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A Note on the Inhomogeneous Schrödinger Equation with Mixed Power Nonlinearity

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Abstract

We investigate the initial value problem for an inhomogeneous nonlinear Schrödinger equation with a combined power nonlinearity. We prove global well-posedness in the defocusing case. In the focusing case, we prove existence of ground state and nonlinear instability of standing waves.

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1 Introduction

Consider the nonlinear Schrödinger equation with an inhomogeneous combined power nonlinearity

$$iu_t + \Delta u + \varepsilon(|x|^b|u|^{p-1}u + |x|^b|u|^{q-1}u) = 0, \tag{1.1}$$

with initial data

$$u(0,.) = u_0. \tag{1.2}$$

Here and hereafter $\varepsilon \in \{\pm 1\}$, $N \geq 2$, $b > 0$, $1 < p < q$ and $u := u(t,x)$ is a complex-valued function of the variable $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$.

When $b = 0$, the equation (1.1) models the propagation of intense laser beams in an homogeneous bulk medium with a Kerr nonlinearity. It was suggested that stable high power propagation can be achieved in plasma by sending a preliminary laser beam that creates a channel with a reduced electron density, and thus reduces the nonlinearity inside the channel [9]. Equation (1.1) describes the beam propagation in an inhomogeneous medium, where

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u is the electric field in laser optics and $|x|^b$ is proportional to the electric density [12]. A basic physical question is when can the condensate be unstable to collapse or exist for all time?

A solution u to (1.1) formally satisfies conservation of the mass and the energy

$$M(t) = M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^N)} = M(0);$$

$$E(t) = E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)} - \epsilon \int_{\mathbb{R}^N} |x|^b \Big(\frac{|u(t)|^{p+1}}{p+1} + \frac{|u(t)|^{q+1}}{q+1} \Big) dx = E(0).$$

If $\varepsilon = -1$, the energy is always positive and we say that (1.1) is *defocusing*. Otherwise, (1.1) is said to be focusing.

The Cauchy problem and the stability of standing waves for the inhomogeneous nonlinear Schrödinger equation (INLS-equation) have been studied extensively, in particular Merle [14] proved the existence and nonexistence of blow-up solutions to

$$iu_t + \Delta u + V(x)|u|^{p-2}u = 0, \quad x \in \mathbb{R}^N, \tag{1.3}$$

in the case of critical power $p = 2 + \frac{4}{N}$ and where V is bounded. Later on Fibich, Liu and Wang [7], proved the stability and instability of standing waves for (1.3) under the assumptions $p \geq 2 + \frac{4}{N}$, $V(x) = V(\epsilon|x|)$ with small $\epsilon > 0$ and $V \in C^4 \cap L^\infty$. In the same context, Fukuizumi and Ohta [8] obtained the instability of standing waves for the equation (1.3) when the inhomogeneity V behaves like $|x|^{-b}$ at infinity with $0 < b < 2$.

When V is unbounded, for example, $V(|x|) = |x|^b$, $b > 0$ it seems that the standard Gagliardo-Nirenberg inequality cannot be used any more. Recently, Chen and Guo [3, 4], established a variant of interpolation inequality and used it to study (1.3).

In the present paper we study well-posedness issues of the inhomogeneous nonlinear Schrödinger equation (1.1) with combined power-type nonlinearity. Our aim is to establish local and global existence of solution, then we prove existence of ground state solution in the focusing case and study the nonlinear instability of standing waves.

The plan of the paper is as follows. In the second section, we derive the main results and some useful tools . Section three contains a proof of local and global existence of a solution to the equation (1.1). Section four is devoted to establish existence of a ground state. In the last section, we prove instability of standing waves.

