L_g > 0$, $\quad k = 1,2,\ldots,p$ such that

i

$$L_g q M < M\sum_{k=1}^{p} d_k < \frac{1}{2}, \quad \|J_k(y) - J_k(z)\|_\alpha \leq d_k\|y-z\|_\alpha, \quad y,z \in Z^\alpha,$$

for all $t \in [-r,0]$. And M is as in (3.2) ii We have $g(0) = 0$,

$$\|g(y)(t) - g(v)(t)\|_\alpha \leq L_g \sum_{i=1}^{q} \|y_i(t) - v_i(t)\|_\alpha, \quad \forall y, v \in C([-r,0]; Z_q^\alpha),$$

and for $s_1, s_2 \in (0, \tau]$ with $s_1 < s_2$ the following inequality

$$\|g(y)(s_2) - g(y)(s_2)\|_\alpha \leq \theta(|s_2 - s_1|), \quad \forall y \in C([-r,0]; Z_q^\alpha),$$

where $\theta : [0,\tau] \to \mathbb{R}_+$ is a continuous function such that $\theta(0) = 0$.

iii For all $t \in [-r, 0]$, the mapping $y \in C([-r,0]; Z_q^\alpha) \to (g(y))(t) \in Z^\alpha$ is completely continuous.

(H2) The function $F : [0,\tau] \times \mathcal{D}_\alpha \to Z$ satisfies the following conditions.

$$\|F(t,\phi_1) - F(t,\phi_2)\| \leq \mathcal{K}(\|\phi_1\|_d, \|\phi_2\|_d)\|\phi_1 - \phi_2\|_d, \quad \phi_1, \phi_2 \in \mathcal{D}_\alpha.$$

$$\|F(t,\phi)\| \leq \Psi(\|\phi\|), \quad \phi \in \mathcal{D}_\alpha,$$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and nondecreasing functions of their arguments.
(H3) Assume the following relation holds for ρ, τ:

$$\left(M L_g q + M \sum_{k=1}^{p} d_k \right) (\|\tilde{\phi}\|_d + \rho) + \frac{\tau^{1-\alpha}}{1-\alpha} M_\alpha \Psi(\|\tilde{\phi}\|_d + \rho) \leq \rho,$$

where the function $\tilde{\phi}$ is define as follows

$$\tilde{\phi}(t) = \begin{cases} T(t)\phi(0), & t \in [0,\tau] \\ \phi(t), & t \in [-r,0] \end{cases} \tag{4.2}$$

(H4) Assume the following relation holds for ρ, τ:

$$\frac{\tau^{1-\alpha}}{1-\alpha} M_\alpha \mathcal{K}(\|\tilde{\phi}\|_d + \rho, \|\tilde{\phi}\|_d + \rho) + M \sum_{k=1}^{p} d_k < 1.$$

Theorem 4.1. Suppose that (H1)-(H3) hold. Then problem (2.8) has a least one mild solution on $[-r, \tau]$.

Proof We shall transform problem (2.8) into a fixed point problem. Define the following two operators:

$$\mathcal{T} : PC([-r,\tau]; Z^\alpha) \times PC([-r,\tau]; Z^\alpha) \to PC([-r,\tau]; Z^\alpha) \text{ and}$$
$$\mathcal{B} : PC([-r,\tau]; Z^\alpha) \to PC([-r,\tau]; Z^\alpha),$$

defined by

$$\mathcal{T}(z,y)(t) = \begin{cases} y(t) + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k)), & t \in [0,\tau] \\ \phi(t) - (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(t), & t \in [-r,0] \end{cases}$$

$$\mathcal{B}(y)(t) = \begin{cases} T(t)\{\phi(0) - (g(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))(0)\} + \int_0^t T(t-s)F(s,y_s)ds, & t \in [0,\tau] \\ \phi(t), & t \in [-r,0] \end{cases}$$

The problem of finding the solution of problem (2.8) is reduced to the problem of finding solutions of the operator equation $\mathcal{T}(z,\mathcal{B}(z)) = z$. First we shall prove that the operator $\mathcal{B}$ is compact. After that, we shall prove that family $\{\mathcal{T}(\cdot,y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive, where D is the closed convex set given by (4.4). So, by applying Theorem 3.1 we get the result. The proof of this Theorem will be given by steps:

Step 1: $\mathcal{B}$ is continuous.

In fact, using the hypothesis H1), if we consider $z, y \in PC([-r,\tau];Z^\alpha)$, we have following estimate

$$\begin{aligned} \|\mathcal{B}(z)(t) - \mathcal{B}(y)(t)\|_\alpha &\leq L_g qM\|z-y\| + \int_0^t \|A^\alpha T(t-s)(F(s,z_s) - F(s,y_s))\|ds \\ &\leq L_g qM\|z-y\| + \int_0^t \frac{M_\alpha}{(t-s)^\alpha}\mathcal{K}(\|z_s\|_d, \|y_s\|_d)\|z_s - y_s\|_d ds \\ &\leq L_g qM\|z-y\| + M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha}\mathcal{K}(\|z\|,\|y\|)\|z-y\|. \end{aligned}$$

Therefore,

$$\|\mathcal{B}(z) - \mathcal{B}(y)\| \leq \left(L_g qM + M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha}\mathcal{K}(\|z\|,\|y\|)\right)\|z-y\|.$$

So, $\mathcal{B}$ is continuous. Moreover, $\mathcal{B}$ is locally Lipschitz.

