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Lebedev's Type Index Transforms with the Modified Bessel Functions

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Abstract

New index transforms of the Lebedev type are investigated. It involves the real part of the product of the modified Bessel functions as the kernel. Boundedness properties are examined for these operators in the Lebesgue weighted spaces. Inversion theorems are proved. Important particular cases are exhibited. The results are applied to solve an initial value problem for the fourth order PDE, involving the Laplacian. Finally, it is shown that the same PDE has another fundamental solution, which is associated with the generalized Lebedev index transform, involving the square of the modulus of Macdonald's function, recently considered by the author.

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1 Introduction

Let $\alpha \in \mathbb{R}$. The objects of this paper are the following index transforms [7], [6]

$$(F_\alpha f)(\tau) = \frac{2\sqrt{\pi}}{\cosh(\pi\tau)} \int_0^\infty \mathrm{Re}\Big[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x})\Big] f(x)dx, \quad \tau \in \mathbb{R}, \tag{1.1}$$

$$(G_\alpha g)(x) = 2\sqrt{\pi} \int_{-\infty}^\infty \mathrm{Re}\Big[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x})\Big] \frac{g(\tau)}{\cosh(\pi\tau)} d\tau, \quad x \in \mathbb{R}_+, \tag{1.2}$$

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where i is the imaginary unit, Re denotes the real part of the complex-valued function and $K_\mu(z), I_\mu(z)$ [1], Vol. II are modified Bessel functions, satisfying the differential equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2+\mu^2)u=0. \tag{1.3}$$

The asymptotic behaviour at infinity is given by the formulas

$$K_\mu(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.4}$$

$$I_\mu(z)=\left(\frac{1}{2\pi z}\right)^{1/2}e^{z}[1+O(1/z)], \qquad z\to\infty. \tag{1.5}$$

Near zero we have, correspondingly, the relations

$$K_\mu(z)=O(z^{-|\mathrm{Re}\mu|}),\ \mu\neq 0, \quad K_0(z)=O(\log z),\ z\to 0, \tag{1.6}$$

$$I_\mu(z)=O(z^{|\mathrm{Re}\mu|}),\ z\to 0. \tag{1.7}$$

The product of the modified Bessel functions can be represented by the following integrals (see relations (2.12.14.1) in [4], Vol. II and (8.4.23.25) in [4], Vol. III)

$$\mathrm{Re}\Big[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x})\Big]=\int_0^\infty J_{2\alpha}\Big(2\sqrt{x}\,\sinh y\Big)\cos(2y\tau)dy,\ x>0, \tag{1.8}$$

$$\mathrm{Re}\Big[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x})\Big]=\frac{\cosh(\pi\tau)}{4\pi\sqrt{\pi}i}$$

$$\times\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}x^{-s}ds,\ x>0, \tag{1.9}$$

where $\max(-\alpha,0)<\gamma<1/2$ and $J_\mu(z)$ is the Bessel function of the first kind [1], Vol. II. The latter integral is a key ingredient to derive the differential equation for the kernel of the index transforms (1.1), (1.2). In fact, denoting by

$$\Phi_{\alpha,\tau}(x)=\frac{2\sqrt{\pi}}{\cosh(\pi\tau)}\mathrm{Re}\Big[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x})\Big], \tag{1.10}$$

we have

Lemma 1.1. *The kernel $\Phi_{\alpha,\tau}(x)$ is a fundamental solution of the following fourth order differential equation with variable coefficients*

$$x^3\frac{d^4\Phi_{\alpha,\tau}}{dx^4}+6x^2\,\frac{d^3\Phi_{\alpha,\tau}}{dx^3}+x\Big(7+\tau^2-\alpha^2-x\Big)\frac{d^2\Phi_{\alpha,\tau}}{dx^2}+\left(1+\tau^2-\alpha^2-\frac{5}{2}\,x\right)\frac{d\Phi_{\alpha,\tau}}{dx}$$

$$-\left(\frac{(\alpha\tau)^2}{x}+\frac{1}{2}\right)\Phi_{\alpha,\tau}=0. \tag{1.11}$$

Proof. Indeed, recalling the representation (1.9), we appeal to the Stirling asymptotic formula for the gamma-function [1], Vol. I, to write for each $\tau \in \mathbb{R}$ and $s = \gamma + it$

$$\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)} = O\left(e^{-\pi|t|}|t|^{2\gamma-3/2}\right), \quad |t| \to \infty. \tag{1.12}$$

This means that the repeated differentiation with respect to x under the integral sign in (1.9) is allowed, and with the use of the reduction formula for the gamma-function [1], Vol. I, we obtain

$$\left(x\frac{d}{dx}\right)^2 \Phi_{\alpha,\tau}(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{s^2\,\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}x^{-s}ds$$

$$= -\tau^2\Phi_{\alpha,\tau}(x) + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(1+s+i\tau)\Gamma(1+s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}x^{-s}ds$$

$$= -\tau^2\Phi_{\alpha,\tau}(x) + \frac{1}{2\pi i}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s-1+\alpha)\Gamma(3/2-s)}{\Gamma(2+\alpha-s)\Gamma(s-1)}x^{1-s}ds$$

$$= -\tau^2\Phi_{\alpha,\tau}(x) + \frac{1}{2\pi i}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)(1/2-s)(s-1)}{(s-1+\alpha)(1+\alpha-s)\Gamma(1+\alpha-s)\Gamma(s)}x^{1-s}ds.$$

