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VECTOR INEQUALITIES FOR TWO PROJECTIONS IN HILBERT SPACES AND APPLICATIONS

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Abstract

In this paper we establish some vector inequalities related to Schwarz and Buzano results. We show amongst others that in an inner product space H we have the inequality

$$\frac{1}{4}\left[\|x\|\|y\|+|\langle x,y\rangle-2\langle Px,y\rangle-2\langle Qx,y\rangle|\right]\geq|\langle QPx,y\rangle|$$

for any vectors x,y and P,Q two orthogonal projections on H. If $PQ=0$ we also have

$$\frac{1}{2}\left[\|x\|\|y\|+|\langle x,y\rangle|\right]\geq|\langle Px,y\rangle+\langle Qx,y\rangle|$$

for any $x,y\in H$.

Applications for norm and numerical radius inequalities of two bounded operators are given as well.

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1 Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\|y\|\geq|\langle x,y\rangle| \text{ for any } x,y\in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

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In 1985 the author [5] (see also [23]) established the following refinement of (1.1):

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| + |\langle x,e\rangle\langle e,y\rangle| \geq |\langle x,y\rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \geq |\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|$$

and by (1.2) we get

$$\begin{aligned}\|x\|\,\|y\| &\geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| + |\langle x,e\rangle\langle e,y\rangle| \\ &\geq 2\,|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle|,\end{aligned}$$

which implies the *Buzano inequality* [2]

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle| \tag{1.3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

The following results provides inequalities for two unitary vectors [19]:

Theorem 1.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned}&\|x\|\,\|y\| - |\langle x,e\rangle\langle f,y\rangle| \\ &\geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle|.\end{aligned} \tag{1.4}$$

Moreover, if $e \perp f$, i.e. $\langle e,f\rangle = 0$, then

$$\|x\|\,\|y\| - |\langle x,e\rangle\langle f,y\rangle| \geq |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle|. \tag{1.5}$$

and

Theorem 1.2. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned}\|x\|\,\|y\| &\geq \max\{|\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle|, |\langle x,y\rangle - 2\langle x,f\rangle\langle f,y\rangle|\} \\ &\geq \begin{cases} |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle|, \\ \\ |\langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \end{cases}\end{aligned} \tag{1.6}$$

and

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle - 2\langle x,f\rangle\langle f,y\rangle + 4\langle x,e\rangle\langle f,y\rangle\langle e,f\rangle|. \tag{1.7}$$

The inequalities (2.1) and (1.7) are sharp.

Remark 1.3. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$ and $f \perp e$ then by (1.7) we get

$$\|x\|\,\|y\| \geq |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle - 2\langle x,f\rangle\langle f,y\rangle|. \tag{1.8}$$

Corollary 1.4. *If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\frac{1}{4}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq \left|\frac{1}{2}\left[\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle\right] - \langle x,e\rangle\langle f,y\rangle\langle e,f\rangle\right|. \tag{1.9}$$

In particular, if $f \perp e$, then

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle|. \tag{1.10}$$

The following result for two vectors also holds [20]:

Theorem 1.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. If $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$, then*

$$\begin{aligned} &\left|\langle x,y\rangle - \frac{1}{2}\left[\langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle e,y\rangle\right]\right| \\ &\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2 - |\langle y,f\rangle|^2\right)^{1/2} + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\|y\|\right] \end{aligned} \tag{1.11}$$

Remark 1.6. If we use the triangle inequality

$$\begin{aligned} &\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle| \\ &\leq \left|\langle x,y\rangle - \frac{1}{2}\left[\langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle e,y\rangle\right]\right|, \end{aligned}$$

then we get from (2.1)

$$\begin{aligned} &\frac{1}{2}|\langle x,f\rangle\langle f,y\rangle + \langle x,e\rangle\langle e,y\rangle| \\ &\leq |\langle x,y\rangle| + \frac{1}{2}\left[\|x\|\left(\|y\|^2 - |\langle y,f\rangle|^2\right)^{1/2} + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\|y\|\right] \end{aligned} \tag{1.12}$$

for any $x, y, e, f \in H$ with $\|e\| = \|f\| = 1$.

Now, let us recall some basic facts on *orthogonal projection* that will be used in the sequel.

If K is a subset of a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, the set of *vectors orthogonal* to K is defined by

$$K^{\perp} := \{x \in H : \langle x,k\rangle = 0 \text{ for all } k \in K\}.$$

We observe that $K^{\perp}$ is a *closed subspace* of H and so forms itself a Hilbert space. If V is a closed subspace of H, then $V^{\perp}$ is called the *orthogonal complement* of V. In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in $K^{\perp}$. Therefore, H is the *internal Hilbert direct sum* of V and $V^{\perp}$, and we denote that as $H = V \oplus V^{\perp}$.

The linear operator $P_V : H \to H$ that maps x to v is called *the orthogonal projection* onto V. There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all *bounded self-adjoint* operators P such that $P^2 = P$. Specifically, the orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V, where V is the range of E. For every x in H, $P_V(x)$ is the unique element v of

V, which minimizes the distance $\|x-v\|$. This provides the geometrical interpretation of $P_V(x)$: it is *the best approximation* to x by elements of V.

Projections P_U and P_V are called *mutually orthogonal* if $P_UP_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H. The sum of the two projections P_U and P_V is a projection only if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$.The composite P_UP_V is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_UP_V = P_{U\cap V}$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j,k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j,k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

It is well known that for any orthonormal family $\{e_j\}_{j\in J}$ we have *Bessel's inequality*

$$\sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2 \leq \|x\|^2 \text{ for any } x \in H.$$

This becomes *Parseval's identity*

$$\sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2 = \|x\|^2 \text{ for any } x \in H,$$

when $\{e_j\}_{j\in J}$ an othonormal basis in H.