Step 2: $\mathcal{B}$ maps bounded sets into bounded sets in $PC([-r,\tau];Z^\alpha)$.

$\mathcal{B}$ maps bounded sets of PC_α into bounded set of PC_α. It is enough to show that for any $R > 0$ there exists $l > 0$ such that for each $y \in B_R = \{z \in PC_\alpha : \|z\| \leq R\}$ we have that $\|\mathcal{B}y\| \leq l$. In fact, choose $y \in B_R$, then the following estimate holds.

$$\begin{aligned} \|\mathcal{B}(y)(t)\|_\alpha &\leq \|A^\alpha T(t)\{\phi(0) - (g(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))(0)\}\| + \int_0^t \|A^\alpha F(s,y_s))\|ds \\ &\leq M\{\|\phi(0)\|_\alpha + L_g q\|y\|\} + M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha}\Psi(\|y\|) \\ &\leq M\{\|\phi(0)\|_\alpha + L_g qR\} + M_\alpha \frac{\tau^{1-\alpha}}{1-\alpha}\Psi(R) = l. \end{aligned}$$

Step 3: $\mathcal{B}$ maps bounded sets into equicontinuous sets of $PC([-r,\tau];Z^\alpha)$.

In fact, consider B_R as in the foregoing Claim. Then we shall prove that the family of functions $\mathcal{B}(B_R)$ is equicontinuous on the interval $[-r,\tau]$. Clearly, it is sufficient to prove this on $(0,\tau]$. Let $0 < s_1 < s_2 < \tau$ and consider the following estimate for $y \in B_q$

$$\|A^\alpha[T(s_2)\phi(0) - T(s_1)\phi(0)]\| \leq \|T(s_2) - T(s_1)\|\|\phi(0)\|_\alpha,$$

and the following bound:

$$\begin{aligned}
& \|T(s_2)(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(s_2) - T(s_1)(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(s_1)\|_\alpha \\
\leq \; & \|T(s_2) - T(s_1)\|\|(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(s_2)\|_\alpha \\
+ \; & \|T(s_1)\|\|(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(s_2) - (g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(s_1)\|_\alpha \\
\leq \; & \|T(s_2) - T(s_1)\|L_g qR + M\theta(s_2 - s_1).
\end{aligned}$$

Now, using these two estimates we get the following bound:

$$
\begin{aligned}
\|\mathcal{B}(y)(s_2)-\mathcal{B}(y)(s_1)\|_\alpha &\leq \|T(s_2)-T(s_1)\|[L_gqR+\|\phi(0)\|_\alpha]+M\theta(s_2-s_1)\\
&+ \int_0^{s_1-\epsilon}\|(A^\alpha T(s_2-s)-A^\alpha T(s_1-s))F(s,y_s)\|ds\\
&+ \int_{s_1-\epsilon}^{s_1}\|(A^\alpha T(s_2-s)-A^\alpha T(s_1-s))F(s,y_s)\|ds\\
&+ \int_{s_1}^{s_2}\|A^\alpha T(s_2-s)F(s,y_s)\|ds\\
&\leq \|T(s_2)-T(s_1)\|[L_gqR+\|\phi(0)\|_\alpha]+M\theta(s_2-s_1)\\
&+ \|T(s_2-s_1+\epsilon)-T(\epsilon)\|\int_0^{s_1-\epsilon}\|A^\alpha T(s_1-s-\epsilon)F(s,y_s)\|ds\\
&+ \frac{M_\alpha\Psi(\|y\|)}{1-\alpha}\{(s_2-s_1+\epsilon)^{1-\alpha}-(s_2-s_1)^{1-\alpha}+(\epsilon)^{1-\alpha}\}\\
&+ \frac{M_\alpha\Psi(\|y\|)}{1-\alpha}(s_2-s_1)^{1-\alpha}\\
&\leq \|T(s_2)-T(s_1)\|[L_gqR+\|\phi(0)\|_\alpha]+M\theta(s_2-s_1)\\
&+ \|T(s_2-s_1+\epsilon)-T(\epsilon)\|\frac{M_\alpha\Psi(R)}{1-\alpha}(s_1-\epsilon)^{1-\alpha}\\
&+ \frac{M_\alpha\Psi(R)}{1-\alpha}\{(s_2-s_1+\epsilon)^{1-\alpha}-(s_2-s_1)^{1-\alpha}+\epsilon^{1-\alpha}\}\\
&+ \frac{M_\alpha\Psi(R)}{1-\alpha}(s_2-s_1)^{1-\alpha}.
\end{aligned}
$$

Since $T(t)$ is a compact operator for $t>0$, then $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup, which implies that $\|\mathcal{B}(y)(s_2)-\mathcal{B}(y)(s_1)\|_\alpha$ goes to zero uniformly on y as $s_2-s_1\to 0$, and therefore $\mathcal{B}(B_q)$ is equicontinuous.