Hence, moving the contour to the left by Cauchy's theorem, we multiply the latter equality by x^α and differentiate with respect to x again. Then

$$\frac{d}{dx}\left[x^\alpha\left(x\frac{d}{dx}\right)^2\Phi_{\alpha,\tau}(x)\right] = -\tau^2\frac{d}{dx}\left[x^\alpha\Phi_{\alpha,\tau}(x)\right]$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)(1/2-s)(s-1)}{(s-1+\alpha)\Gamma(1+\alpha-s)\Gamma(s)}x^{\alpha-s}ds.$$

In a similar manner we continue to reduce the denominator of the integrand, multiplying by $x^{1-2\alpha}$ and fulfilling again the differentiation. Hence,

$$\frac{d}{dx}\left[x^{1-2\alpha}\frac{d}{dx}\left[x^\alpha\left(x\frac{d}{dx}\right)^2\Phi_{\alpha,\tau}(x)\right]\right] = -\tau^2\frac{d}{dx}\left[x^{1-2\alpha}\frac{d}{dx}\left[x^\alpha\Phi_{\alpha,\tau}(x)\right]\right]$$

$$-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)(1/2-s)(s-1)}{\Gamma(1+\alpha-s)\Gamma(s)}x^{-\alpha-s}ds.$$

Now multiplying by x^α and accounting (1.9), we find

$$x^\alpha\frac{d}{dx}\left[x^{1-2\alpha}\frac{d}{dx}\left[x^\alpha\left(x\frac{d}{dx}\right)^2\Phi_{\alpha,\tau}(x)\right]\right] = -\tau^2\,x^\alpha\frac{d}{dx}\left[x^{1-2\alpha}\frac{d}{dx}\left[x^\alpha\Phi_{\alpha,\tau}(x)\right]\right]$$

$$+\frac{1}{2\pi i}\frac{d}{dx}\int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}x^{1-s}ds$$

$$= -\tau^2\,x^\alpha\frac{d}{dx}\left[x^{1-2\alpha}\frac{d}{dx}\left[x^\alpha\Phi_{\alpha,\tau}(x)\right]\right] - \frac{3}{2}\,\frac{d}{dx}\left[x\,\Phi_{\alpha,\tau}(x)\right] + \frac{d^2}{dx^2}\left[x^2\,\Phi_{\alpha,\tau}(x)\right].$$

Finally, fulfilling the differentiation, we end up with the equation (1.11), completing the proof of Lemma 1.1. □

2 Mapping Properties for Index Transform (1.1)

Our approach to examine the boundedness properties for the introduced index transforms and to prove the corresponding inversion theorems is based on the Mellin transform technique developed in [6], and an extensive use of the Marichev table for the Mellin transform in [3], [4], Vol. III. Appealing to the classical Titchmarsh monograph [5], the Mellin transform is defined, for instance, in $L_{\nu,p}(\mathbb{R}_+)$, $1 < p \le 2$ by the integral

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{2.1}$$

where the convergence is understood in mean with respect to the norm in $L_q(\nu - i\infty, \nu + i\infty)$, $q = p/(p-1)$. Moreover, the Parseval equality holds for $f \in L_{\nu,p}(\mathbb{R}_+)$, $g \in L_{1-\nu,q}(\mathbb{R}_+)$

$$\int_0^\infty f(x)g(x)dx = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)g^*(1-s)ds. \tag{2.2}$$

The inverse Mellin transform is given accordingly

$$f(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)x^{-s}ds, \tag{2.3}$$

where the integral converges in mean with respect to the norm in $L_{\nu,p}(\mathbb{R}_+)$

$$\|f\|_{\nu,p} = \left(\int_0^\infty |f(x)|^p x^{\nu p-1}dx\right)^{1/p}. \tag{2.4}$$

In particular, letting $\nu = 1/p$ we get the usual space $L_1(\mathbb{R}_+)$. Further, denoting by $C_b(\mathbb{R})$ the space of bounded continuous functions, we prove the following result.

Theorem 2.1. *Let $\alpha \ge -1/4$. The index transform* (1.1) *is well-defined as a bounded operator $F_\alpha : L_{3/4,1}(\mathbb{R}_+) \to C_b(\mathbb{R})$. Moreover, if in addition $f \in L_{1-\nu,p}(\mathbb{R}_+)$, $1 < p \le 2$, $\max(-\alpha, 0) < \nu < 1/2$, then*

$$(F_\alpha f)(\tau) = \frac{2\sqrt{\pi}}{\cosh(\pi\tau)}\int_0^\infty K_{i\tau}(\sqrt{x})\mathrm{Re}\left[I_{i\tau}(\sqrt{x})\right]\varphi_\alpha(x)dx, \tag{2.5}$$

where the integral converges absolutely,

$$\varphi_\alpha(x) = \frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} \frac{\Gamma(1-s+\alpha)\Gamma(s)}{\Gamma(s+\alpha)\Gamma(1-s)}\, f^*(s)x^{-s}ds, \tag{2.6}$$

and integral (2.6) *converges in mean with respect to the norm in $L_{1-\nu,p}(\mathbb{R}_+)$.*

Proof. Recalling the integral representation (1.8) of the index kernel in (1.1) and elementary inequality for the Bessel function of the first kind, which is a straightforward consequence of its asymptotic behaviour [1], Vol. II, namely,

$$\sqrt{x}\left|J_\mu(x)\right| < C, \quad \mathrm{Re}\,\mu \ge -1/2,$$

where C is a positive constant, depending on μ, we have the estimate (we will keep the same notation for different positive constants)

$$|(F_\alpha f)(\tau)| \le \frac{2\sqrt{\pi}}{\cosh(\pi\tau)} \int_0^\infty \int_0^\infty \left| J_{2\alpha}\left(2\sqrt{x}\,\sinh y\right)\right| |f(x)| dy\, dx$$

$$\le C \int_0^\infty \frac{dy}{\sqrt{\sinh y}} \int_0^\infty x^{-1/4} |f(x)|\, dx = C\, \|f\|_{3/4,1},\ \alpha \ge -\frac{1}{4}.$$