Here and hereafter, we denote the Lebesgue space $L^p := L^p(\mathbb{R}^N)$, the Sobolev space $H^1 := H^1(\mathbb{R}^N)$ and $\int .dx := \int_{\mathbb{R}^N} .dx$. If X is an abstract space, we denote $X_{rd} := \{u \in X,\ u(x) = u(|x|)\}$,. We mention that C is an absolute positive constant which may vary from line to line. If A and B are nonnegative real numbers, $A \lesssim B$ means that $A \leq CB$. Finally, we denote $p_c := 1 + \frac{2b+4}{N}$,$p_0 := max(1 + \frac{2b}{N-1}, p_c)$ and

$$\widetilde{p} := \begin{cases} \frac{N+2}{N-2} + \frac{2b}{N-1}, & if \quad N \geq 3; \\ +\infty, & if \quad N = 2. \end{cases}$$

2 Background and Main results

In this section we list the main results proved in this paper and some technical tools needed in the sequel. First, let us give some notations. For $\phi \in H^1_{rd}$ and $(\alpha,\beta) \in \mathbb{R}^2$, we define the quantities

$$J(\phi) := \frac{1}{2}\|\phi\|^2_{H^1} - \frac{1}{p+1}\int |x|^b|\phi|^{p+1}\,dx - \frac{1}{q+1}\int |x|^b|\phi|^{q+1}dx;$$

$$\phi^\lambda := e^{\alpha\lambda}\phi(e^{-\beta\lambda}),\quad \mathcal{L}_{\alpha,\beta}J(\phi) := \partial_\lambda(J(\phi^\lambda))_{|\lambda=0},\quad K_{\alpha,\beta} := \mathcal{L}_{\alpha,\beta}J,\quad H_{\alpha,\beta} := J - \frac{1}{2\alpha+N\beta}K_{\alpha,\beta}.$$

With a direct computation, we have

$$\begin{aligned} K_{\alpha,\beta}(\phi) = {} & \frac{2\alpha+(N-2)\beta}{2}\|\nabla\phi\|^2_{L^2} + \frac{2\alpha+N\beta}{2}\|\phi\|^2_{L^2} - \frac{\alpha(p+1)+(N+b)\beta}{p+1}\int |x|^b|\phi|^{p+1}\,dx \\ & - \frac{\alpha(q+1)+(N+b)\beta}{q+1}\int |x|^b|\phi|^{q+1}\,dx; \end{aligned}$$

$$H_{\alpha,\beta}(\phi) = \frac{\beta}{2\alpha+N\beta}\|\nabla\phi\|^2_{L^2} + \frac{\alpha(p-1)+b\beta}{(p+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{p+1}\,dx + \frac{\alpha(q-1)+b\beta}{(q+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{q+1}\,dx.$$

We denote the quadratic and nonlinear parts of $K_{\alpha,\beta}$,

$$K^Q_{\alpha,\beta}(\phi) := \frac{2\alpha+(N-2)\beta}{2}\|\nabla\phi\|^2_{L^2} + \frac{2\alpha+N\beta}{2}\|\phi\|^2_{L^2},\quad K^N_{\alpha,\beta} := K_{\alpha,\beta} - K^Q_{\alpha,\beta}.$$

Now, we list the main result proved in this paper.

2.1 Main results

Let us start with local well-posedness of the problem (1.1).

Theorem 2.1. *Let $N \geq 2$, $b > 0$ and $1 + \frac{2b}{N-1} + \frac{4}{N} \leq p < q < \widetilde{p}$. For any initial data $u_0 \in H^1_{rd}$, there exists $T > 0$ and a unique solution u to the Cauchy problem* (1.1)-(1.2) *in the energy space*

$$C([0,T], H^1_{rd}).$$

Moreover,

1. *$u \in L^\alpha((0,T), W^{1,\beta})$, where (α,β) is an admissible pair in the meaning of Definition 2.9;*
2. *u satisfies conservation of the energy and the mass;*
3. *if $q < p_c$, then u is global.*

In order to study the focusing problem associated to (1.1), we consider the stationary equation

$$\Delta\phi - \phi + |x|^b\phi|\phi|^{p-1} + |x|^b\phi|\phi|^{q-1} = 0,\quad 0 \neq \phi \in H^1_{rd}. \tag{2.1}$$