Step4: The set $W=\{\mathcal{B}(y): y\in B_R\}$ is relatively compact in $PC([-r,\tau];Z^\alpha)$. To prove that, it is enough to prove that the corresponding set $\tilde{W}_i$ is relatively compact in $C([t_i,t_{i+1}];Z^\alpha)$ for $i=0,1,2,\ldots,p$ with $t_0=0$ and $t_{p+1}=\tau$. According with Arzela-Ascoli Theorem in infinite dimensional Banach spaces it is sufficient to prove that $\tilde{W}_i(t)=\{\tilde{\mathcal{B}(y)}_i(t): y\in B_q\}$ is relatively compact in Z^α for each $t\in[t_i,t_{i+1}]$.
In fact, for the case $t\in[-r,0]$ we have that

$$W(t)=\{\phi(t)-(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(t): y\in B_R\},$$

which is relatively compact from the hypothesis H1-iii).

Now, suppose $t\in[t_i,t_{i+1}]$, with $t>0$, then

$$\tilde{W}_i(t)=T(t)\phi(0)+\tilde{V}_i(t),$$

where

$$\tilde{V}_i(t)=\{v_i(t)=-T(t)(g(y_{\tau_1},y_{\tau_2},\ldots,y_{\tau_q}))(0)+\int_0^t T(t-s)F(s,\tilde{y}_{i,s})ds: y\in B_R\}.$$

It is sufficient to prove that $\tilde{V}_i(t)$ is relatively compact in Z^α. In fact, consider ϵ, with $0 < \epsilon < t$, and the set

$$\tilde{V}_{i,\epsilon}(t) = \{v_{i,\epsilon}(t) = -T(t)(g(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))(0) + \int_0^{t-\epsilon} T(t-s)F(s, \tilde{y}_{i,s})ds : y \in B_R\}$$

$$= \{v_{i,\epsilon}(t) = -T(t)(g(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))(0) + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)F(s, \tilde{y}_{i,s})ds : y \in B_R\}.$$

From H1-iii) and the compactness of $T(\epsilon)$ for $\epsilon > 0$, we get that $\tilde{V}_{i,\epsilon}(t)$ is relatively compact in Z^α for any ϵ, with $0 < \epsilon < t$.
Since

$$\begin{aligned} \|v_i(t) - v_{i,\epsilon}(t)\|_\alpha &\le \int_{t-\epsilon}^t \|A^\alpha T(t-s)F(s, \tilde{y}_{is}))\|ds \\ &\le \int_{t-\epsilon}^t \frac{M_\alpha}{(t-s)^\alpha}\|F(s, \tilde{y}_{is})\|ds \\ &\le \frac{M_\alpha \Psi(R)}{1-\alpha}\epsilon^{1-\alpha}. \end{aligned}$$

Hence, we have a sequence of relatively compact sets arbitrarily close to $v_i(t)$, which implies that $v_i(t)$ is relatively compact in Z^α.

Step 5: The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive and the conditions of Theorem 3.1 are satisfied for the following closed and convex set

$$D = D(\rho, \tau, \phi) = \{y \in PC([-r, \tau]; Z^\alpha) : \|y - \tilde{\phi}\| \le \rho\}, \tag{4.3}$$

where function $\tilde{\phi}$ is define as follows

$$\tilde{\phi}(t) = \begin{cases} T(t)\phi(0), & t \in [0, \tau] \\ \phi(t), & t \in [-r, 0] \end{cases}$$

In fact, for $z, x \in PC([-r, \tau]; Z^\alpha)$ and $t \in [0, \tau]$ we have the following estimate

$$\begin{aligned} \|\mathcal{T}(z, \mathcal{B}(y))(t) - \mathcal{T}(x, \mathcal{B}(y))(t)\|_\alpha &\le \sum_{0<t_k<t} \|A^\alpha T(t-t_k)(J_k(z(t_k)) - J_k(x(t_k)))\| \\ &\le M\sum_{k=1}^p \|A^\alpha(J_k(z(t_k)) - J_k(x(t_k)))\| \\ &\le M\sum_{k=1}^p d_k\|z(t_k)) - x(t_k)))\|_\alpha \\ &\le M\sum_{k=1}^p d_k\|(z - x\|. \end{aligned}$$

On the other hand, for $t \in [-r, 0]$ we have

$$\begin{aligned} \|\mathcal{T}(z, \mathcal{B}(y))(t) - \mathcal{T}(x, \mathcal{B}(y))(t)\|_\alpha &\le \|(g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}))(t) - (g(x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_q}))(t)\|_\alpha \\ &\le L_g q\|(z - x\|. \end{aligned}$$