For an othonormal family $\{e_j\}_{j\in J}$ we define the operator $P_J : H \to H$ by

$$P_J x := \sum_{j\in J} \left\langle x, e_j\right\rangle e_j\,,\; x \in H. \tag{1.13}$$

We know that P_J is an *orthogonal projection* and

$$\langle P_J x, y\rangle = \sum_{j\in J} \left\langle x, e_j\right\rangle \left\langle e_j, y\right\rangle,\; x,y \in H \text{ and } \langle P_J x, x\rangle = \sum_{j\in J} \left|\left\langle x, e_j\right\rangle\right|^2,\; x \in H.$$

The particular case when the family reduces to one vector $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle\,,\; x,y \in H \tag{1.14}$$

and Buzano's inequality can be written as

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x, y\rangle|\right] \geq |\langle P_e x, y\rangle| \tag{1.15}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

Consider the othonormal family $\{e_j\}_{j\in J}$ and assume that $\emptyset \neq K, L \subset J$. If we consider the projections

$$P_K x := \sum_{k\in K} \langle x, e_k\rangle e_k\,,\; P_L x := \sum_{\ell\in L} \langle x, e_\ell\rangle e_\ell\,,\; x \in H$$

we have

$$P_K P_L x = \sum_{k\in K} \langle P_L x, e_k\rangle e_k = \sum_{k\in K} \left\langle \sum_{\ell\in L} \langle x, e_\ell\rangle e_\ell , e_k \right\rangle e_k$$
$$= \sum_{k\in K}\sum_{\ell\in L} \langle x, e_\ell\rangle \langle e_\ell , e_k\rangle e_k = \sum_{k\in K\cap L} \langle x, e_k\rangle e_k,\ x\in H.$$

We observe that if $K\cap L=\emptyset$, then $P_K P_L = 0$ and $P_K + P_L$ is a projection with

$$P_K x + P_L x = \sum_{k\in K\cup L} \langle x, e_k\rangle e_k\ ,\ x\in H.$$

Motivated by the above results we establish in this paper some similar inequalities incorporating two projections and apply then to obtain some norm and numerical radius inequalities for two bounded linear operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$.

2 Vector Inequalities for Two Projections

We have the following result for two projections.

Theorem 2.1. *Let P, Q be two orthogonal projections on H. For any $x, y\in H$ we have the inequalities*

$$\|x\|\,\|y\| - \langle Px, x\rangle^{1/2} \langle Qy, y\rangle^{1/2} \geq |\langle x, y\rangle - \langle Px, y\rangle - \langle Qx, y\rangle + \langle QPx, y\rangle| \tag{2.1}$$

and

$$\|x\|\,\|y\| - \langle (1_H - P)x, x\rangle^{1/2} \langle (1_H - Q)y, y\rangle^{1/2} \geq |\langle QPx, y\rangle| . \tag{2.2}$$

Proof. Using Schwarz inequality we get

$$\|x-Px\|^2 \|y-Qy\|^2 \geq |\langle x-Px, y-Qy\rangle|^2 \tag{2.3}$$

for any $x, y\in H$.

Observe that

$$\|x-Px\|^2 = \langle x-Px, x-Px\rangle = \|x\|^2 - \langle Px, x\rangle - \langle x, Px\rangle + \langle Px, Px\rangle$$
$$= \|x\|^2 - \langle Px, x\rangle - \langle x, Px\rangle + \left\langle P^2 x, x\right\rangle = \|x\|^2 - \langle Px, x\rangle$$

and, similarly

$$\|y-Qy\|^2 = \|y\|^2 - \langle Qy, y\rangle ,$$

for any $x, y\in H$.

Also, we have

$$|\langle x-Px, y-Qy\rangle| = |\langle x, y\rangle - \langle Px, y\rangle - \langle Qx, y\rangle + \langle QPx, y\rangle|$$

for any $x, y\in H$.

Inequality (2.3) can be then written as

$$\left(\|x\|^2-\langle Px,x\rangle\right)\left(\|y\|^2-\langle Qy,y\rangle\right) \tag{2.4}$$
$$\geq |\langle x,y\rangle-\langle Px,y\rangle-\langle Qx,y\rangle+\langle QPx,y\rangle|^2$$

for any $x,y\in H$.

Using the elementary inequality that holds for any real numbers a,b,c,d

$$(ac-bd)^2\geq\left(a^2-b^2\right)\left(c^2-d^2\right),$$

we have

$$\left(\|x\|\,\|y\|-\langle Px,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}\right)^2\geq\left(\|x\|^2-\langle Px,x\rangle\right)\left(\|y\|^2-\langle Qy,y\rangle\right) \tag{2.5}$$

for any $x,y\in H$.

Since

$$\|x\|\geq\langle Px,x\rangle^{1/2},\ \ \|y\|\geq\langle Qy,y\rangle^{1/2},$$

then

$$\|x\|\,\|y\|-\langle Px,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}\geq 0,$$

for any $x,y\in H$.

Making use of (2.4) and (2.5) we get

$$\left(\|x\|\,\|y\|-\langle Px,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}\right)^2 \tag{2.6}$$
$$\geq |\langle x,y\rangle-\langle Px,y\rangle-\langle Qx,y\rangle+\langle QPx,y\rangle|^2$$

and by taking the square root in (2.6) we get the desired result (2.1).

Now, if we replace P with 1_H-P and Q with 1_H-Q in (2.1) we get

$$\|x\|\,\|y\|-\langle(1_H-P)\,x,x\rangle^{1/2}\langle(1_H-Q)\,y,y\rangle^{1/2} \tag{2.7}$$
$$\geq |\langle x,y\rangle-\langle(1_H-P)\,x,y\rangle-\langle(1_H-Q)\,x,y\rangle+\langle(1_H-Q)\,(1_H-P)\,x,y\rangle|$$
$$=|\langle QPx,y\rangle|$$

for any $x,y\in H$ and the inequality (2.2) is proved. □

Remark 2.2. If $QP=0$, then we have

$$\|x\|\,\|y\|-\langle Px,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}\geq|\langle x,y\rangle-\langle Px,y\rangle-\langle Qx,y\rangle| \tag{2.8}$$

for any $x,y\in H$.

If we use the triangle inequality then we get from (2.8) that

$$\|x\|\,\|y\|+|\langle x,y\rangle|\geq\langle Px,x\rangle^{1/2}\langle Qy,y\rangle^{1/2}+|\langle Px,y\rangle+\langle Qx,y\rangle|$$

for any $x,y\in H$.

