Communications in Mathematical Analysis
Volume 21, Number 1, pp. 23–34 (2018) www.math-res-pub.org/cma
ISSN 1938-9787

A C R F A E H C

S S B *
Department of Mathematics
Pierre and Marie Curie University
Paris, 75005, France

(Communicated by Jin Liang)

Abstract

We give blow-up behavior for a Brezis and Merle's problem with Dirichlet and Hölderian conditions. Also we derive a compactness criterion as in the work of Brezis and Merle.

AMS Subject Classification: 35J60, 35B44, 35B45.

Keywords: Blow-up, boundary, Dirichlet condition, compactness, Holderian condition.

1 Introduction

We set $\Delta = -(\partial_{11} + \partial_{22})$ on open set Ω of $\mathbb{R}^2$ with analytic boundary.

We consider the following equation:

$$(P)\begin{cases} \Delta u = V(1+|x|^{2\beta})e^u & \text{in } \ \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \ \partial\Omega. \end{cases}$$

Here, we assume that:

$$0 \in \partial\Omega, \ \beta \in [0, 1/2).$$

and,

$$0 \le V \le b < +\infty, \ e^u \in L^1(\Omega) \text{ and } u \in W_0^{1,1}(\Omega).$$

We can see in [8] a nice formulation of this problem (P) in the sense of the distributions. This Problem arises from geometrical and physical problems, see for example [1, 3, 21, 24]. The above equation was studied by many authors, with or without the boundary condition, also for Riemannian surfaces, see [1-23], where one can find some existence and compactness results. In [7] we have the following important Theorem,

*E-mail address: samybahoura@gmail.com

Theorem A*(Brezis-Merle [7]).For* $(u_i)_i$ *and* $(V_i)_i$ *two sequences of functions relative to* (P) *with,*

$$0 < a \leq V_i \leq b < +\infty$$

then it holds,

$$\sup_K u_i \leq c,$$

with c depending on a, b, β, K *and* Ω*.*

One can find in [7] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}, namely, we have:

Theorem B*(Brezis-Merle [7]).For* $(u_i)_i$ *and* $(V_i)_i$ *two sequences of functions relative to the problem* (P) *with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,$$

then it holds;

$$\sup_K u_i \leq c,$$

with c depending on b, β, C, K *and* Ω*.*

We look to the uniform boundedness in all $\bar{\Omega}$ of the solutions of the Problem (P). When $a = 0$, the boundedness of $\int_\Omega e^{u_i}$ is a necessary condition in the problem (P) as showed in [7] by the following counterexample.

Theorem C*(Brezis-Merle [7]).There are two sequences* $(u_i)_i$ *and* $(V_i)_i$ *of the problem* (P) *with,*

$$0 \leq V_i \leq b < +\infty \text{ and } \int_\Omega e^{u_i} dy \leq C,$$

such that,

$$\sup_\Omega u_i \to +\infty.$$

To obtain the two first previous results (Theorems A and B) Brezis and Merle used an inequality (Theorem 1 of [7]) obtained by an approximation argument and used Fatou's lemma and applied the maximum principle in $W_0^{1,1}(\Omega)$ which arises from Kato's inequality. Also this weak form of the maximum principle is used to prove the local uniform boundedness result by comparing a certain function and the Newtonian potential. We refer to [6] for a topic about the weak form of the maximum principle.

When $\beta = 0$, the above equation has many properties in the constant and the Lipschitzian cases:

Note that for the problem (P) $(\beta = 0)$, by using the Pohozaev identity, we can prove that $\int_\Omega e^{u_i}$ is uniformly bounded when $0 < a \leq V_i \leq b < +\infty$ and $||\nabla V_i||_{L^\infty} \leq A$ and Ω starshaped, when $a = 0$ and $\nabla \log V_i$ is uniformly bounded, we can bound uniformly $\int_\Omega V_i e^{u_i}$. In [20], Ma-Wei have proved that those results stay true for all open sets not necessarily starshaped.

In [10] $(\beta = 0)$ Chen-Li have proved that if $a = 0$, $\nabla \log V_i$ is uniformly bounded and u_i is locally uniformly bounded in L^1, then the functions are uniformly bounded near the boundary.

In [10] ($\beta = 0$) Chen-Li have proved that if $a = 0$ and $\int_\Omega e^{u_i}$ is uniformly bounded and $\nabla \log V_i$ is uniformly bounded, then we have the compactness result directly. Ma-Wei in [20], extend this result in the case where $a > 0$.