Since $L_g qM \leq M\sum_{k=1}^{p} d_k < \frac{1}{2}$, then

$$\|\mathcal{T}(z,\mathcal{B}(y)) - \mathcal{T}(x,\mathcal{B}(y))\| \leq (M\sum_{k=1}^{p} d_k)\|(z-x\|,$$

is a contraction independently of $y \in \overline{\mathcal{B}(D)}$.
Finally, we shall prove that

$$\mathcal{T}(\cdot,\mathcal{B})D(\rho,\tau,\phi) \subset D(\rho,\tau,\phi).$$

In fact, let us consider $z \in D(\rho,\tau,\phi)$ and $t \in [0,\tau]$. Then $\mathcal{T}(z,\mathcal{B}(z))(t) =$
$T(t)[\phi(0) - (g(z_{\tau_1}, z_{\tau_2},\ldots,z_{\tau_q}))(0)] + \int_0^t T(t-s)F(s,z_s)ds + \sum_{0<t_k<t} T(t-t_k)J_k(z(t_k))$.
On the other hand, for $t \in [-r,0]$, we have
$\mathcal{T}(z,\mathcal{B}(z))(t) = \phi(t) - (g(z_{\tau_1}, z_{\tau_2},\ldots,z_{\tau_q}))(t)$.
Therefore, for $t \in [0,\tau]$ we have

$$\begin{aligned}
\|\mathcal{T}(z,\mathcal{B}(z))(t) - \tilde{\phi}(t)\|_\alpha &\leq M\|(g(z_{\tau_1}, z_{\tau_2},\ldots,z_{\tau_q}))(0)\|_\alpha + \\
&+ \int_0^t \|A^\alpha T(t-s)F(s,z_s)\|ds + \sum_{0<t_k<t}\|A^\alpha T(t-t_k)J_k(z(t_k))\| \\
&\leq ML_g q\|z\| + \int_0^t \frac{M_\alpha}{(t-s)^\alpha}\|F(s,z_s)\| + M\sum_{k=1}^{p}\|A^\alpha(J_k(z(t_k)) - J_k(0))\|ds \\
&\leq ML_g q(\|\phi\|_d + \rho) + \frac{\tau^{1-\alpha}}{1-\alpha}M_\alpha\Psi(\|z\|) + M\sum_{k=1}^{p} d_k\|z(t_k))\|_\alpha \\
&\leq ML_g q(\|\phi\|_d + \rho) + \frac{\tau^{1-\alpha}}{1-\alpha}M_\alpha\Psi(\|\phi\|_d + \rho) + (M\sum_{k=1}^{p} d_k)(\|\phi\|_d + \rho) \\
&= \left(ML_g q + M\sum_{k=1}^{p} d_k\right)(\|\tilde{\phi}\|_d + \rho) + \frac{\tau^{1-\alpha}}{1-\alpha}M_\alpha\Psi(\|\tilde{\phi}\|_d + \rho) \leq \rho.
\end{aligned}$$

On the other hand, for $t \in [-r,0]$ we have

$$\begin{aligned}
\|\mathcal{T}(z,\mathcal{B}(z))(t) - \tilde{\phi}(t)\|_\alpha &= \|(g(z_{\tau_1}, z_{\tau_2},\ldots,z_{\tau_q}))(t)\|_\alpha \\
&\leq L_g q\|z\| \leq ML_g q\|z\| \leq ML_g q(\|\phi\|_d + \rho) < \rho.
\end{aligned}$$

From the hypothesis (H3) we get that $\mathcal{T}(\cdot,\mathcal{B})D(\rho,\tau,\phi) \subset D(\rho,\tau,\phi)$. Hence, as a consequence of Theorem 3.1 it follows that the equation $\mathcal{T}(z,\mathcal{B}(z)) = z$, has a solution, which is a mild solution of problem (2.8).

□

Theorem 4.2. In addition to the conditions of Theorem 4.1, suppose that (H4) holds. Then problem (2.8) has only one mild solution on $[-r,\tau]$.

Proof . Let z_1 and z_2 be two solutions of problem (2.8). Then consider the following estimate:

$$\begin{aligned}
\|z_1(t)-z_2(t)\|_\alpha &\le \|(g(z_{1,\tau_1},z_{1,\tau_2},\dots,z_{1,\tau_q}))(0)-(g(z_{2,\tau_1},z_{2,\tau_2},\dots,z_{2,\tau_q}))(0)\|_\alpha \\
& \int_0^t \|A^\alpha T(t-s)(F(s,z_{1s})-F(s,z_{2s}))\|ds \\
&+ \sum_{0<t_k<t} \|A^\alpha T(t-t_k)(J_k(z_1(t_k))-J_k(z_2(t_k)))\| \\
&\le \left(ML_g q+\frac{M_\alpha \tau^{1-\alpha}}{1-\alpha}\mathcal{K}(\|\tilde{\phi}\|_d+\rho),\|\tilde{\phi}\|_d+\rho)+M\sum_{k=1}^{p} d_k\right)\|(z_1-z_2\|.
\end{aligned}$$

From the hypotheses (H4) we know that

$$ML_g q+\frac{M_\alpha \tau^{1-\alpha}}{1-\alpha}\mathcal{K}(\|\phi\|_d+\rho),\|\phi\|_d+\rho)+M\sum_{k=1}^{p} d_k<1,$$

which implies that $z_1=z_2$.