Hence via the absolute and uniform convergence it follows the continuity of $(F_\alpha f)(\tau)$ and the boundedness of the operator (1.1), namely

$$\sup_{\tau\in\mathbb{R}} |(F_\alpha f)(\tau)| \equiv \|F_\alpha f\|_{C(\mathbb{R})} \le C\|f\|_{3/4,1}.$$

Further, from the condition $f \in L_{1-\nu,p}(\mathbb{R}_+)$, asymptotic behaviour (1.12) and the Parseval equality (2.2) we derive the representation

$$(F_\alpha f)(\tau) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau) \frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)} f^*(1-s) ds. \tag{2.7}$$

Meanwhile, by virtue of the Stirling formula for the gamma-function

$$\frac{\Gamma(1-s+\alpha)\Gamma(s)}{\Gamma(s+\alpha)\Gamma(1-s)} = O(1),\ s = 1-\nu+it,\ |t| \to \infty. \tag{2.8}$$

Therefore, employing relation (8.4.23.23) in [4], Vol. III, we apply again the Parseval equality (2.2) to the right-hand side of (2.7). This leads to the formula (2.5), which is, in turn, the Lebedev index transform with the modified Bessel functions as the kernel [2] of the function φ_α defined by (2.6). Theorem 2.1 is proved.

□

The inversion formula for the Lebedev type transform (1.1) is established by

Theorem 2.2. *Let $\alpha > 1/4$, $f \in L_{1-\nu,p}(\mathbb{R}_+)$, its Mellin transform $f^*(s) \in L_1(1-\nu-i\infty, 1-\nu+i\infty)$ and be analytic in the open strip $\mathrm{Re}\, s = 1-\nu \in (0, \alpha+1)$. Then under the integrability condition $(F_\alpha f)(\tau) \in L_1(\mathbb{R}_+; \tau e^{2\pi\tau} d\tau)$ the following inversion formula holds for all $x > 0$*

$$f(x) = -\frac{4}{\pi\sqrt{\pi}} \int_0^\infty \left[\alpha \left[\frac{1}{2x} + \frac{d}{dx}\left(\mathrm{Re}\, \left[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x}) \right] \right) - \sqrt{x}\, \mathrm{Re}\, \left[K_{\alpha+i\tau}(\sqrt{x})\, I'_{\alpha-i\tau}(\sqrt{x}) \right] \right] \right.$$

$$\left. + \tau \frac{d}{dx}\left(\mathrm{Im}\, \left[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x}) \right] \right) \right] \cosh(\pi\tau) (F_\alpha f)(\tau) d\tau, \tag{2.9}$$

where $'$ is the symbol for derivative, Im *denotes the imaginary part of a complex -valued function and the corresponding integral converges absolutely.*

Proof. Since $f \in L_{1-\nu,p}(\mathbb{R}_+)$ we have from Theorem 86 in [5] that $f^*(s) \in L_q(1-\nu-i\infty, 1-\nu+i\infty)$, $q = p/(p-1)$. Recalling (2.5), our goal now is to verify conditions of the Lebedev expansion theorem in [2] in order to prove the following inversion formula

$$\int_x^\infty \varphi_\alpha(y)dy = \frac{2}{\pi^2\sqrt{\pi}} \int_0^\infty \tau \sinh(2\pi\tau) K^2_{i\tau}(\sqrt{x})(F_\alpha f)(\tau)d\tau, \ x > 0. \tag{2.10}$$

To do this, we show that $\varphi_\alpha \in L_{3/4,1}((0,1)) \cap L_{5/4,1}((1,\infty))$. Indeed, the integrability condition, estimate (2.8) and analyticity of $f^*(s)$ in the strip $\mathrm{Re}s = 1-\nu \in (0, \alpha+1)$ allow to to move the contour in (2.6) by Cauchy's theorem , keeping the same value. Therefore, choosing $\nu \in (1/4, 1)$, we use the Hölder inequality to find

$$||\varphi_\alpha||_{3/4,1} = \int_0^1 |\varphi_\alpha(x)| x^{-1/4} dx \le ||\varphi_\alpha||_{1-\nu,p} \left(\int_0^1 x^{(\nu-1/4)q-1} dx \right)^{1/q} = \frac{||\varphi_\alpha||_{1-\nu,p}}{[(\nu-1/4)q]^{1/q}} < \infty.$$

In the meantime, taking $\nu \in (-\alpha, -1/4)$, we have

$$||\varphi_\alpha||_{5/4,1} = \int_1^\infty |\varphi_\alpha(x)| x^{1/4} dx \le ||\varphi_\alpha||_{1-\nu,p} \left(\int_1^\infty x^{(\nu+1/4)q-1} dx \right)^{1/q} = \frac{||\varphi_\alpha||_{1-\nu,p}}{[-(\nu+1/4)q]^{1/q}} < \infty.$$

Therefore, substituting (2.6) in the left-hand side of (2.10) and making the integration with respect to y via Fubini's theorem and a simple substitution, we obtain

$$-\frac{1}{2\pi i} \int_{-\nu-i\infty}^{-\nu+i\infty} \frac{\Gamma(\alpha-s)\Gamma(s+1)}{\Gamma(1+s+\alpha)\Gamma(1-s)} f^*(1+s) x^{-s} ds$$

$$= \frac{2}{\pi^2\sqrt{\pi}} \int_0^\infty \tau \sinh(2\pi\tau) K^2_{i\tau}(\sqrt{x})(F_\alpha f)(\tau)d\tau. \tag{2.11}$$