If we assume V more regular, we can have another type of estimates, a sup + inf type inequalities. It was proved by Shafrir see [23], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right)\sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

One can see in [11] an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality, see [3]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brezis-Li-Shafrir [5]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [11]. Also, one can see in [18], an extension of the Brezis-Li-Shafrir result to compact Riemannian surfaces without boundary. One can see in [19] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used.

In [9] we have some a priori estimates on the 2 and 3-spheres $\mathbb{S}_2$, $\mathbb{S}_3$.

Here we give the behavior of the blow-up points on the boundary and a proof of Brezis-Merle Problem when $\beta \geq 0$.

The Brezis-Merle Problem (see [7]) is:

Problem. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i \leq b$ for some positive constant b. Also, we consider a sequence of solutions (u_i) of (P) relative to (V_i) such that,

$$\int_\Omega e^{u_i} dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C = C(b, \beta, C, V, \Omega)?$$

Here we give a blow-up analysis for a sequence of solutions of the Problem (P) and a proof of compactness result for the Brezis-Merle's Problem when $\beta \geq 0$. We extend the result of Chen-Li [10]. For the blow-up analysis we assume that:

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary, but for the proof of the compactness result we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

Our mains results are:

2 Main Results

Theorem 2.1. *Assume that* $\max_\Omega u_i \to +\infty$, *where* (u_i) *are solutions of the problem* (P) *with:*

$$\beta \in [0,1/2),\ 0 \leq V_i \leq b \text{ and } \int_\Omega e^{u_i} dx \leq C,\ \forall\, i,$$

then, after passing to a subsequence, there is a finction u, *there is a number* $N \in \mathbb{N}$ *and* N *points* $x_1, \dots, x_N \in \partial\Omega$, *such that,*

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^N \alpha_j \delta_{x_j},\ \alpha_j \geq 4\pi, \text{ weakly in the sens of measures on } \partial\Omega.$$

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

Theorem 2.2. *Assume that* (u_i) *are solutions of* (P) *relative to* (V_i) *with the following conditions:*

$$0 \in \partial\Omega,\ \beta \in [0,1/2),$$

and,

$$0 \leq V_i \leq b,\ ||\nabla V_i||_{L^\infty} \leq A \text{ and } \int_\Omega e^{u_i} \leq C,$$

we have,

$$||u_i||_{L^\infty} \leq c(b,\beta,A,C,\Omega),$$

In the last theorem we extend the result of Chen-Li ($\beta = 0$). The proof of Chen-Li and Ma-Wei [10,20], use the moving-plane method ($\beta = 0$).

3 Proof of theorem 2.1

We have:

$$u_i \in W_0^{1,1}(\Omega).$$

Since $e^{u_i} \in L^1(\Omega)$ by the corollary 1 of Brezis-Merle's paper (see [7]) we have $e^{u_i} \in L^k(\Omega)$ for all $k > 2$ and the elliptic estimates of Agmon and the Sobolev embedding (see [1]) imply that:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

We denote by $\partial_\nu u_i$ the inner normal derivative. By the maximum principle we have, $\partial_\nu u_i \geq 0$.

By the Stokes formula we have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

We use the weak convergence in the space of Radon measures to have the existence of a nonnegative Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \phi d\sigma \to \mu(\phi), \ \forall \ \phi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. For $\epsilon > 0$ small enough set $I_\epsilon = B(x_0,\epsilon)\cap\partial\Omega$ on the unit disk or one can assume it as an interval. We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{on } I_\epsilon, \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{outside } I_{2\epsilon}, \\ 0 \le \eta_\epsilon \le 1, \\ ||\nabla\eta_\epsilon||_{L^\infty(I_{2\epsilon})} \le \dfrac{C_0(\Omega,x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} \Delta\tilde{\eta}_\epsilon = 0 & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \tilde{\eta}_\epsilon = \eta_\epsilon & \text{in } \ \partial\Omega. \end{cases}$$

Remark: We use the following steps in the construction of $\tilde{\eta}_\epsilon$:

We take a cutoff function η_0 in $B(0,2)$ or $B(x_0,2)$:

1- We set $\eta_\epsilon(x) = \eta_0(|x-x_0|/\epsilon)$ in the case of the unit disk it is sufficient.