□

Now, we shall consider the following subset $\tilde{D}_\alpha$ of Z^α:

$$\tilde{D}_\alpha=\{y\in Z^\alpha : \|y\|_\alpha\le R\}, \quad \text{with} \quad R=\|\tilde{\phi}\|+\rho. \tag{4.4}$$

Therefore, for all $z\in D$ we have $z(t)\in\tilde{D}_\alpha$ for $-r\le t\le\tau$.

THEOREM 4.3. Suppose that the conditions of Theorem 4.2 hold. If z is a solution of problem (2.8) on $[-r,s_1)$ and s_1 is maximal, so there is no solution of (2.8) on $[-r,s_2)$ if $s_2>s_1$, then either $s_1=+\infty$ or else there exists a sequence $\tau_n\to s_1$ as $n\to\infty$ such that $z(\tau_n)\to\partial\tilde{D}_\alpha$.

Proof . Suppose that $s_1<\infty$ and $z(t)$ doesn't enter in a neighborhood N of $\tilde{D}_\alpha$ for $0<s_2\le t<s_1$. Let us take $N=\tilde{D}_\alpha\backslash B$ where B is a closed subset of $\tilde{D}_\alpha$, and $z(t)\in B$ for $0<s_2\le t<s_1$. We shall prove the existence of $z_1\in B$ such that $z(t)\to z_1$ in Z^α as $t\to s_1^-$, which implies the solution may be extended beyond time s_1 using Theorem 4.2, contradicting the maximality of s_1.
In fact, if we consider $0<t_p<s_2\le l<t<s_1$, then for $\epsilon>0$ small enough we have that

$$
\begin{aligned}
\|z(t)-z(l)\|_\alpha &\leq \|T(t)-T(l)\|\|\phi(0)\|_\alpha + \|T(t)-T(l)\|\|g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\|_\alpha \\
&+ \int_0^{l-\epsilon} \|(A^\alpha T(t-s) - A^\alpha T(l-s))F(s,z_s)\|ds \\
&+ \int_{l-\epsilon}^{l} \|(A^\alpha T(t-s) - A^\alpha T(l-s))F(s,z_s)\|ds \\
&+ \int_l^t \|A^\alpha T(t-s)F(s,z_s)\|ds \\
&+ \|T(t-l+\epsilon) - T(\epsilon)\| \sum_{k=1}^{p} \|T(l-t_k-\epsilon)A^\alpha J_k(z(t_k)\| \\
&\leq \|T(t)-T(l)\|[\|\phi(0)\|_\alpha + L_g q]R \\
&+ \|T(t-l+\epsilon) - T(\epsilon)\| \int_0^{l-\epsilon} \|A^\alpha T(l-s-\epsilon)F(s,z_s)\|ds \\
&+ \frac{M_\alpha \Psi(R)}{1-\alpha}\{(t-l+\epsilon)^{1-\alpha} - (t-l)^{1-\alpha} + (\epsilon)^{1-\alpha}\} \\
&+ \frac{M_\alpha \Psi(R)}{1-\alpha}(t-l)^{1-\alpha} + M\|T(t-l+\epsilon) - T(\epsilon)\| \sum_{k=1}^{p} \|J_k(z(t_k)\|_\alpha
\end{aligned}
$$

Since $T(t)$ is a compact operator for $t > 0$, then $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semi-group, which implies that $\|z(t)-z(l)\|_\alpha$ goes to zero as $l < t \to s_1$. Therefore, $\lim_{t\to s_1} z(t) = z_1$ exists in Z^α, and since B is closed, z_1 belongs to B. This completes the proof.

$\square$

Corollary 4.1. In the conditions of Theorem 4.2, if the second part of hypothesis (H2) is changed to

$$\|F(t,\phi)\| \leq h(t)(1 + \|\phi(0)\|_\alpha), \quad \phi \in \mathcal{D}_\alpha,$$

where $h(\cdot)$ is a continuous function on $[-r,\infty)$, then a unique solution of problem (2.8) exists on $[-r,\infty)$.