Further, taking the Mellin transform from both sides of (2.11), basing on the condition $(F_\alpha f)(\tau) \in L_1(\mathbb{R}_+; \tau e^{2\pi\tau} d\tau)$ and the uniform estimate $|K_{i\tau}(\sqrt{x})| \le K_0(\sqrt{x})$ for the modified Bessel function, we change the order of integration due to the absolute and uniform convergence and appeal to relation (8.4.23.27) in [4], Vol. III to calculate the inner integral with respect to x in the right-hand side of the obtained equality. Hence, recalling the reduction formula for the gamma-function, we end up with the equality

$$sf^*(1+s) = -\frac{1}{\pi^2} \frac{\Gamma(1+s+\alpha)\Gamma(1-s)}{\Gamma(\alpha-s)\Gamma(s+1/2)} \int_0^\infty \tau \sinh(2\pi\tau) \Gamma(s+i\tau)\Gamma(s-i\tau)(F_\alpha f)(\tau)d\tau. \tag{2.12}$$

Then, reciprocally, via formula (2.3) and properties of the Mellin transform we find

$$xf(x) = \frac{1}{\pi^2} \int_0^\infty \tau \sinh(2\pi\tau) S(x,\tau)(F_\alpha f)(\tau)d\tau, \tag{2.13}$$

where

$$S(x,\tau) = -\frac{1}{2\pi i} \int_x^\infty \int_{\gamma-i\infty}^{\gamma+i\infty} \Gamma(s+i\tau)\Gamma(s-i\tau) \frac{\Gamma(1+s+\alpha)\Gamma(1-s)}{\Gamma(\alpha-s)\Gamma(s+1/2)} y^{-s-1} ds dy \tag{2.14}$$

and $\gamma \in (0,1)$. The kernel $S(y,\tau)$ can be calculated, recalling relation (8.4.23.25) in [4], Vol. III, and using repeated differentiation under the integral sign. In fact, via relation (8.4.23.25) we have that

$$\frac{\sqrt{\pi}}{i\sinh(\pi\tau)}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right]$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1-s)}{\Gamma(1+\alpha-s)\Gamma(s+1/2)}x^{-s}ds,\ x>0,\ \tau\in\mathbb{R}\backslash\{0\}. \tag{2.15}$$

Hence, involving the repeated differentiation and the reduction formula for the gamma-function, it is not difficult to verify the equalities

$$-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(1+s+\alpha)\Gamma(1-s)}{\Gamma(\alpha-s)\Gamma(s+1/2)}x^{-s-1}ds$$

$$= \frac{\sqrt{\pi}}{i\sinh(\pi\tau)}x^{\alpha}\frac{d}{dx}\,x^{1-2\alpha}\frac{d}{dx}\,x^{\alpha}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right]$$

$$= \frac{\sqrt{\pi}}{i\sinh(\pi\tau)}\left[\frac{d}{dx}\,x\frac{d}{dx}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right]\right.$$

$$\left.-\frac{\alpha^2}{x}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right]\right].$$

Therefore, taking into account the asymptotic behaviour of the function

$$x\frac{d}{dx}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right] = o(1),\ x\to+\infty,$$

which can be established, for instance, from the integral representation (2.15) and Stirling asymptotic formula for the gamma-function, we return to (2.14) to obtain

$$S(x,\tau) = -\frac{\sqrt{\pi}}{i\sinh(\pi\tau)}\left[2\alpha^2\int_{\sqrt{x}}^{\infty}\left[K_{\alpha+i\tau}(y)\,I_{\alpha-i\tau}(y) - K_{\alpha-i\tau}(y)\,I_{\alpha+i\tau}(y)\right]\frac{dy}{y}\right.$$

$$\left.+x\frac{d}{dx}\left[K_{\alpha+i\tau}(\sqrt{x})\,I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\,I_{\alpha+i\tau}(\sqrt{x})\right]\right]. \tag{2.16}$$

However, the integral in (2.16) can be treated, employing relations (1.12.4.4), (2.16.28.3) in [4], Vol. II and asymptotic formulae (1.4), (1.5), (1.6), (1.7) for the modified Bessel functions. Then we derive

$$\int_{\sqrt{x}}^{\infty}\left[K_{\alpha+i\tau}(y)\,I_{\alpha-i\tau}(y) - K_{\alpha-i\tau}(y)\,I_{\alpha+i\tau}(y)\right]\frac{dy}{y}$$

$$= \lim_{\varepsilon\to0+}\left(\int_0^{\infty} - \int_0^{\sqrt{x}}\right)\left[K_{\alpha+i\tau}(y)\,I_{\alpha+\varepsilon-i\tau}(y) - K_{\alpha-i\tau}(y)\,I_{\alpha+\varepsilon+i\tau}(y)\right]\frac{dy}{y}$$

$$= \frac{i}{2\alpha\tau} - \lim_{\varepsilon\to0+}\int_0^{\sqrt{x}}\left[K_{\alpha+i\tau}(y)\,I_{\alpha+\varepsilon-i\tau}(y) - K_{\alpha-i\tau}(y)\,I_{\alpha+\varepsilon+i\tau}(y)\right]\frac{dy}{y}$$

$$= \frac{i}{2\alpha\tau} + \frac{i\sqrt{x}}{4\alpha\tau}\Big[K'_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x}) - K_{\alpha+i\tau}(\sqrt{x})\, I'_{\alpha-i\tau}(\sqrt{x})\Big]$$

$$+\frac{i\sqrt{x}}{4\alpha\tau}\Big[K'_{\alpha-i\tau}(\sqrt{x})\, I_{\alpha+i\tau}(\sqrt{x}) - K_{\alpha-i\tau}(\sqrt{x})\, I'_{\alpha+i\tau}(\sqrt{x})\Big]$$

$$= \frac{i}{2\alpha\tau} + \frac{ix}{2\alpha\tau}\frac{d}{dx}\Big[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x}) + K_{\alpha-i\tau}(\sqrt{x})\, I_{\alpha+i\tau}(\sqrt{x})\Big]$$

$$-\frac{i\sqrt{x}}{2\alpha\tau}\Big[K_{\alpha+i\tau}(\sqrt{x})\, I'_{\alpha-i\tau}(\sqrt{x}) + K_{\alpha-i\tau}(\sqrt{x})\, I'_{\alpha+i\tau}(\sqrt{x})\Big].$$