Corollary 2.3. *Let P be an orthogonal projection on H. For any $x,y\in H$ we have the following inequalities*

$$\|x\|\,\|y\|\geq\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}+|\langle x,y\rangle-\langle Px,y\rangle| \tag{2.9}$$
$$\geq|\langle Px,y\rangle|+|\langle x,y\rangle-\langle Px,y\rangle|.$$

Proof. The first inequality follows by (2.1) for $Q = P$. The second inequality follows by Schwarz inequality for the nonnegative operator P, namely

$$\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} \geq |\langle Px,y\rangle|$$

for any $x,y \in H$. □

Remark 2.4. By the triangle inequality for modulus we get the following refinement of Schwarz inequality

$$\begin{aligned} \|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq |\langle Px,y\rangle| + |\langle x,y\rangle - \langle Px,y\rangle| \geq |\langle x,y\rangle| \end{aligned} \tag{2.10}$$

for any $x,y \in H$.

Using the triangle inequality, then by (2.9) we have

$$\begin{aligned} \|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq |\langle Px,y\rangle| + |\langle x,y\rangle - \langle Px,y\rangle| \geq 2\,|\langle Px,y\rangle| - |\langle x,y\rangle| \end{aligned} \tag{2.11}$$

for any $x,y \in H$.

If we use the inequality between the first and last term in (2.11) we get

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq |\langle Px,y\rangle| \tag{2.12}$$

for any $x,y \in H$.

This is a generalization of Buzano's inequality (1.3) for a projection P.

We also observe that from the first inequality in (2.10) we have

$$\begin{aligned} \|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle| - |\langle Px,y\rangle|, \end{aligned}$$

which implies that

$$\|x\|\,\|y\| - |\langle x,y\rangle| \geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} - |\langle Px,y\rangle| \geq 0 \tag{2.13}$$

for any $x,y \in H$.

Similarly

$$\begin{aligned} \|x\|\,\|y\| &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle x,y\rangle - \langle Px,y\rangle| \\ &\geq \langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle Px,y\rangle| - |\langle x,y\rangle|, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] &\geq \frac{1}{2}\left[\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2} + |\langle Px,y\rangle|\right] \\ &\geq |\langle Px,y\rangle| \end{aligned} \tag{2.14}$$

for any $x,y \in H$, that is better than (2.12).

Consider the orthonormal families $E = \{e_i\}_{i\in I}$ and $F = \left\{f_j\right\}_{j\in J}$ and the projections

$$P_E x := \sum_{i\in I} \langle x, e_i\rangle e_i\,,\ P_F x := \sum_{j\in J} \left\langle x, f_j\right\rangle f_j\,,\ x\in H.$$

Using the inequality (2.1) for P_E and P_F defined above, we have the inequalities

$$\begin{aligned} &\|x\|\,\|y\| - \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{1/2} \left(\sum_{j\in J} \left|\left\langle y, f_j\right\rangle\right|^2\right)^{1/2} \\ &\geq \left|\langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle - \sum_{j\in J} \left\langle x, f_j\right\rangle \left\langle f_j, y\right\rangle \right. \\ &\left. + \sum_{j\in J}\sum_{i\in I} \langle x, e_i\rangle \left\langle e_i, f_j\right\rangle \left\langle f_j, y\right\rangle \right| \end{aligned} \tag{2.15}$$

for any $x, y\in H$.

If $E = \{e\}$ and $F = \{f\}$, then from (2.15) we recapture the inequality (1.4).

If $E \perp F$, namely $e_i \perp f_j$ for any $i\in I,\ j\in J$, then we get from (2.15)

$$\begin{aligned} &\|x\|\,\|y\| - \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{1/2} \left(\sum_{j\in J} \left|\left\langle y, f_j\right\rangle\right|^2\right)^{1/2} \\ &\geq \left|\langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle - \sum_{j\in J} \left\langle x, f_j\right\rangle \left\langle f_j, y\right\rangle \right| \end{aligned} \tag{2.16}$$

for any $x, y\in H$.

By the triangle inequality for modulus we have

$$\begin{aligned} &\left|\langle x, y\rangle - \sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle - \sum_{j\in J} \left\langle x, f_j\right\rangle \left\langle f_j, y\right\rangle \right| \\ &\geq \left|\sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle + \sum_{j\in J} \left\langle x, f_j\right\rangle \left\langle f_j, y\right\rangle \right| - |\langle x, y\rangle| \end{aligned}$$

for any $x, y\in H$.

Making use of (2.16) we deduce the following inequality

$$\begin{aligned} \|x\|\,\|y\| + |\langle x, y\rangle| &\geq \left(\sum_{i\in I} |\langle x, e_i\rangle|^2\right)^{1/2} \left(\sum_{j\in J} \left|\left\langle y, f_j\right\rangle\right|^2\right)^{1/2} \\ &\quad + \left|\sum_{i\in I} \langle x, e_i\rangle \langle e_i, y\rangle + \sum_{j\in J} \left\langle x, f_j\right\rangle \left\langle f_j, y\right\rangle \right| \end{aligned} \tag{2.17}$$

for any $x, y\in H$.

If $E = \{e\}$ and $F = \{f\}$ with $e \perp f$ then from (2.16) we get (1.5) while from (2.17) we obtain

$$\|x\|\|y\| + |\langle x,y\rangle| \geq |\langle x,e\rangle\langle y,f\rangle| + |\langle x,e\rangle\langle e,y\rangle + \langle x,f\rangle\langle f,y\rangle| \tag{2.18}$$

for any $x, y \in H$.

Theorem 2.5. *Let P, Q be two orthogonal projections on H. For any $x, y \in H$ we have the inequalities*

$$\begin{cases} \left|\langle x,y\rangle - \frac{1}{2}\left[\langle Qx,y\rangle + \langle Px,y\rangle\right]\right| \\ \\ \frac{1}{2}|\langle Px,y\rangle - \langle Qx,y\rangle| \end{cases} \tag{2.19}$$
$$\leq \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \langle Qy,y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\|\right]$$
$$\leq \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\|x\|^2 + \|y\|^2 - \langle Px,x\rangle - \langle Qy,y\rangle\right)^{1/2}$$

for any $x, y \in H$.