We prove that the previous elliptic problem has a ground state in the meaning that it has a nontrivial positive radial solution which minimizes the action J when $K_{\alpha,\beta}$ vanishes. Let the minimizing problem

$$m_{\alpha,\beta} := \inf_{0\neq\phi\in H^1_{rd}}\Big\{J(\phi),\ s.t\quad K_{\alpha,\beta} = 0\Big\}. \tag{2.2}$$

Theorem 2.2. *Take a couple of real numbers* $(\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, $b > 0$ *and* $N \geq 2$, *such that* $p_0 < p < q < \widetilde{p}$. *Then,*

(1) $m := m_{\alpha,\beta}$ *is nonzero and independent of* (α,β);

(2) there is a ground state solution to (1.1) *in the following meaning*

$$\Delta\phi - \phi + |x|^b\phi|\phi|^{p-1} + |x|^b\phi|\phi|^{q-1} = 0, \quad 0 \neq \phi \in H^1_{rd}, \quad m = J(\phi). \tag{2.3}$$

The last result established in this paper is about nonlinear instability of standing waves.

Theorem 2.3. *Take* $p_0 < p < q < \widetilde{p}$, *then the standing wave solution of* (1.1) *given by the previous Theorem is nonlinearly unstable.*

2.2 Technical tools

In what follows, we give some standard results needed in the sequel. Let us start some the Sobolev injections [6].

Lemma 2.4. *Let* $N \geq 1$ *and* $2 < p < \frac{2N}{N-2}$ *when* $N \geq 3$, $2 < p < \infty$ *when* $N \in \{1,2\}$. *Then, the following embedding is compact*

$$H^1_{rd} \hookrightarrow L^p. \tag{2.4}$$

Recall the so-called Gagliardo-Nirenberg inequality [1].

Lemma 2.5. *Let* $N \geq 2$ *and* $2 < q < q^*$, *where* $q^* = \frac{2N}{N-2}$ *when* $N \geq 3$ *and* $q^* = +\infty$ *when* $N = 2$. *Then there is a positive constant* $C_{N,q}$ *depending of* N, q *such that for any* $u \in H^1_{rd}$,

$$\int |u|^q\,dx \leq C_{N,q}\Big(\int |\nabla u|^2\,dx\Big)^{\frac{N(q-2)}{4}}\Big(\int |u|^2\,dx\Big)^{\frac{2q-N(q-2)}{4}}. \tag{2.5}$$

The following Strauss'inequality [16], will be useful along this paper.

Lemma 2.6. *Let* $N \geq 2$. *There is a constant* $C_N > 0$ *depending of* N *such that for any* $u \in H^1_{rd}$,

$$|x|^{\frac{N-1}{2}}|u(x)| \leq C_N\Big(\int |u|^2\,dx\Big)^{\frac{1}{4}}\Big(\int |\nabla u|^2\,dx\Big)^{\frac{1}{4}} \quad \textit{for almost any} \quad x \in \mathbb{R}^N. \tag{2.6}$$

Using the previous inequalities, we have [4].

Proposition 2.7. *Take* $N \geq 2$, $b > 0$ *and* $1 + \frac{2b}{(N-1)} < p < \widetilde{p}$. *Then there is a constant* $C_{N,p,b} > 0$ *depending only on* N, p *and* b *such that for any* $u \in H^1_{rd}$,

$$\int |x|^b|u|^{p+1}\,dx \leq C_{N,p,b}\Big(\int |\nabla u|^2\,dx\Big)^{\frac{N(p-1)-2b}{4}}\Big(\int |u|^2\,dx\Big)^{\frac{2(p+1)-[N(p-1)-2b]}{4}}. \tag{2.7}$$

Now, we give some estimates about solutions to semilinear Schrödinger equation. Let us start with the so-called Virial identity [10].