Hence, combining with (2.16), we find

$$S(x,\tau) = -\frac{\alpha\sqrt{\pi}}{\tau\sinh(\pi\tau)}\Bigg[1 + 2x\frac{d}{dx}\Big(\mathrm{Re}\,\Big[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x})\Big]\Big)$$

$$-2\sqrt{x}\,\mathrm{Re}\,\Big[K_{\alpha+i\tau}(\sqrt{x})\, I'_{\alpha-i\tau}(\sqrt{x})\Big]\Bigg] - \frac{2x\sqrt{\pi}}{\sinh(\pi\tau)}\frac{d}{dx}\Big(\mathrm{Im}\,\Big[K_{\alpha+i\tau}(\sqrt{x})\, I_{\alpha-i\tau}(\sqrt{x})\Big]\Big).$$

Substituting this expression of $S(x,\tau)$ into (2.13), we come up with inversion formula (2.9), completing the proof of Theorem 2.2.

□

Remark 2.3. Letting formally $\alpha = 0$ in (2.9), we appeal to the relation (cf. [1], Vol. II) for the Macdonald function

$$K_{i\tau}(\sqrt{x}) = \frac{\pi}{\sinh(\pi\tau)}\mathrm{Im}\,\Big[I_{-i\tau}(\sqrt{x})\Big], \tag{2.17}$$

and making simple substitutions, we arrive at the Lebedev inversion formula (2.10), where $\varphi_0(x) = f(x)$ via (2.3) .

3 Index Transform (1.2)

In this section we investigate the boundedness and establish the inversion theorem for the Lebedev type transform (1.2).

Theorem 3.1. *Let* $\alpha \geq -1/4$, $g \in L_1(\mathbb{R};\ [\cosh(\pi\tau)]^{-1}d\tau)$. *Then* $x^{1/4}(G_\alpha g)(x)$ *is bounded continuous on* $\mathbb{R}_+$ *and*

$$\sup_{x>0}\ x^{1/4}|(G_\alpha g)(x)| \leq C\|g\|_{L_1(\mathbb{R};\ [\cosh(\pi\tau)]^{-1}d\tau)}. \tag{3.1}$$

Besides, if $g \in L_1(\mathbb{R})$ *and* $(G_\alpha g)(x) \in L_{\nu,1}((0,1))$, $\max(-\alpha,0) < \nu < 1/2$, *then for all* $y > 0$

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(s)\Gamma(1+\alpha-s)}{\Gamma(s+\alpha)}(G_\alpha g)^*(s)y^{-s}ds \;=\; \sqrt{\pi}\int_{-\infty}^{\infty}e^{y/2}\,K_{i\tau}\Big(\frac{y}{2}\Big)\frac{g(\tau)}{\cosh(\pi\tau)}d\tau. \tag{3.2}$$

Proof. Doing similarly as in the proof of Theorem 2.1, we employ (1.8) to have an immediate estimate

$$|(G_\alpha g)(x)| \le C\, x^{-1/4} \int_{-\infty}^{\infty} \frac{|g(\tau)|}{\cosh(\pi\tau)} d\tau = C\|g\|_{L_1(\mathbb{R};\, [\cosh(\pi\tau)]^{-1}d\tau)},$$

which yields (3.1). In order to prove the equality (3.2), we use the uniform estimate for the kernel (1.10), which can be obtained from the Mellin-Barnes representation (1.9). Indeed, by definition of the Euler beta-function [1], Vol. I and Stirling asymptotic formula for the gamma-function we have for $x > 0$

$$\begin{aligned}\left|\Phi_{\alpha,\tau}(x)\right| &\le \frac{x^{-\gamma}}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left|\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}\right| ds \\ &\le \frac{x^{-\gamma}B(\gamma,\gamma)}{2\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} \left|\Gamma(2s)\frac{\Gamma(s+\alpha)\Gamma(1/2-s)}{\Gamma(1+\alpha-s)\Gamma(s)}\right| ds = C\, x^{-\gamma} \end{aligned} \tag{3.3}$$

where $\max(-\alpha, 0) < \gamma < 1/2$. Hence applying the Mellin transform (2.1) to both sides of (1.2), we change the order of integration in the right-hand side of the obtained equality by Fubini's theorem. The Mellin transform of its left-hand side exists under the condition $(G_\alpha g)(x) \in L_{\nu,1}((0,1))$, $\max(-\alpha, 0) < \nu < 1/2$ and estimate (3.3), which guarantees the integrability of $(G_\alpha g)(x)$ over $(1, \infty)$. Therefore we end up with the equality

$$\frac{\Gamma(s)\Gamma(1+\alpha-s)}{\Gamma(s+\alpha)}(G_\alpha g)^*(s) = \Gamma(1/2-s) \int_{-\infty}^{\infty} g(\tau)\Gamma(s+i\tau)\Gamma(s-i\tau)d\tau.$$

Finally, the inverse Mellin transform (2.3) and relation (8.4.23.5) in [4], Vol. III will drive us to the equality (3.2), completing the proof of Theorem 3.1.