Proof. Using Schwarz inequality we have

$$\|x\|\|y - Qy\| \geq |\langle x, y - Qy\rangle| = |\langle x,y\rangle - \langle x,Qy\rangle| = |\langle x,y\rangle - \langle Qx,y\rangle| \tag{2.20}$$

and

$$\|x - Px\|\|y\| \geq |\langle x - Px, y\rangle| = |\langle x,y\rangle - \langle Px,y\rangle| \tag{2.21}$$

for any $x, y \in H$.

We have

$$\|y - Qy\| = \left(\|y\|^2 - \langle Qy,y\rangle\right)^{1/2}$$

and

$$\|x - Px\| = \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}$$

for any $x, y \in H$.

If we add the inequalities (2.20) and (2.21) and use the triangle inequality we have

$$\|x\|\left(\|y\|^2 - \langle Qy,y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\|$$
$$\geq \begin{cases} |2\langle x,y\rangle - \langle Qx,y\rangle - \langle Px,y\rangle| \\ \\ |\langle Px,y\rangle - \langle Qx,y\rangle| \end{cases}$$

for any $x, y \in H$, which proved the first inequality in (2.19).

By the Cauchy-Bunyakovsky-Schwarz inequality

$$ac + bd \leq \left(a^2 + b^2\right)^{1/2}\left(c^2 + d^2\right)^{1/2} \text{ for } a,b,c,d \geq 0 \tag{2.22}$$

we have

$$\|x\|\left(\|y\|^2 - \langle Qy,y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\| \tag{2.23}$$
$$\leq \left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\|x\|^2 + \|y\|^2 - \langle Px,x\rangle - \langle Qy,y\rangle\right)^{1/2}$$

for any $x, y \in H$, which proves the last inequality in (2.19). □

Corollary 2.6. *Let P be an orthogonal projection on H. For any $x,y \in H$ we have the following inequalities*

$$\begin{aligned} &|\langle x,y\rangle - \langle Px,y\rangle| \\ &\le \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \langle Py,y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\le \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2 - \langle Px,x\rangle - \langle Py,y\rangle\right)^{1/2} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &\begin{cases} \frac{1}{2}|\langle x,y\rangle| \\ \\ \left|\langle Px,y\rangle - \frac{1}{2}\langle x,y\rangle\right| \end{cases} \\ &\le \frac{1}{2}\left[\|x\|\langle Py,y\rangle^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\le \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2 - \langle Px,x\rangle + \langle Py,y\rangle\right)^{1/2} \end{aligned} \tag{2.25}$$

for any $x,y \in H$.

Proof. The inequality (2.24) follows from (2.19) by taking $Q = P$ while (2.25) follows from (2.19) by taking $Q = 1_H - P$. □

If we consider the orthonormal families $E = \{e_i\}_{i\in I}$ and $F = \{f_j\}_{j\in J}$, then from (2.19) we have the inequalities

$$\begin{aligned} &\begin{cases} \left|\langle x,y\rangle - \frac{1}{2}\left[\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle + \sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle\right]\right| \\ \\ \frac{1}{2}\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle - \sum_{j\in J}\langle x,f_j\rangle\langle f_j,y\rangle\right| \end{cases} \\ &\le \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \sum_{j\in J}\left|\langle f_j,y\rangle\right|^2\right)^{1/2} + \left(\|x\|^2 - \sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{1/2}\|y\|\right] \\ &\le \frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2 - \sum_{i\in I}|\langle x,e_i\rangle|^2 - \sum_{j\in J}\left|\langle f_j,y\rangle\right|^2\right)^{1/2} \end{aligned} \tag{2.26}$$

for any $x,y \in H$.

If $E = \{e\}$ and $F = \{f\}$, then from (2.26) we get (1.11) and the inequality

$$\begin{aligned} &|\langle x,e\rangle\langle e,y\rangle - \langle x,f\rangle\langle f,y\rangle| \\ &\le \|x\|\left(\|y\|^2 - |\langle f,y\rangle|^2\right)^{1/2} + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\|y\| \end{aligned} \tag{2.27}$$

for any $x,y \in H$.

The following result holds:

Theorem 2.7. *Let P,Q be two orthogonal projections on H. For any $x,y \in H$ we have the inequalities*

$$\|x\| \|y\| \geq |\langle x,y\rangle - 2\langle Px,y\rangle - 2\langle Qx,y\rangle + 4\langle QPx,y\rangle|, \tag{2.28}$$

$$\begin{aligned} \|x\| \|y\| &\geq \max\{|\langle x,y\rangle - 2\langle Qx,y\rangle|, |\langle x,y\rangle - 2\langle Px,y\rangle|\} \\ &\geq \begin{cases} |\langle x,y\rangle - \langle Qx,y\rangle - \langle Px,y\rangle|, \\ \\ |\langle Qx,y\rangle - \langle Px,y\rangle| \end{cases} \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)^{1/2}\left(\|x\|^2 + \|y\|^2 - \langle Px,x\rangle - \langle Qy,y\rangle\right)^{1/2} \\ &\geq \frac{1}{2}\left[\|x\|\left(\|y\|^2 - \langle Qy,y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\geq \left|\langle x,y\rangle - \frac{3}{2}\langle Px,y\rangle - \frac{3}{2}\langle Qx,y\rangle + 2\langle QPx,y\rangle\right|. \end{aligned} \tag{2.30}$$

Proof. Observe that if R is a projection, then for any $z \in H$ we have

$$\begin{aligned} \|z - 2Rz\|^2 &= \langle z - 2Rz, z - 2Rz\rangle \\ &= \|z\|^2 - 2\langle Rz,z\rangle - 2\langle z,Rz\rangle + 4\langle Rz,Rz\rangle \\ &= \|z\|^2 - 4\langle Rz,z\rangle + 4\left\langle R^2z,z\right\rangle = \|z\|^2. \end{aligned}$$

Using Schwarz inequality we have

$$\begin{aligned} \|x\| \|y\| &= \|x - 2Px\| \|y - 2Qy\| \geq |\langle x - 2Px, y - 2Qy\rangle| \\ &= |\langle x,y\rangle - 2\langle Px,y\rangle - 2\langle x,Qy\rangle + 4\langle Px,Qy\rangle| \\ &= |\langle x,y\rangle - 2\langle Px,y\rangle - 2\langle Qx,y\rangle + 4\langle QPx,y\rangle| \end{aligned}$$

for any $x,y \in H$.