2- Or, in the general case: we use a chart $(f,\tilde{\Omega})$ with $f(0) = x_0$ and we take $\mu_\epsilon(x) = \eta_0(f(|x|/\epsilon))$ to have connected sets I_ϵ and we take $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$. Because f, f^{-1} are Lipschitz, $|f(x)-x_0| \le k_2|x| \le 1$ for $|x| \le 1/k_2$ and $|f(x)-x_0| \ge k_1|x| \ge 2$ for $|x| \ge 2/k_1 > 1/k_2$, the support of η is in $I_{(2/k_1)\epsilon}$.

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{on } f(I_{(1/k_2)\epsilon}), \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{outside } f(I_{(2/k_1)\epsilon}), \\ 0 \le \eta_\epsilon \le 1, \\ ||\nabla\eta_\epsilon||_{L^\infty(I_{(2/k_1)\epsilon})} \le \dfrac{C_0(\Omega,x_0)}{\epsilon}. \end{cases}$$

3- Also, we can take: $\mu_\epsilon(x) = \eta_0(|x|/\epsilon)$ and $\eta_\epsilon(y) = \mu_\epsilon(f^{-1}(y))$, we extend it by 0 outside $f(B_1(0))$. We have $f(B_1(0)) = D_1(x_0)$, $f(B_\epsilon(0)) = D_\epsilon(x_0)$ and $f(B_\epsilon^+) = D_\epsilon^+(x_0)$ with f and f^{-1} smooth diffeomorphism.

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{on a the connected set } J_\epsilon = f(I_\epsilon), \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{outside } J'_\epsilon = f(I_{2\epsilon}), \\ 0 \le \eta_\epsilon \le 1, \\ ||\nabla\eta_\epsilon||_{L^\infty(J'_\epsilon)} \le \dfrac{C_0(\Omega,x_0)}{\epsilon}. \end{cases}$$

And, $H_1(J'_\epsilon) \le C_1 H_1(I_{2\epsilon}) = C_1 4\epsilon$, since f is Lipschitz. Here H_1 is the Hausdorff measure.

We solve the Dirichlet Problem:

$$\begin{cases} \Delta\bar{\eta}_\epsilon = \Delta\eta_\epsilon & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \bar{\eta}_\epsilon = 0 & \text{in } \ \partial\Omega. \end{cases}$$

and finaly we set $\tilde\eta_\epsilon = -\bar\eta_\epsilon + \eta_\epsilon$. Also, by the maximum principle and the elliptic estimates we have :

$$||\nabla\tilde\eta_\epsilon||_{L^\infty} \leq C(||\eta_\epsilon||_{L^\infty} + ||\nabla\eta_\epsilon||_{L^\infty} + ||\Delta\eta_\epsilon||_{L^\infty}) \leq \frac{C_1}{\epsilon^2},$$

with C_1 depends on Ω.

We use the following estimate, see [4, 8, 14, 25],

$$||\nabla u_i||_{L^q} \leq C_q, \ \forall\ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} \Delta u = V(1+|x|^{2\beta})e^u \in L^1(\Omega) & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \ u = 0 & \text{in } \ \partial\Omega. \end{cases}$$

According to the corollary 1 of Brezis-Merle's result, see [7], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar\Omega)$.

For two vectors f and g we denote by $f \cdot g$ the inner product of f and g.

We can write:

$$\Delta((u_i - u)\tilde\eta_\epsilon) = (1+|x|^{2\beta})(V_i e^{u_i} - Ve^u)\tilde\eta_\epsilon - 2\nabla(u_i - u)\cdot\nabla\tilde\eta_\epsilon. \tag{3.1}$$

We use the interior esimate of Brezis-Merle, see [7],

<u>*Step 1:*</u> Estimate of the integral of the first term of the right hand side of (3.1).

We use the Green formula between $\tilde\eta_\epsilon$ and u, we obtain,

$$\int_\Omega (1+|x|^{2\beta})Ve^u\tilde\eta_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq C'\epsilon||\partial_\nu u||_{L^\infty} = C\epsilon \tag{3.2}$$

We have,

$$\begin{cases} \Delta u_i = (1+|x|^{2\beta})V_i e^{u_i} & \text{in } \ \Omega \subset \mathbb{R}^2, \\ \ u_i = 0 & \text{in } \ \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde\eta_\epsilon$ to have:

$$\int_\Omega (1+|x|^{2\beta})V_i e^{u_i}\tilde\eta_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(J'_\epsilon) \leq 4\pi - \epsilon_0, \ \epsilon_0 > 0. \tag{3.3}$$

From (3.2) and (3.3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_\Omega |(1+|x|^{2\beta})(V_i e^{u_i} - Ve^u)\tilde\eta_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon. \tag{3.4}$$

<u>*Step 2:*</u> Estimate of integral of the second term of the right hand side of (3.1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \leq meas(\Omega - \Omega_{\epsilon^3}) = \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$.