Proposition 2.8. *Let* $u_0 \in \Sigma := \{u \in H^1 \ s.t\ |x|u \in L^2\}$ *and* u *be the solution of the equation*

$$iu_t + \Delta_x u = f(u), \quad u(0,.) = u_0.$$

Then $u(t) \in \Sigma$ *for all* $t \in [0, T^*)$. *Moreover, the function*

$$h : t \mapsto \frac{1}{8}\|xu(t)\|_2^2$$

is of class C^2 *and satisfies for* $0 \le t < T^*$,

$$h''(t) = \int \Big[|\nabla u(t)|^2 - \frac{N(p-1)-2b}{2(p+1)}|x|^b|u(t)|^{p+1} - \frac{N(q-1)-2b}{2(q+1)}|x|^b|u(t)|^{q+1}\Big]dx. \tag{2.8}$$

Definition 2.9. A couple of real numbers (q,r) is admissible if

$$q, r \ge 2, \quad (q,r,N) \neq (2,\infty,2) \quad \text{and} \quad \frac{2}{q} + \frac{N}{r} = \frac{N}{2}.$$

Strichartz estimate [2] is a classical tool to study Schrödinger equation.

Proposition 2.10. *Let* $T > 0$ *and* $(q,r), (\alpha,\beta)$ *two admissible couples. Then, there exists a positive real number* C *such that*

$$\|u\|_{L^q_T(L^r)} \le C\Big(\|u(0,.)\|_{H^1_{rd}} + \|iu_t + \Delta_x u\|_{L^{\alpha'}_T(L^{\beta'})}\Big), \tag{2.9}$$

where (α', β') *is the Hölder conjugate of* (α, β).

In particular we have the following energy estimate.

Proposition 2.11. *For any* $T > 0$ *there exists* $C > 0$ *such that*

$$\sup_{t\in[0,T]} \|u(t,.)\|_{H^1_{rd}} \le C\Big(\|u(0,.)\|_{H^1_{rd}} + \|iu_t + \Delta_x u\|_{L^1_T(H^1_{rd})}\Big). \tag{2.10}$$

We end this section with the following classical Pohozaev [15] result.

Proposition 2.12. *Let* $\phi \in H^1_{rd}$ *a solution to* (2.1). *Then*

$$K_{\alpha,\beta}(\phi) = 0, \quad \text{for any} \quad \alpha, \beta \in \mathbb{R}.$$

3 Proof of Theorem 2.1

This section contains three parts, local existence, uniqueness and global existence in the defocusing case.

3.1 Local Existence

Using the fact that

$$1+\frac{2b}{N-1}+\frac{4}{N}<p<q<\tilde{p},$$

there exist $\sigma\in(0,\frac{2}{N-2})$ such that

$$\frac{2}{p-1-\frac{2b}{N-1}}<\frac{N}{2}<1+\frac{1}{\sigma_1}<\frac{2N}{(N-2)(q-1-\frac{2b}{N-1})}.$$

For $T>0$, we denote the space

$$\mathcal{E}_T:=C([0,T],H^1_{rd})\cap L^\alpha([0,T],W^{1,\beta})$$

endowed with the complete norm

$$\|u\|_T=\|u\|_{L^\infty_T(L^2)\cap L^\alpha_T(L^\beta)}+\|\nabla u\|_{L^\infty_T(L^2)\cap L^\alpha_T(L^\beta)},$$

where $\alpha:=\frac{4\sigma+4}{N\sigma}$ and $\beta:=2\sigma+2$. We denote by $\mathcal{B}_T(r)$ the closed ball in $\mathcal{E}_T$ with center zero and radius $r>0$. Let w to be the solution of the free Schrödinger equation

$$i\partial_t w+\Delta w=0,\quad w(0,.)=u_0.$$

We consider the map ψ defined on $\mathcal{B}_T(1)$ by $\psi(v)=:\widetilde{v}$, where

$$i\partial_t\widetilde{v}+\Delta\widetilde{v}=f(w+v),\quad \widetilde{v}(0,.)=0. \tag{3.1}$$

The source term stands for $f(u):=\epsilon|x|^b(u|u|^{p-1}+u|u|^{q-1})$. We prove that for $T>0$ sufficiently small, the map ψ is a contraction which leaves $\mathcal{B}_T(1)$ stable.
Let $u:=v+w$, applying Strichartz estimate (2.9), we get

$$\|\widetilde{v}\|_{L^\infty_T(L^2)\cap L^\alpha_T(L^\beta)}\lesssim\|f(u)\|_{L^{\alpha'}_T(L^{\beta'})}.$$