Proof

$$
\begin{aligned}
&\|z(t)\|_\alpha \le M[\|\phi(0)\|_\alpha + \|g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)\|_\alpha] + \int_0^t \|A^\alpha T(t-s)F(s, z_s)\|ds \\
+ & \sum_{0<t_k<t} \|A^\alpha T(t-t_k)J_k(z(t_k))\| \le M[\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] \\
+ & \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}\|F(s, z_s)\| + M\sum_{k=1}^{p} \|A^\alpha(J_k(z(t_k)) - J_k(0))\| \\
\le & M[\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] + \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}(1 + \|z(s)\|_\alpha) \\
+ & M\sum_{k=1}^{p} d_k\|z(t_k))\|_\alpha \le M[\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] \\
+ & \frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}} M_\alpha + \int_0^t \frac{M_\alpha}{(t-s)^\alpha} e^{-\eta(t-s)}\|z(s)\|_\alpha ds \\
+ & M\sum_{k=1}^{p} d_k\|z(t_k))\|_\alpha.
\end{aligned}
$$

Then applying Lemma 3.2 we get the following estimate

$$
\|z(t)\|_\alpha \le \left(M[\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] + \frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}} M_\alpha \right) \prod_{t_0<t_k<t} (1 + Md_k) e^{\frac{\Gamma(1-\alpha)}{\eta^{1-\alpha}} M_\alpha},
$$

$\tilde{z} = (z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})^T$. This implies that $\|z(t)\|_\alpha$ remains bounded as $t \to s_1$ and applying Theorem 4.3 we get the result. □

THEOREM 4.4. Under the conditions of Theorem 4.1, if z is a solution of problem (2.8) on $[-r, \infty)$ with $\|z(t)\|_\alpha$ bounded as $t \to \infty$, then $\{z(t, \phi)\}_{t>0}$ is a compact set in Z^α

Proof . Observe that for $0 < \alpha < \beta < 1$, we have the following estimate for $t > t_p$

$$
\begin{aligned}
&\|A^\beta z(t)\| \le \|A^{\beta-\alpha} T(t) A^\alpha [\phi(0) + g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0)]\| \\
+ & \int_0^t \|A^\beta T(t-s)F(s, z_s)\|ds + \sum_{0<t_k<t} \|A^{\beta-\alpha} T(t-t_k) A^\alpha J_k(z(t_k))\| \\
\le & \frac{M_\beta}{t^{\beta-\alpha}} [\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] + \int_0^t \frac{M_\beta}{(t-s)^\beta}\|F(s, z_s)\|ds \\
+ & \sum_{k=1}^{p} \frac{M_\beta}{(t-t_k)^{\beta-\alpha}} \|A^\alpha(J_k(z(t_k)) - J_k(0))\| \\
\le & \frac{M_\beta}{t^{\beta-\alpha}} [\|\phi(0)\|_\alpha + L_g\|\tilde{z}(0)\|_q^\alpha] \\
+ & \frac{t^{1-\beta}}{1-\beta} M_\beta \Psi(\|z\|) + \frac{M_\beta}{(t-t_p)^{\beta-\alpha}} \sum_{k=1}^{p} d_k\|z(t_k))\|_\alpha.
\end{aligned}
$$

Which implies that $\{A^\beta z(t) : t \in [-r, \infty)\}$ is bounded in Z. On the other hand, we know that $A^{-\beta} : Z \to Z^\alpha$ is a compact operator since the imbedding $Z^\beta \hookrightarrow Z^\alpha$ is compact.Therefore, $\{z(t) : t \in [-r, \infty)\}$ is compact in Z^α □

5 Application to The Burgers Equation

In this section we shall apply our previous results to the Burgers equation with impulses, delay and nonlocal conditions (2.9). To this end, we make the following hypotheses:
The nonlinear functions $f, J_k : \mathbb{R} \to \mathbb{R}$ are smooth enough and $h : \mathbb{R}^q \to \mathbb{R}$ is a globally Lipschitz function, with $h(0) = 0$, and there exist constants $L > 0$, L_k such that.

$$|f(t,z) - f(t,w)| \le L|z-w|, \quad t \in [0,\tau], z, w \in \mathbb{R}. \tag{5.1}$$

$$|J_k(z) - J_k(w)| \le L_k|z-w|, \quad t \in [0,\tau], z, w \in \mathbb{R}, k = 1,2,\dots,p. \tag{5.2}$$

$$|f(t,z,u)| \le a(t)|z| + b(t), \quad t \in [0,\tau] \quad \text{and} \quad z, u \in \mathbb{R}, \quad a(\cdot), b(\cdot) \in L_\infty[0,\tau]. \tag{5.3}$$

We shall denote $\Omega = [0,1]$ and by C the space of continuous functions:

$$C = \{\phi : [-r,0] \to H_0^1(\Omega) = Z^{1/2} : \phi \quad \text{is continuous}\},$$

endowed with the norm

$$\|\phi\| = \sup_{-r\le s\le 0} \|\phi(s)\|_{Z^{1/2}}, \ \text{and} \ \phi(s)(x) = \phi(s,x), x \in \Omega = [0,1].$$

Now, we choose a Hilbert space where system (2.9) can be written as an abstract differential equation(See [2]); to this end, we consider the following notations:
Let us consider the Hilbert space $Z = L_2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \longrightarrow \infty$ the eigenvalues of operator $A\phi = -\nu\phi_{xx}$. Then we have the following well known properties (i) There exists a complete orthonormal set $\{\phi_j\}$ of eigenvectors of A. (ii) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j < \xi, \phi_j > \phi_j = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{5.4}$$

where $< \cdot, \cdot >$ is the inner product in Z and

$$E_n z = < z, \phi_j > \phi_j. \tag{5.5}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z, \ z \in Z$. (iii) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z \ \text{and} \ \|T(t)\| \le e^{-\lambda_1 t}, \quad t \ge 0. \tag{5.6}$$