By Schwarz inequality we also have

$$\|x\| \|y\| = \|x\| \|y - 2Qy\| \geq |\langle x, y - 2Qy\rangle| = |\langle x,y\rangle - 2\langle Qx,y\rangle|$$

and

$$\|x\| \|y\| = \|x - 2Px\| \|y\| \geq |\langle x - 2Px, y\rangle| = |\langle x,y\rangle - 2\langle Px,y\rangle|$$

for any $x,y \in H$, which produce the first inequality in (2.29).

By the elementary inequality $\max\{a,b\} \geq \frac{1}{2}(a+b)$ and the triangle inequality for the modulus we have

$$\begin{aligned} &\max\{|\langle x,y\rangle - 2\langle Qx,y\rangle|, |\langle x,y\rangle - 2\langle Px,y\rangle|\} \\ &\geq \frac{1}{2}\left[|\langle x,y\rangle - 2\langle Qx,y\rangle| + |\langle x,y\rangle - 2\langle Px,y\rangle|\right] \\ &\geq \begin{cases} |\langle x,y\rangle - \langle Qx,y\rangle - \langle Px,y\rangle| \\ \\ |\langle Qx,y\rangle - \langle Px,y\rangle| \end{cases} \end{aligned}$$

for any $x, y \in H$, which proves the last part of (2.29).

We also have

$$\begin{aligned} \|x - Px\| \|y\| &= \|x - Px\| \|y - 2Qy\| \geq |\langle x - Px, y - 2Qy \rangle| \\ &= |\langle x, y \rangle - \langle Px, y \rangle - 2\langle Qx, y \rangle + 2\langle QPx, y \rangle| \end{aligned}$$

and

$$\begin{aligned} \|x\| \|y - Py\| &= \|x - 2Px\| \|y - Qy\| \geq |\langle x - 2Px, y - Qy \rangle| \\ &= |\langle x, y \rangle - 2\langle Px, y \rangle - \langle Qx, y \rangle + 2\langle QPx, y \rangle| \end{aligned}$$

and by addition and the triangle inequality we have

$$\begin{aligned} &\|x - Px\| \|y\| + \|x\| \|y - Py\| \\ &\geq |\langle x, y \rangle - \langle Px, y \rangle - 2\langle Qx, y \rangle + 2\langle QPx, y \rangle| \\ &+ |\langle x, y \rangle - 2\langle Px, y \rangle - \langle Qx, y \rangle + 2\langle QPx, y \rangle| \\ &\geq |2\langle x, y \rangle - 3\langle Px, y \rangle - 3\langle Qx, y \rangle + 4\langle QPx, y \rangle| \end{aligned}$$

and dividing by 2 we get

$$\begin{aligned} &\frac{1}{2}\left[\|x - Px\| \|y\| + \|x\| \|y - Py\|\right] \\ &\geq \left|\langle x, y \rangle - \frac{3}{2}\langle Px, y \rangle - \frac{3}{2}\langle Qx, y \rangle + 2\langle QPx, y \rangle\right| \end{aligned}$$

for any $x, y \in H$.

This proves the second inequality in (2.30).

The first inequality in (2.30) was proved before. □

Remark 2.8. Using the triangle inequality and (2.28) we have

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - 2\langle Px, y \rangle - 2\langle Qx, y \rangle + 4\langle QPx, y \rangle| \\ &\geq 4|\langle QPx, y \rangle| - |\langle x, y \rangle - 2\langle Px, y \rangle - 2\langle Qx, y \rangle|, \end{aligned}$$

which implies that

$$\frac{1}{4}\left[\|x\| \|y\| + |\langle x, y \rangle - 2\langle Px, y \rangle - 2\langle Qx, y \rangle|\right] \geq |\langle QPx, y \rangle| \tag{2.31}$$

for any $x, y \in H$.

From (2.30) we also have

$$\begin{aligned} &\frac{1}{4}\left[\|x\| \left(\|y\|^2 - \langle Qy, y \rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px, x \rangle\right)^{1/2} \|y\|\right] \\ &+ \frac{1}{2}\left|\langle x, y \rangle - \frac{3}{2}\langle Px, y \rangle - \frac{3}{2}\langle Qx, y \rangle\right| \\ &\geq |\langle QPx, y \rangle| \end{aligned} \tag{2.32}$$

for any $x, y \in H$.

Corollary 2.9. *Let P be an orthogonal projection on H. For any $x,y \in H$ we have the inequalities*

$$\|x\| \|y\| \geq |\langle x,y\rangle - 2\langle Px,y\rangle| \tag{2.33}$$

and

$$\begin{aligned} &\frac{1}{2}\left(\|x\|^2+\|y\|^2\right)^{1/2}\left(\|x\|^2+\|y\|^2-\langle Px,x\rangle-\langle Py,y\rangle\right)^{1/2} \\ &\geq \frac{1}{2}\left[\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2}+\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\geq |\langle x,y\rangle-\langle Px,y\rangle|. \end{aligned} \tag{2.34}$$

Remark 2.10. By the triangle inequality we have

$$\|x\| \|y\| \geq |\langle x,y\rangle - 2\langle Px,y\rangle| \geq 2|\langle Px,y\rangle| - |\langle x,y\rangle|$$

for any $x,y \in H$, which also implies the inequality (2.12).

From (2.34) we have

$$\begin{aligned} &|\langle x,y\rangle| + \frac{1}{2}\left[\|x\|\left(\|y\|^2-\langle Py,y\rangle\right)^{1/2}+\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\geq |\langle Px,y\rangle| \end{aligned} \tag{2.35}$$

for any $x,y \in H$.