Remark: For the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1-\epsilon^3)$.

(Proof of this fact; let's consider $d(x, \partial\Omega) = d(x, z_0), z_0 \in \partial\Omega$, this imply that $(d(x, z_0))^2 \leq (d(x,z))^2$ for all $z \in \partial\Omega$ which it is equivalent to $(z - z_0) \cdot (2x - z - z_0) \leq 0$ for all $z \in \partial\Omega$, let's consider a chart around z_0 and $\gamma(t)$ a curve in $\partial\Omega$, we have;

$(\gamma(t) - \gamma(t_0) \cdot (2x - \gamma(t) - \gamma(t_0)) \leq 0$ if we divide by $(t - t_0)$ (with the sign and tend t to t_0), we have $\gamma'(t_0) \cdot (x - \gamma(t_0)) = 0$, this imply that $x = z_0 - s\nu_0$ where ν_0 is the outward normal of $\partial\Omega$ at z_0)).

With this fact, we can say that $S = \{x, d(x, \partial\Omega) \leq \epsilon\} = \{x = z_0 - s\nu_{z_0}, z_0 \in \partial\Omega, \ -\epsilon \leq s \leq \epsilon\}$. It is sufficient to work on $\partial\Omega$. Let's consider a charts $(z, D = B(z, 4\epsilon_z), \gamma_z)$ with $z \in \partial\Omega$ such that $\cup_z B(z, \epsilon_z)$ is cover of $\partial\Omega$. One can extract a finite cover $(B(z_k, \epsilon_k)), k = 1, ..., m$, by the area formula the measure of $S \cap B(z_k, \epsilon_k)$ is less than a $k\epsilon$ (a ϵ-rectangle). For the reverse inequality, it is sufficient to consider one chart around one point of the boundary).

We write,

$$\int_\Omega |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx. \tag{3.5}$$

<u>*Step 2.1:*</u> Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $\|\nabla\tilde{\eta}_\epsilon\|_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle's theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_\Omega |\nabla u_i| f dx \to \int_\Omega |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have:

$$\text{for } \epsilon > 0 \ \exists \ i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \ \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq meas(\Omega - \Omega_{\epsilon^3}) \|\nabla u\|_{L^\infty} + \epsilon^3 = \epsilon^3(k_1\|\nabla u\|_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1\|\nabla u\|_{L^\infty} + 1) \tag{3.6}$$

The constant C_1 does not depend on ϵ but on Ω.

<u>*Step 2.2:*</u> Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon| dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^3})$. We have,

$$||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon| dx \leq ||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^3})} ||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1\epsilon \text{ for } i \geq i_3,$$

For $\epsilon > 0$, we have for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_\Omega |\nabla(u_i-u)\cdot\nabla\tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty}+2) \tag{3.7}$$

From (3.4) and (3.7), we have, for $\epsilon > 0$, there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that,

$$\int_\Omega |\Delta[(u_i-u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon 2C_1(2k_1||\nabla u||_{L^\infty}+2+C) \tag{3.8}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (3.1).
Indeed, we have:

$$\begin{cases} \Delta[(u_i-u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega \subset \mathbb{R}^2, \\ (u_i-u)\tilde{\eta}_\epsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \dfrac{\epsilon_0}{2}$.
We can use Theorem 1 of [7] to conclude that there are $q \geq \tilde{q} > 1$ such that:

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_\Omega e^{q|u_i-u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon,\Omega).$$

where, $V_\epsilon(x_0)$ is a neighberhood of x_0 in $\bar{\Omega}$. Here we have used that in a neighborhood of x_0 by the elliptic estimates, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$. (We can take $B(x_0, \epsilon^3)$).
Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that:

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \ \forall \ i. \tag{3.9}$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$\Delta(u_i\eta) = (1+|x|^{2\beta})V_i e^{u_i}\eta - 2\nabla u_i \cdot \nabla\eta + u_i\Delta\eta.$$

By the elliptic estimates (see [15]) $(u_i)_i$ is uniformly bounded in $W^{2,q_1}(V_\epsilon(x_0))$ and also, in $C^1(V_\epsilon(x_0))$. Finaly, we have, for some $\epsilon > 0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \ \forall \ i.$$

We have proved that, there is a finite number of points $\bar{x}_1,\dots,\bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded (in $C^{1,\theta}, \theta > 0$) in $\bar{\Omega} - \{\bar{x}_1,\dots,\bar{x}_m\}$.