□

Now we are ready to prove the inversion formula for the index transform (1.2).

Theorem 3.2. *Let $g(z/i)$ be an even analytic function in the strip*

$$D = \{z \in \mathbb{C} :\ |\mathrm{Re} z| < \beta < 1/2\}, g(0) = g'(0) = 0$$

and $g(z/i)$ is absolutely integrable over any vertical line in D. Then under conditions of Theorem 3 for all $x \in \mathbb{R}$ the following inversion formula holds

$$\begin{aligned} g(x) = \frac{\cosh(\pi x)}{\pi\sqrt{\pi}} \lim_{\varepsilon\to 0} \int_0^\infty \Bigg[y^\alpha \left|\frac{\Gamma(\varepsilon-1-\alpha+ix)}{\Gamma(ix)}\right|^2 \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)\Gamma(\varepsilon-1/2-\alpha)} \\ \times {}_2F_3\left(1+\alpha,\ \frac{3}{2}+\alpha-\varepsilon;\ 1+2\alpha,\ 2+\alpha-\varepsilon-ix,\ 2+\alpha-\varepsilon+ix;\ y\right) \\ + \frac{y^{\varepsilon-1}}{\sqrt{\pi}} \mathrm{Re}\left[\left(\frac{y}{4}\right)^{ix} \frac{\Gamma(\varepsilon+ix)}{\Gamma(ix)}\, \Gamma(1-\varepsilon+\alpha-ix)\right. \\ \left. \times {}_2F_3\left(\varepsilon+ix,\ \frac{1}{2}+ix;\ 1+2ix,\ \varepsilon-\alpha+ix,\ \varepsilon+\alpha+ix;\ y\right)\right]\Bigg] (G_\alpha g)(y)dy, \end{aligned} \tag{3.4}$$

where ${}_2F_3(a_1,\ a_2;\ b_1,\ b_2,\ b_3; z)$ is the generalized hypergeometric function [1], Vol. I.

Proof. In fact, multiplying both sides of (3.2) by $e^{-y/2}K_{ix}(y/2)\,y^{\varepsilon-1}$ for some positive $\varepsilon \in (0,1)$ we integrate with respect to y over $(0,\infty)$. Hence, since under conditions of the theorem $(G_\alpha g)^*(s)$ is bounded, we change the order of integration in the left-hand side of the obtained equality and appeal to the relation (8.4.23.3) in [4], Vol. III. Therefore, for $\nu \in (0, \min(\varepsilon,\ 1+\alpha))$

$$\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(\varepsilon-s+ix)\Gamma(\varepsilon-s-ix)\Gamma(s)\Gamma(1+\alpha-s)}{\Gamma(1/2+\varepsilon-s)\Gamma(s+\alpha)}(G_\alpha g)^*(s)ds$$

$$=\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-\infty}^{\infty}K_{i\tau}\left(\frac{y}{2}\right)\frac{g(\tau)}{\cosh(\pi\tau)}d\tau dy. \tag{3.5}$$

In the meantime, the right-hand side of (3.5) can be treated, taking into account the evenness of g and representation (2.17) for the Macdonald function. Indeed, we have

$$\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-\infty}^{\infty}K_{i\tau}\left(\frac{y}{2}\right)\frac{g(\tau)}{\cosh(\pi\tau)}d\tau dy$$

$$=2\pi i\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-i\infty}^{i\infty}I_z\left(\frac{y}{2}\right)\frac{g(z/i)}{\sin(2\pi z)}dz\,dy. \tag{3.6}$$

On the other hand, according to our assumption $g(z/i)$ is analytic in the vertical strip $0 \le \mathrm{Re}z < \beta < 1/2$, $g(0) = g'(0) = 0$ and integrable in the strip. Hence, appealing to the inequality for the modified Bessel function of the first kind (see [6], p. 93)

$$|I_z(y)| \le I_{\mathrm{Re}z}(y)\ e^{\pi|\mathrm{Im}z|/2},\ 0 < \mathrm{Re}z < \beta,$$

one can move the contour to the right in the latter integral in (3.6). Then

$$2\pi i\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-i\infty}^{i\infty}I_z\left(\frac{y}{2}\right)\frac{g(z/i)}{\sin(2\pi z)}dz\,dy$$

$$=2\pi i\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{\beta-i\infty}^{\beta+i\infty}I_z\left(\frac{y}{2}\right)\frac{g(z/i)}{\sin(2\pi z)}dz\,dy.$$

Now $\mathrm{Re}z > 0$, and it is possible to pass to the limit under the integral sign when $\varepsilon \to 0$ and to change the order of integration due to the absolute and uniform convergence. Recalling the relation (2.16.28.3) in [4], Vol. II, we find

$$\lim_{\varepsilon\to0}2\pi i\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-i\infty}^{i\infty}I_z\left(\frac{y}{2}\right)\frac{g(z/i)}{\sin(2\pi z)}dz\,dy$$

$$=2\pi i\int_{\beta-i\infty}^{\beta+i\infty}\frac{g(z/i)}{(x^2+z^2)\sin(2\pi z)}dz=\pi i\left(\int_{-\beta+i\infty}^{-\beta-i\infty}+\int_{\beta-i\infty}^{\beta+i\infty}\right)\frac{g(z/i)\,dz}{(z-ix)\,z\sin(2\pi z)}. \tag{3.7}$$