Corollary 2.11. *Let P,Q be two orthogonal projections on H with $QP = 0$. For any $x,y \in H$ we have the inequalities*

$$\|x\| \|y\| \geq |\langle x,y\rangle - 2\langle Px,y\rangle - 2\langle Qx,y\rangle| \tag{2.36}$$

and

$$\begin{aligned} &\frac{1}{2}\left[\|x\|\left(\|y\|^2-\langle Qy,y\rangle\right)^{1/2}+\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\geq \left|\langle x,y\rangle - \frac{3}{2}\langle Px,y\rangle - \frac{3}{2}\langle Qx,y\rangle\right|. \end{aligned} \tag{2.37}$$

Remark 2.12. Let P,Q be two orthogonal projections on H with $QP = 0$. Using the triangle inequality we have from (2.36)

$$\|x\| \|y\| \geq |\langle x,y\rangle - 2\langle Px,y\rangle - 2\langle Qx,y\rangle| \geq 2|\langle Px,y\rangle + \langle Qx,y\rangle| - |\langle x,y\rangle|,$$

which implies

$$\frac{1}{2}\left[\|x\| \|y\| + |\langle x,y\rangle|\right] \geq |\langle Px,y\rangle + \langle Qx,y\rangle| \tag{2.38}$$

for any $x,y \in H$.

From (2.37) we get

$$\begin{aligned} &\frac{1}{2}\left[\|x\|\left(\|y\|^2-\langle Qy,y\rangle\right)^{1/2}+\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\|y\|\right] \\ &\geq \left|\langle x,y\rangle - \frac{3}{2}\langle Px,y\rangle - \frac{3}{2}\langle Qx,y\rangle\right| \\ &\geq \frac{3}{2}|\langle Px,y\rangle + \langle Qx,y\rangle| - |\langle x,y\rangle|, \end{aligned}$$

which implies

$$\frac{1}{3}\left[\|x\|\left(\|y\|^2-\langle Qy,y\rangle\right)^{1/2}+\left(\|x\|^2-\langle Px,x\rangle\right)^{1/2}\|y\|+|\langle x,y\rangle|\right] \tag{2.39}$$
$$\geq |\langle Px,y\rangle+\langle Qx,y\rangle|$$

for any $x,y\in H$.

If we consider the orthonormal families $E=\{e_i\}_{i\in I}$ and $F=\left\{f_j\right\}_{j\in J}$, then from (2.28) and (2.31) we have the inequalities

$$\|x\|\,\|y\| \geq \left|\langle x,y\rangle-2\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-2\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle\right. \tag{2.40}$$
$$\left.+4\sum_{j\in J}\sum_{i\in I}\langle x,e_i\rangle\left\langle e_i,f_j\right\rangle\left\langle f_j,y\right\rangle\right|,$$

and

$$\frac{1}{4}\|x\|\,\|y\|+\frac{1}{4}\left|\langle x,y\rangle-2\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-2\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle\right| \tag{2.41}$$
$$\geq\left|\sum_{j\in J}\sum_{i\in I}\langle x,e_i\rangle\left\langle e_i,f_j\right\rangle\left\langle f_j,y\right\rangle\right|$$

for any $x,y\in H$.

From (2.29) we also have

$$\|x\|\,\|y\|\geq\begin{cases}\left|\langle x,y\rangle-\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle-\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle\right|,\\[2ex]\left|\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle-\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle\right|\end{cases} \tag{2.42}$$

for any $x,y\in H$.

If $E\perp F$ then by (2.38) and (2.39) we have

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle+\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle\right| \tag{2.43}$$

and

$$\frac{1}{3}\|x\|\left(\|y\|^2-\sum_{j\in J}\left|\left\langle f_j,y\right\rangle\right|^2\right)^{1/2}+\frac{1}{3}\left(\|x\|^2-\sum_{i\in I}|\langle x,e_i\rangle|^2\right)^{1/2}\|y\| \tag{2.44}$$
$$+\frac{1}{3}|\langle x,y\rangle|$$
$$\geq\left|\sum_{i\in I}\langle x,e_i\rangle\langle e_i,y\rangle+\sum_{j\in J}\left\langle x,f_j\right\rangle\left\langle f_j,y\right\rangle\right|$$

for any $x, y \in H$.

If $E = \{e\}$ and $F = \{f\}$, then from (2.40)-(2.44) we get some results stated in the introduction. We omit the details.

3 Inequalities for Norm and Numerical Radius

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [26, p. 1]:

$$W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by [26, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T) \le \|T\| \le 2w(T), \text{ for any } T \in B(H).$$

Utilising Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [13] or [14]:

Theorem 3.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{3.1}$$

The constant $\frac{1}{2}$ is best possible in (3.1).

The following general result for the product of two operators holds [26, p. 37]:

Theorem 3.2. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$, then $w(AB) \le 4w(A)w(B)$. In the case that $AB = BA$, then $w(AB) \le 2w(A)w(B)$. The constant 2 is best possible here.*

The following results are also well known [26, p. 38].

Theorem 3.3. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{3.2}$$

If A is an isometry and $AB = BA$, then (3.2) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. The following result holds [26, p. 38].

Theorem 3.4. *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|. \tag{3.3}$$

As a consequence of the above, we have [26, p. 39]:

Corollary 3.5. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)\,w(B). \tag{3.4}$$

A related problem with the inequality (3.3) is to find the best constant c for which the inequality

$$w(AB) \le cw(A)\,\|B\|$$

holds for any two commuting operators $A, B \in B(H)$. It is known that $1.064 < c < 1.169$, see [3], [34] and [35].

In relation to this problem, it has been shown in [24] that:

Theorem 3.6. *For any $A, B \in B(H)$ we have*

$$w\left(\frac{AB+BA}{2}\right) \le \sqrt{2}w(A)\,\|B\|. \tag{3.5}$$

For other numerical radius inequalities see the recent monograph [18] and the references therein.

We recall that the *absolute value* of an operator T is defined by $|T| = (T^*T)^{1/2}$.