Consequently, systems (2.9) can be written as an abstract functional differential equations with memory in Z:

$$\begin{cases} z' = -Az + f^e(t, z_t(-r)), & z \in Z \quad t \ge 0, \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r,0], \\ z(t_k^+) = z(t_k^-) + J_k^e(z(t_k)), & k = 1,2,3,\dots,p, \end{cases} \tag{5.7}$$

where $z_t \in C([-r,0]; Z^{1/2})$ and is defined by $z_t(s) = z(t+s), -r \le s \le 0$ and the functions $J_k^e : Z \to Z$, $f^e : [0,\tau] \times C \to Z$ and $g : C([-r,0]; Z_q^{1/2}) \to C([-r,0]; Z^{1/2})$ are defined for $k = 1,2,\dots,p$ by

$$J_k^e(z)(x) = J_k(z(x)), \quad \forall x \in \Omega,$$

$$f^e(t,\phi)(x) = \phi(-r,x)\phi_x(-r,x) + f(t,\phi(-r,x)) \ \ \forall x \in \Omega,$$

and

$$g(z)(s)(x) = h(z_1(s,x), z_2(s,x), \ldots, z_q(s,x)), \quad \forall x \in \Omega, -r \leq s \leq 0.$$

PROPOSITION 5.1. The function f^e is locally Lipschitz in the second variable. Moreover, the following estimate holds:

$$\|f^e(t,\phi_1) - f^e(t,\phi_2)\| \leq \{\|\phi_1 - \phi_2\|_C + L\}\|\phi_1 - \phi_2\|_C. \tag{5.8}$$

$$\begin{aligned} \|f^e(t,\phi)\| &\leq \|\phi(-r)\|^2 + 4\|a\|_{L_\infty}\|\phi(-r)\| + 4\|b\|_{L_\infty}\sqrt{\mu(\Omega)} \\ &\leq \|\phi\|_C^2 + 4\|a\|_{L_\infty}\|\phi\|_C + 4\|b\|_{L_\infty}\sqrt{\mu(\Omega)} \end{aligned} \tag{5.9}$$

Proof Clearly that the following estimate holds:

$$\begin{aligned} &\|f^e(t,\phi) - f^e(t,\psi)\|_Z \leq \\ &\|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z + L\|\phi(-r) - \psi(-r)\|_Z \end{aligned} \tag{5.10}$$

On the other hand

$$\begin{aligned} &\|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z \leq \\ &\|\phi(-r)[\phi_x(-r) - \psi_x(-r)]\|_Z + \|[\phi(-r) - \psi(-r)]\psi_x(-r)\|_Z \leq \\ &\|\phi(-r)\|_{L_\infty}\|[\phi_x(-r) - \psi_x(-r)]\|_Z + \|[\phi(-r) - \psi(-r)]\|_{L_\infty}\|\psi_x(-r)\|_Z \end{aligned}$$

Then, for all $z \in Z^1 = H_0^1(\Omega)$, by the Sobolev Theorem and Poincare Inequality we have that:

$$\|z\|_{L_\infty}^2 \leq 2\|z\|_Z\|z_x\|_Z \leq \|z\|_Z^2 + \|z_x\|_Z^2 = \|z\|_{Z^{1/2}}^2$$

and

$$\begin{aligned} &\|\phi(-r)\phi_x(-r) - \psi(-r)\psi_x(-r)\|_Z \leq \\ &\|\phi(-r)\|_{Z^{1/2}}\|[\phi(-r) - \psi(-r)]\|_{Z^{1/2}} + \|[\phi(-r) - \psi(-r)]\|_{Z^1}\|\psi(-r)\|_{Z^{1/2}}. \end{aligned}$$

Using this estimate and (5.10) we get the result.

□

In this case the functions $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ are given by

$$\mathcal{K}(v,w) = v + w + L \ \text{ and } \ \Psi(v) = v^2 + 4\|a\|_\infty v + 4\|b\|_\infty,$$

since $\sqrt{\mu(\Omega)} = 1$. Therefore, we have the following result for the impulsive Burgers Equation with delay.

THEOREM 5.1. For d_k small enough there exist $\tau > 0$ such that the system (2.9) has only one mild solution define on $[-r,\tau]$.