Hence, using the Cauchy formula in the right-hand side of the latter equality in (3.7) under conditions of the theorem, we derive

$$\lim_{\varepsilon\to0}2\pi i\int_0^\infty K_{ix}\left(\frac{y}{2}\right)y^{\varepsilon-1}\int_{-i\infty}^{i\infty}I_z\left(\frac{y}{2}\right)\frac{g(z/i)}{\sin(2\pi z)}dz\,dy=\frac{2\pi^2\,g(x)}{x\sinh(2\pi x)},\quad x\in\mathbb{R}\backslash\{0\}. \tag{3.8}$$

Thus, returning to (3.5), employing the Parseval identity (2.2) and passing to the limit when $\varepsilon \to 0$, we come up with the equality

$$\frac{2\pi^2\, g(x)}{x\sinh(2\pi x)} = \lim_{\varepsilon\to 0}\int_0^\infty S_\varepsilon(x,y)(G_\alpha g)(y)dy, \tag{3.9}$$

where

$$S_\varepsilon(x,y) = \frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} \frac{\Gamma(\varepsilon-1+s+ix)\Gamma(\varepsilon-1+s-ix)\Gamma(s+\alpha)\Gamma(1-s)}{\Gamma(\varepsilon-1/2+s)\Gamma(1-s+\alpha)} y^{-s}ds. \tag{3.10}$$

Meanwhile, integral (3.10) can be calculated with the use of Slater's theorem [3] in terms of the generalized hypergeometric functions ${}_2F_3$. Namely, it involves the left-hand simple poles of the gamma-functions $s = 1-\varepsilon \pm ix - n,\ s = -\alpha - n,\ n \in \mathbb{N}_0$. Consequently, after straightforward calculations we express the kernel $S_\varepsilon(x,y)$ in the form

$$S_\varepsilon(x,y) = y^\alpha \frac{\Gamma(1+\alpha)|\Gamma(\varepsilon-1-\alpha+ix)|^2}{\Gamma(1+2\alpha)\Gamma(\varepsilon-1/2-\alpha)}$$

$$\times {}_2F_3\left(1+\alpha,\ \frac{3}{2}+\alpha-\varepsilon;\ 1+2\alpha,\ 2+\alpha-\varepsilon-ix,\ 2+\alpha-\varepsilon+ix;\ y\right)$$

$$+\frac{y^{\varepsilon-1}}{\sqrt{\pi}}\mathrm{Re}\left[\left(\frac{y}{4}\right)^{ix}\Gamma(1-\varepsilon+\alpha-ix)\Gamma(\varepsilon+ix)\Gamma(-ix)\right.$$

$$\left.\times {}_2F_3\left(\varepsilon+ix,\ \frac{1}{2}+ix;\ 1+2ix,\ \varepsilon-\alpha+ix,\ \varepsilon+\alpha+ix;\ y\right)\right].$$

Substituting this value in (3.9) and using the reduction formula for the gamma-function, we end up with the inversion formula (3.4), completing the proof.

□

4 Initial value problem

In this section the index transform (1.2) is employed to investigate the solvability of an initial value problem for the following fourth order partial differential equation, involving the Laplacian

$$\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2\right)^2+\alpha^2\right]\Delta u - \frac{\sqrt{x^2+y^2}+2\alpha^2}{x^2+y^2}\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]^2 u$$

$$-\frac{3}{2\sqrt{x^2+y^2}}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]+\frac{u}{2\sqrt{x^2+y^2}}=0,\ (x,y)\in\mathbb{R}^2\backslash\{0\}, \tag{4.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplacian in $\mathbb{R}^2$. In fact, writing (4.1) in polar coordinates (r,θ), we end up with the equation

$$r^3\frac{\partial^4 u}{\partial r^4}+r\,\frac{\partial^4 u}{\partial r^2\partial\theta^2}+6r^2\,\frac{\partial^3 u}{\partial r^3}+\frac{\partial^3 u}{\partial r\,\partial\theta^2}+r\,(7-\alpha^2-r)\frac{\partial^2 u}{\partial r^2}$$

$$+\frac{\alpha^2}{r}\,\frac{\partial^2 u}{\partial\theta^2}+\left(1-\alpha^2-\frac{5}{2r}\right)\frac{\partial u}{\partial r}+\frac{u}{2}=0. \tag{4.2}$$

Lemma 4.1. *Let* $\alpha \geq -1/4, g(\tau) \in L_1\left(\mathbb{R}; e^{(\beta-\pi)|\tau|}d\tau\right)$, $\beta \in (0, 2\pi)$. *Then the function*

$$u_\alpha(r,\theta) = 2\sqrt{\pi}\int_{-\infty}^{\infty} e^{\theta\tau}\,\mathrm{Re}\left[K_{\alpha+i\tau}(\sqrt{r})\,I_{\alpha-i\tau}(\sqrt{r})\right]\frac{g(\tau)}{\cosh(\pi\tau)}d\tau \tag{4.3}$$

satisfies the partial differential equation (4.2) *on the wedge* $(r,\theta) : r > 0,\ 0 \leq \theta < \beta$, *vanishing at infinity.*

Proof. The proof is straightforward by substitution (4.3) into (4.2) and the use of (1.11). The necessary differentiation with respect to r and θ under the integral sign is allowed via the absolute and uniform convergence, which can be verified using inequality (3.1) and the integrability condition $g \in L_1\left(\mathbb{R}; e^{(\beta-\pi)|\tau|}d\tau\right)$, $\beta \in (0, 2\pi)$ of the lemma. Finally, the condition $u(r,\theta) \to 0,\ r \to \infty$ is again due to (3.1). □

We are ready to formulate the initial value problem for equation (4.2) and give its solution.