4 Proof of theorem 2.2.

Without loss of generality, we can assume that 0 is a blow-up point. Since the boundary is an analytic curve $\gamma(t)$, there is a neighborhood of 0 such that the curve γ can be extend to a holomorphic map such that $\gamma'(0) \neq 0$ (series) and by the inverse mapping one can assume that this map is univalent around 0. In the case when the boundary is a simple Jordan curve the domain is simply connected, see [24]. In the case that the domains has a finite number of holes it is conformally equivalent to a disk with a finite number of disks removed, see [17]. Here we consider a general domain. Without loss of generality one can assume that $\gamma(B_1^+) \subset \Omega$ and also $\gamma(B_1^-) \subset (\bar{\Omega})^c$ and $\gamma(-1,1) \subset \partial\Omega$ and γ is univalent. This means that (B_1,γ) is a local chart around 0 for Ω and γ univalent. (This fact holds if we assume that we have an analytic domain, in the sense of Hofmann see [16], (below a graph of an analytic function), we have necessary the condition $\partial\bar{\Omega} = \partial\Omega$ and the graph is analytic, in this case $\gamma(t) = (t,\phi(t))$ with ϕ real analytic and an example of this fact is the unit disk around the point $(0,1)$ for example).

By this conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which not contain 0 and on which u_i converge in the C^1 norm to u. Let us consider B_ϵ^+, the half ball with radius $\epsilon > 0$. Also, one can consider a C^1 domain (a rectangle between two half disks) and by charts its image is a C^1 domain).

We know that:

$$u_i \in C^{2,\epsilon}(\bar{\Omega}).$$

Thus we can use integrations by parts (Gauss-Green-Riemann-Stokes formula). The second Pohozaev identity applied around the blow-up 0 see for example [2, 20, 22] gives :

$$\int_{B_\epsilon^+} \Delta u_i (x\cdot\nabla u_i) dx = -\int_{\partial^+ B_\epsilon^+} g(\nabla u_i) d\sigma, \tag{4.1}$$

with,

$$g(\nabla u_i) = (\nu\cdot\nabla u_i)(x\cdot\nabla u_i) - x\cdot\nu\frac{|\nabla u_i|^2}{2}.$$

Thus,

$$\int_{B_\epsilon^+} V_i(1+|x|^{2\beta})e^{u_i}(x\cdot\nabla u_i) dx = -\int_{\partial^+ B_\epsilon^+} g(\nabla u_i) d\sigma. \tag{4.2}$$

After integration by parts, we obtain:

$$\int_{B_\epsilon^+} 2V_i(1+(1+\beta)|x|^{2\beta})e^{u_i}dx + \int_{B_\epsilon^+} x\cdot\nabla V_i(1+|x|^{2\beta})e^{u_i}dx - \int_{\partial B_\epsilon^+} \nu\cdot x(1+|x|^{2\beta})V_i e^{u_i}d\sigma =$$

$$= \int_{\partial^+ B_\epsilon^+} g(\nabla u_i)d\sigma. \tag{4.3}$$

Also, for u we have:

$$\int_{B_\epsilon^+} 2V(1+(1+\beta)|x|^{2\beta})e^{u}dx + \int_{B_\epsilon^+} x\cdot\nabla V(1+|x|^{2\beta})e^{u}dx - \int_{\partial B_\epsilon^+} \nu\cdot x(1+|x|^{2\beta})V e^{u}d\sigma =$$

$$= \int_{\partial^+ B_\epsilon^+} g(\nabla u)d\sigma. \tag{4.4}$$

We use the fact that $u_i = u = 0$ on $\{x_1 = 0\}$ and u_i, u are bounded in the C^1 norm outside a neighborhood of 0 and we tend i to $+\infty$ and then ϵ to 0 to obtain:

$$\int_{B_\epsilon^+} V_i(1+|x|^{2\beta})e^{u_i}dx = o(1) + O(\epsilon), \tag{4.5}$$

however

$$\int_{\gamma(B_\epsilon^+)} V_i(1+|x|^{2\beta})e^{u_i}dx = \int_{\partial\gamma(B_\epsilon^+)} \partial_\nu u_i d\sigma = \alpha_1 + O(\epsilon) + o(1) > 0, \tag{4.6}$$

which is a contradiction.

Here we have used a theorem of Hofmann see [16], which gives the fact that $\gamma(B_\epsilon^+)$ is a Lipschitz domain. Also, we can see that $\gamma((-\epsilon,\epsilon))$ and $\gamma(\partial^+ B_\epsilon^+)$ are submanifolds.