Theorem 3.7. *Let P, Q be two orthogonal projections on H. If A, B are two bounded linear operators on H then we have the inequalities*

$$w\left(B\left[1_H - \frac{1}{2}(Q+P)\right]A\right) \le \frac{1}{2}\left\||A|^2 + |(1_H-P)A|^2\right\|^{1/2}\left\|\left|B^*\right|^2 + \left|(1_H-Q)B^*\right|^2\right\|^{1/2} \tag{3.6}$$

and

$$w\left(B\left[1_H - \frac{1}{2}(Q+P)\right]A\right) \le \frac{1}{4}\left\||A|^2 + |(1_H-P)A|^2 + \left|B^*\right|^2 + \left|(1_H-Q)B^*\right|^2\right\|. \tag{3.7}$$

Proof. By the inequality (2.19) we have

$$\left|\left\langle\left[1_H - \frac{1}{2}(Q+P)\right]x, y\right\rangle\right| \le \frac{1}{2}\left[\|x\|\,\|(1_H-Q)y\| + \|(1_H-P)x\|\,\|y\|\right] \tag{3.8}$$

for any $x, y \in H$.

By taking $y = B^*u$, $x = Au$ with $u \in H$ in (3.8) we get

$$\begin{aligned}
&\left|\left\langle B\left[1_H - \frac{1}{2}(Q+P)\right]Au, u\right\rangle\right| \\
&\le \frac{1}{2}\left[\|Au\|\left\|(1_H-Q)B^*u\right\| + \|(1_H-P)Au\|\left\|B^*u\right\|\right] \\
&\le \frac{1}{2}\left[\|Au\|^2 + \|(1_H-P)Au\|^2\right]^{1/2}\left[\left\|B^*u\right\|^2 + \left\|(1_H-Q)B^*u\right\|^2\right]^{1/2}
\end{aligned} \tag{3.9}$$

for any $u \in H$, where for the last inequality we used the elementary Cauchy-Bunyakovsky-Schwarz inequality

$$ac + bd \leq \left(a^2 + b^2\right)^{1/2} \left(c^2 + d^2\right)^{1/2} \text{ for } a,b,c,d \geq 0.$$

Observe that, by using the notation absolute value of an operator T we have

$$\|Au\|^2 + \|(1_H - P)Au\|^2 = \left\langle \left[|A|^2 + |(1_H - P)A|^2\right] u, u\right\rangle$$

and

$$\left\|B^* u\right\|^2 + \left\|(1_H - Q)B^* u\right\|^2 = \left\langle \left[\left|B^*\right|^2 + \left|(1_H - Q)B^*\right|^2\right] u, u\right\rangle$$

for any $u \in H$.

By (3.9) we have

$$\begin{aligned} &\left|\left\langle B\left[1_H - \frac{1}{2}(Q+P)\right] Au, u\right\rangle\right| \qquad (3.10)\\ &\leq \frac{1}{2}\left[\left\langle \left[|A|^2 + |(1_H - P)A|^2\right] u, u\right\rangle\right]^{1/2} \\ &\times \left[\left\langle \left[\left|B^*\right|^2 + \left|(1_H - Q)B^*\right|^2\right] u, u\right\rangle\right]^{1/2} \end{aligned}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.10) we get the desired result (3.6).

By the arithmetic mean-geometric mean inequality we also have

$$\begin{aligned} &\left|\left\langle B\left[1_H - \frac{1}{2}(Q+P)\right] Au, u\right\rangle\right| \qquad (3.11)\\ &\leq \frac{1}{2}\left[\|Au\| \left\|(1_H - Q)B^* u\right\| + \|(1_H - P)Au\| \left\|B^* u\right\|\right] \\ &\leq \frac{1}{4}\left[\|Au\|^2 + \left\|(1_H - Q)B^* u\right\|^2 + \|(1_H - P)Au\|^2 + \left\|B^* u\right\|^2\right] \\ &= \frac{1}{4}\left\langle \left[|A|^2 + |(1_H - P)A|^2 + \left|B^*\right|^2 + \left|(1_H - Q)B^*\right|^2\right] u, u\right\rangle \end{aligned}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.11) we get the desired result (3.7). □

Remark 3.8. If we take $Q = P$ and then replace $1_H - P$ by P we get from (3.6) and (3.7) that

$$w(BPA) \leq \frac{1}{2}\left\||A|^2 + |PA|^2\right\|^{1/2} \left\|\left|B^*\right|^2 + \left|PB^*\right|^2\right\|^{1/2} \qquad (3.12)$$

and

$$w(BPA) \leq \frac{1}{4}\left\||A|^2 + |PA|^2 + \left|B^*\right|^2 + \left|PB^*\right|^2\right\|. \qquad (3.13)$$

If we take in (3.6) and (3.7) $Q = 1_H - P$ we get

$$w(BA) \leq \left\||A|^2 + |(1_H - P)A|^2\right\|^{1/2} \left\|\left|B^*\right|^2 + \left|PB^*\right|^2\right\|^{1/2} \qquad (3.14)$$

and

$$w(BA) \le \frac{1}{4} \left\| |A|^2 + |(1_H - P)A|^2 + \left|B^*\right|^2 + \left|PB^*\right|^2 \right\| \quad (3.15)$$

for any projection P.

Remark 3.9. Using a similar argument and the inequality

$$\begin{aligned} &|\langle Px, y\rangle - \langle Qx, y\rangle| \\ &\le \|x\| \left(\|y\|^2 - \langle Qy, y\rangle\right)^{1/2} + \left(\|x\|^2 - \langle Px, x\rangle\right)^{1/2} \|y\| \end{aligned}$$

for any $x, y \in H$, we get

$$\begin{aligned} &w(B(P-Q)A) \\ &\le \left\| |A|^2 + |(1_H - P)A|^2 \right\|^{1/2} \left\| \left|B^*\right|^2 + \left|(1_H - Q)B^*\right|^2 \right\|^{1/2} \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} &w(B(P-Q)A) \\ &\le \frac{1}{2} \left\| |A|^2 + |(1_H - P)A|^2 + \left|B^*\right|^2 + \left|(1_H - Q)B^*\right|^2 \right\| \end{aligned} \quad (3.17)$$

for P, Q two orthogonal projections on H and A, B two bounded linear operators on H.