Theorem 4.2. *Let* $g(x)$ *be given by formula* (3.4) *and its transform* $(G_\alpha g)(t) \equiv G_\alpha(t)$ *satisfies conditions of Theorem 3.1. Then* $u(r,\theta),\ r > 0,\ 0 \leq \theta < \beta$ *by formula* (4.3) *will be a solution of the initial value problem for the partial differential equation* (4.2) *subject to the initial condition*

$$u_\alpha(r,0) = G_\alpha(r).$$

Finally we will pay our attention to the so-called generalized Lebedev index transform recently considered by the author [8], which contains the square modulus of the Macdonald function as the kernel

$$\Psi_\alpha(r,\theta) = \int_{-\infty}^{\infty} e^{\theta\tau}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 g(\tau)d\tau, \tag{4.4}$$

where $\alpha \in \mathbb{R},\ r > 0,\ 0 \leq \theta \leq 2\pi$. Namely, we will show that the kernel $\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2$ satisfies differential equation (1.11) and, correspondingly, the index transform (4.4) is a solution of the PDE (4.2) under the boundary condition

$$\lim_{r\to\infty}\Psi_\alpha(r,\theta) = 0. \tag{4.5}$$

Lemma 4.3. *Let* $\alpha,\ \tau \in \mathbb{R}$. *The kernel* $\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2$ *is a fundamental solution of the differential equation* (1.11).

Proof. Taking the integral representation for the kernel in terms of the Mellin-Barnes integral via relation (8.4.23.31) in [4], Vol. III, we have

$$\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 = \frac{1}{4i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(1/2+s)\Gamma(s)}r^{-s}ds, \tag{4.7}$$

where $\gamma > |\alpha|$. Hence, since the integrand in (4.7) behaves at infinity as

$$\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(1/2+s)\Gamma(s)} = O\left(e^{-\pi|t|}|t|^{2\gamma-3/2}\right),\ s = \gamma+it,\quad |t| \to \infty,$$

the repeated differentiation with respect to r under the integral sign is permitted. Therefore following the same scheme as in the proof of Lemma 1.1, we derive

$$\left(r\frac{d}{dr}\right)^2 \left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 = \frac{1}{4i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{s^2\,\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(1/2+s)\Gamma(s)}r^{-s}ds$$

$$= -\tau^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 + \frac{1}{4i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(1+s+i\tau)\Gamma(1+s-i\tau)\frac{\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(1/2+s)\Gamma(s)}r^{-s}ds$$

$$= -\tau^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 + \frac{1}{4i\sqrt{\pi}}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{\Gamma(s-1+\alpha)\Gamma(s-1-\alpha)}{\Gamma(s-1/2)\Gamma(s-1)}r^{1-s}ds$$

$$= -\tau^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2 + \frac{1}{4i\sqrt{\pi}}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{(s-1/2)(s-1)\Gamma(s+\alpha)\Gamma(s-\alpha)}{(s-1+\alpha)(s-1-\alpha)\Gamma(s+1/2)\Gamma(s)}r^{1-s}ds.$$

Hence, shifting the contour to the left by Cauchy's theorem, we multiply the latter equality by $r^{-\alpha}$ and differentiate it with respect to r again. Then

$$\frac{d}{dr}\left[r^{-\alpha}\left(r\frac{d}{dr}\right)^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right] = -\tau^2\frac{d}{dr}\left[r^{-\alpha}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]$$

$$-\frac{1}{4i\sqrt{\pi}}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{(s-1/2)(s-1)\Gamma(s+\alpha)\Gamma(s-\alpha)}{(s-1-\alpha)\Gamma(s+1/2)\Gamma(s)}r^{-s-\alpha}ds.$$

In the same fashion we deduce the equality

$$\frac{d}{dr}\left[r^{1+2\alpha}\frac{d}{dr}\left[r^{-\alpha}\left(r\frac{d}{dr}\right)^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]\right] = -\tau^2\frac{d}{dr}\left[r^{1+2\alpha}\frac{d}{dr}\left[r^{-\alpha}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]\right]$$

$$+\frac{1}{4i\sqrt{\pi}}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{(s-1/2)(s-1)\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(s+1/2)\Gamma(s)}r^{-s+\alpha}ds.$$

Consequently, minding (4.7), we have

$$r^{-\alpha}\frac{d}{dr}\left[r^{1+2\alpha}\frac{d}{dr}\left[r^{-\alpha}\left(r\frac{d}{dr}\right)^2\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]\right] = -\tau^2 r^{-\alpha}\frac{d}{dr}\left[r^{1+2\alpha}\frac{d}{dr}\left[r^{-\alpha}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]\right]$$

$$-\frac{1}{4i\sqrt{\pi}}\frac{d}{dr}\int_{1+\gamma-i\infty}^{1+\gamma+i\infty}\Gamma(s+i\tau)\Gamma(s-i\tau)\frac{(s-1/2)\Gamma(s+\alpha)\Gamma(s-\alpha)}{\Gamma(s+1/2)\Gamma(s)}r^{1-s}ds$$

$$= -\tau^2 r^{-\alpha}\frac{d}{dr}\left[r^{1+2\alpha}\frac{d}{dr}\left[r^{-\alpha}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right]\right] + \frac{1}{2}\frac{d}{dr}\left[r\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2\right] + \frac{d}{dr}r^2\frac{d}{dr}\left|K_{\alpha+i\tau}(\sqrt{r})\right|^2.$$

Thus fulfilling the differentiation, we come again to (1.11).

□

Analogously to Lemma 4.1 one proves

Theorem 4.4. *Let $\alpha\in\mathbb{R}, g(\tau)\in L_1\left(\mathbb{R};e^{(\beta-\pi)|\tau|}d\tau\right)$, $\beta\in(0,2\pi)$. Then the function* (4.4) *satisfies the partial differential equation* (4.2) *on the wedge $(r,\theta): r>0,\ 0\le\theta<\beta$, vanishing at infinity.*

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