We start with a Lipschitz domain B_ϵ^+ because it is convex and by the univalent and conformal map γ the image of this domain $\gamma(B_\epsilon^+)$ is a Lipschitz domain and thus we can apply the integration by part and here we know the explicit formula of the unit outward normal it is the usual unit outward normal (normal to the tangent space of the boundary which we know explicitly because we have two submanifolds).

In the case of the disk $D = \Omega$, it is sufficient to consider $B(0,\epsilon)\cap D$ which is a Lipschitz domain because it is convex (and not necessarily $\gamma(B_\epsilon^+)$).

There is a version of the integration by part which is the Green-Riemann formula in dimension 2 on a domain Ω. This formula holds if we assume that there is a finite number of points $y_1, ..., y_m$ such that $\partial\Omega - (y_1, ..., y_m)$ is a C^1 manifold and for C^1 tests functions, see [2], for the Gauss-Green-Riemann-Stokes formula, for C^1 domains with singular points (here a finite number of singular points).

Remark: Note that a monograph of Droniou contain a proof of all fact about Sobolev spaces (with Strong Lipschitz property) with only weak Lipschitz property (Lipschitz-Charts), we start with Strong Lipschitz property and by γ we have weak Lipschtz property.

Acknowledgments

I would like to thank the referee forthe many comments on the paper.

References

[1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag, 1998.

[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded variations and Free discontinuity Problems, Oxford Press. 2000.

[3] C. Bandle, Isoperimetric Inequalities and Applications. Pitman, 1980.

[4] L. Boccardo, T. Gallouet. Nonlinear elliptic and parabolic equations involving measure data. J. Funct. Anal. 87 no 1, (1989), 149-169.

[5] H. Brezis, Y. Y. Li and I. Shafrir, A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct. Anal.115 (1993) 344-358.

[6] H. Brezis, M. Marcus, A. C. Ponce, Nonlinear elliptic equations with measures revisited. Mathematical aspects of nonlinear dispersive equations, 55-109, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[7] H. Brezis and F. Merle, Uniform estimates and Blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimension. Commun. in Partial Differential Equations, 16 (8 and 9), 1223-1253(1991).

[8] H. Brezis and W. A. Strauss, Semi-linear second-order elliptic equations in L^1. J. Math. Soc. Japan 25 (1973), 565-590.

[9] S.Y. A. Chang, M. J. Gursky, and P. C. Yang, Scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Differential Equations 1 (1993), no. 2, 205-229.

[10] W. Chen, C. Li, A priori estimates for solutions to nonlinear elliptic equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.

[11] C. C. Chen, C. S. Lin, A sharp sup+inf inequality for a nonlinear elliptic equation in $\mathbb{R}^2$. Commun. Anal. Geom. 6, No.1, 1-19 (1998).

[12] D. G. De Figueiredo, P. L. Lions, R. D. Nussbaum, A priori Estimates and Existence of Positive Solutions of Semilinear Elliptic Equations, J. Math. Pures et Appl., vol 61, 1982, pp.41-63.

[13] J. Droniou, Quelques resultats sur les espaces de Sobolev. Hal 2001.

[14] W. Ding, J. Jost, J. Li, and G. Wang, The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. Asian J. Math. 1 (1997), no. 2, 230-248.

[15] D. Gilbarg, N. S. Trudinger, Elliptic Partial Differential Equations of Second order, Berlin Springer-Verlag.

[16] S. Hofmann, M. Mitrea, and M. Taylor, Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. J. Geom. Anal. 17 (2007), no. 4, 593–647.

[17] S. Krantz, Geometric functions theory. Birkhäuser.

[18] Y. Y. Li, Harnack Type Inequality: the method of moving planes. Commun. Math. Phys. 200,421-444 (1999).

[19] Y. Y. Li, I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.

[20] L. Ma, J-C. Wei, Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.

[21] K. Nagasaki and T. Suzuki, Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities. Asymptotic Anal. 3 (1990), no. 2, 173–188.

[22] J. Necas, Direct Methods in the Theory of Elliptic Equations. Monographs in Mathematics. Springer, Heidelberg, 2012. Springer.

[23] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = Ve^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.

[24] J. Stoker, Differential Geometry.

[25] G. Tarantello, Multiple condensate solutions for the Chern-Simons-Higgs theory. J. Math. Phys. 37 (1996), no. 8, 3769-3796.