If we take in (3.16) and (3.17) $Q = 1 - P$, then we get

$$\begin{aligned} &w\left(B\left(P - \frac{1}{2}1_H\right)A\right) \\ &\le \frac{1}{2} \left\| |A|^2 + |(1_H - P)A|^2 \right\|^{1/2} \left\| \left|B^*\right|^2 + \left|PB^*\right|^2 \right\|^{1/2} \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} &w\left(B\left(P - \frac{1}{2}1_H\right)A\right) \\ &\le \frac{1}{4} \left\| |A|^2 + |(1_H - P)A|^2 + \left|B^*\right|^2 + \left|PB^*\right|^2 \right\|. \end{aligned} \quad (3.19)$$

Theorem 3.10. *Let P, Q be two orthogonal projections on H. If A, B are two bounded linear operators on H then we have the inequalities*

$$\begin{aligned} \|A\| \|B\| &\ge \max\{\|B(Q-P)A\|, \|B(1_H - P - Q)A\|\} \\ &\ge \left\| B\left(\frac{1}{2}1_H - P\right)A \right\| \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} \frac{1}{2} \left\| |A|^2 + \left|B^*\right|^2 \right\| &\ge \max\{w[B(Q-P)A], w[B(1_H - P - Q)A]\} \\ &\ge w\left[B\left(\frac{1}{2}1_H - P\right)A\right]. \end{aligned} \quad (3.21)$$

Proof. By (2.29) we have

$$\|x\|\|y\| \geq \begin{cases} |\langle (1_H - P - Q)x, y\rangle|, \\ \\ |\langle (Q-P)x, y\rangle| \end{cases} \tag{3.22}$$

for any $x, y \in H$.

This implies that

$$\|Au\|\left\|B^*v\right\| \geq \begin{cases} |\langle B(1_H - P - Q)Au, v\rangle|, \\ \\ |\langle B(Q-P)Au, v\rangle| \end{cases} \tag{3.23}$$

for any $u, v \in H$.

Taking the supremum over $u, v \in H$, $\|u\| = \|v\| = 1$ in (3.23) we get the first part of (3.20). The rest is obvious.

From (3.23) and the arithmetic mean -geometric mean inequality we have

$$\frac{1}{2}\left\langle \left(|A|^2 + \left|B^*\right|^2\right)u, u\right\rangle = \frac{1}{2}\left[\|Au\|^2 + \left\|B^*u\right\|^2\right] \geq \|Au\|\left\|B^*u\right\| \tag{3.24}$$
$$\geq \begin{cases} |\langle B(1_H - P - Q)Au, u\rangle|, \\ \\ |\langle B(Q-P)Au, u\rangle| \end{cases}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.23) we get the first part of (3.20). The rest is obvious. □

We also have:

Theorem 3.11. *Let P, Q be two orthogonal projections on H. If A, B are two bounded linear operators on H then we have the inequalities*

$$\frac{1}{4}\|A\|\|B\| + \left\|B\left(\frac{1}{4}1_H - \frac{P+Q}{2}\right)A\right\| \geq \|BQPA\| \tag{3.25}$$

and

$$\frac{1}{8}\left\||A|^2 + \left|B^*\right|^2\right\| + w\left[B\left(\frac{1}{4}1_H - \frac{P+Q}{2}\right)A\right] \geq w(BQPA). \tag{3.26}$$

Proof. By the inequality (2.31) we have

$$\frac{1}{4}\left[\|x\|\|y\| + |\langle (1_H - 2P - 2Q)x, y\rangle|\right] \geq |\langle QPx, y\rangle| \tag{3.27}$$

for any $x, y \in H$.

This implies that

$$\frac{1}{4}\left[\|Au\|\left\|B^*v\right\| + |\langle [B(1_H - 2P - 2Q)A]u, v\rangle|\right] \geq |\langle (BQPA)u, v\rangle| \tag{3.28}$$

for any $u,v \in H$.

Taking the supremum over $u,v \in H$, $\|u\| = \|v\| = 1$ in (3.28) we get the desired result (3.25).

From (3.28) we also have

$$\frac{1}{4}\left[\|Au\|\left\|B^*u\right\| + |\langle[B(1_H - 2P - 2Q)A]u,u\rangle|\right] \geq |\langle(BQPA)u,u\rangle|$$

for any $u \in H$ and since

$$\frac{1}{2}\left\langle\left(|A|^2 + \left|B^*\right|^2\right)u,u\right\rangle \geq \|Au\|\left\|B^*u\right\|$$

we get

$$\frac{1}{4}\left[\frac{1}{2}\left\langle\left(|A|^2 + \left|B^*\right|^2\right)u,u\right\rangle + |\langle[B(1_H - 2P - 2Q)A]u,u\rangle|\right] \geq |\langle(BQPA)u,u\rangle| \tag{3.29}$$

for any $u \in H$.

Taking the supremum over $u \in H$, $\|u\| = 1$ in (3.29) we get the desired result (3.26). □

Remark 3.12. If in (3.25) and (3.26) we take $Q = 1_H$, then we get

$$\frac{1}{4}\|A\|\|B\| + \frac{1}{2}\left\|B\left(P + \frac{1}{2}1_H\right)A\right\| \geq \|BPA\| \tag{3.30}$$

and

$$\frac{1}{8}\left\||A|^2 + \left|B^*\right|^2\right\| + \frac{1}{2}w\left[B\left(P + \frac{1}{2}1_H\right)A\right] \geq w(BPA). \tag{3.31}$$

Also, if in (3.25) and (3.26) we take $Q = P$, then we get

$$\frac{1}{4}\|A\|\|B\| + \left\|B\left(\frac{1}{4}1_H - P\right)A\right\| \geq \|BPA\| \tag{3.32}$$

and

$$\frac{1}{8}\left\||A|^2 + \left|B^*\right|^2\right\| + w\left[B\left(\frac{1}{4}1_H - P\right)A\right] \geq w(BPA). \tag{3.33}$$

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