6 Application to The Benjamin-Bona-Mahony (BBM) equation

In this section we shall apply our previous results to The Benjamin-Bona-Mahony (BBM) equation with impulses, delay and nonlocal conditions (2.11). To this end, we shall consider the hypotheses corresponding to the Burgers equation:

Consequently, systems (2.11) can be written as an abstract functional differential equations with impulses and nonlocal conditions in Z:

$$\begin{cases} z' + aAz' + bAz = f^e(t, z_t(-r)), & z \in Z \quad t > 0, t \neq t_k, \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k^e(t_k, z(t_k)), & k = 1, 2, 3, \dots, p, \end{cases} \tag{6.1}$$

where the functions $J_k^e : [0,\tau] \times Z \to Z$, $f^e : [0,\tau] \times C \to Z$ and $g : C([-r,0]; Z_q^{1/2}) \to C([-r,0]; Z^{1/2})$ are defined for $k = 1, 2, \dots, p$ by

$$J_k^e(t,z)(x) = J_k(t, z(x)), \quad \forall x \in \Omega,$$

$$f^e(t,\phi)(x) = \phi(-r,x)\phi_x(-r,x) + f(t, \phi(-r,x)) \;\; \forall x \in \Omega.$$

and

$$g(z)(s)(x) = h(z_1(s,x), z_2(s,x), \dots, z_q(s,x)), \quad \forall x \in \Omega, -r \le s \le 0.$$

Since $(I + aA) = a(A - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(A)(\rho(A)$ is the resolvent set of A), then the operator:

$$I + aA : D(A) \to Z \tag{6.2}$$

is invertible with bounded inverse

$$(I + aA)^{-1} : Z \to D(A). \tag{6.3}$$

Therefore, the systems (6.1) can be written as follows, for $z \in Z, t \in (0, \tau]$

$$\begin{cases} z' + b(I + aA)^{-1}Az = +(I + aA)^{-1} f^e(t, z(t-r)), & t > 0, t \neq t_k \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \tag{6.4}$$

$(I + aA)$ and $(I + aA)^{-1}$ can be written as follows:

$$(I + aA)z = \sum_{j=1}^{\infty} (1 + a\lambda_j) E_j z \tag{6.5}$$

$$(I + aA)^{-1} z = \sum_{j=1}^{\infty} \frac{1}{1 + a\lambda_j} E_j z. \tag{6.6}$$

$B = (I + aA)^{-1}$ and $F(t, \phi, u) = (I + aA)^{-1} f^e(t, \phi)$, systems (6.4) can be written:

$$\begin{cases} z' + bBAz = +F(t, z_t(-r)), & t > 0, t \neq t_k \\ z(s) + (g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k)), & k = 1, 2, 3, \dots, p. \end{cases} \tag{6.7}$$

and the functions F defined above satisfy:

$$\|F(t,\phi_1)-F(t,\phi_2)\| \leq K\{\|\phi_1-\phi_2\|_C+L\}\|\phi_1-\phi_2\|_C,$$
$$\|F(t,\phi)\| \leq K\|\phi\|_C^2+4K\|a\|_{L_\infty}\|\phi\|_C+4K\|b\|_{L_\infty}\sqrt{\mu(\Omega)}.$$

Proposition 6.1. The operators bBA and $T(t)=e^{-bBAt}$ are given by the following expressions

$$bBAz=\sum_{j=1}^{\infty}\frac{b\lambda_j}{1+a\lambda_j}E_jz \tag{6.8}$$

$$T(t)z=e^{-bBAt}z=\sum_{j=1}^{\infty}e^{\frac{-b\lambda_j}{1+a\lambda_j}t}E_jz. \tag{6.9}$$

$$\| T(t) \| \leq e^{-\beta t}, \quad t\geq 0, \tag{6.10}$$

$$\beta=\inf_{j\geq 1}\left\{\frac{b\lambda_j}{1+a\lambda_j}\right\}=\frac{b\lambda_1}{1+a\lambda_1}. \tag{6.11}$$

If $\mathcal{A}=bBA$, systems (6.7) can be written as

$$\begin{cases} z'=-\mathcal{A}z+F(t,z_t), & t\geq 0, t\neq t_k\\ z(s)+(g(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))(s)=\phi(s), & s\in[-r,0],\\ z(t_k^+)=z(t_k^-)+I_k^e(t_k,z(t_k)), & k=1,2,3,\dots,p.\end{cases} \tag{6.12}$$

Theorem 6.1. For d_k small enough there exist $\tau>0$ such that the system (6.12) has only one mild solution define on $[-r,\tau]$.

7 Final Remark

We are planing to follow Dan Henry's book on Geometric Theory of Semilinear Porabolic Equatios ([5]) in order to investigate the following topics:
1. Existence of bounded solutions for impulsive evolution equations with delay and nonlocal conditions.
2. Dynamical systems and Liapunov stability for impulsive evolution equations with delay and nonlocal conditions.
3. Neighborhood of an equilibrium point for impulsive evolution equations with delay and nonlocal conditions.
4. Stability and instability by linear approximation for impulsive evolution equations with delay and nonlocal conditions.
5. Traveling waves for impulsive evolution equations with delay and nonlocal conditions.
6. Exponential dichotomy for impulsive evolution equations with delay and nonlocal conditions.
7. Existence of attractors for impulsive evolution equations with delay and nonlocal conditions.

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