Taking $\theta:=\frac{2\sigma(2\sigma+2)}{2-(N-2)\sigma}$, yields

$$\frac{1}{\alpha'}=\frac{2\sigma}{\theta}+\frac{1}{\alpha},\quad \frac{1}{\beta'}=\frac{2\sigma}{\beta}+\frac{1}{\beta}.$$

Using Hölder inequality, we obtain

$$\begin{aligned}\|f(u)\|_{L^{\alpha'}_T(L^{\beta'})} &\leq \|u\|_{L^\alpha_T(L^\beta)}\||x|^b|u|^{p-1}+|x|^b|u|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\\ &\leq \|u\|_{L^\alpha_T(L^\beta)}\Big(\||x|^b|u|^{p-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}+\||x|^b|u|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\Big).\end{aligned}$$

Applying Strauss inequality (2.6), we get

$$\begin{aligned}(|x|^{\frac{N-1}{2}}|u|)^{\frac{b\beta}{\sigma(N-1)}} &\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}})^{\frac{b\beta}{\sigma(N-1)}},\\ &\lesssim (1+\|u_0\|_{H^1_{rd}})^{\frac{b\beta}{\sigma(N-1)}}.\end{aligned}$$

In the other side, we have

$$2 < \frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1}) = (1+\frac{1}{\sigma})(p-1-\frac{2b}{N-1}) < \frac{2N}{N-2}.$$

Thus, using Gagliardo-Nirenberg inequality (2.5), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}dx &\lesssim \|\nabla u\|_{L^2}^{\frac{N(\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})-2)}{2}} \|u\|_{L^2}^{\frac{(2\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})-N(\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})-2))}{2}} \\ &\lesssim (1+\|u_0\|_{H^1_{rd}})^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}.\end{aligned}$$

Writing

$$\||x|^b|u|^{p-1}\|_{L^{\frac{\beta}{2\sigma}}}^{\frac{\beta}{2\sigma}} = \int_{\mathbb{R}^N} (|x|^{\frac{N-1}{2}}|u|)^{\frac{b\beta}{\sigma(N-1)}} |u|^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}dx,$$

yields

$$\||x|^b|u|^{p-1}\|_{L^{\frac{\beta}{2\sigma}}} \lesssim (1+\|u_0\|_{H^1_{rd}})^{p-1}.$$

Then

$$\||x|^b|u|^{p-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{p-1}. \tag{3.2}$$

With the same way, we get

$$\||x|^b|u|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}.$$

Thus, we obtain

$$\begin{aligned}\|f(u)\|_{L_T^{\alpha'}(L^{\beta'})} &\lesssim T^{\frac{2\sigma}{\theta}}\Big((1+\|u_0\|_{H^1_{rd}})^{p-1}+(1+\|u_0\|_{H^1_{rd}})^{q-1}\Big)\|u\|_{L_T^{\alpha}(L^{\beta})} \\ &\lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|u\|_{L_T^{\alpha}(L^{\beta})}.\end{aligned}$$

Consequently

$$\|\widetilde{v}\|_{L_T^{\infty}(L^2)\cap L_T^{\alpha}(L^{\beta})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|u\|_{L_T^{\alpha}(L^{\beta})}.$$

On the other hand, since

$$1+\frac{2b}{N-1}+\frac{4}{N}+\frac{2}{N-1} < p < q < \tilde{p},$$

there exist $\sigma_1 \in (0, \frac{2}{N-2})$ such that

$$\frac{2}{p-1-\frac{2(b-1)}{N-1}} < \frac{N}{2} < 1+\frac{1}{\sigma_1} < \frac{2N}{(N-2)(q-1-\frac{2(b-1)}{N-1})}.$$

We denote also

$$\alpha_1 := \frac{4\sigma_1+4}{N\sigma_1}, \quad \beta_1 := 2\sigma_1+2, \quad \theta_1 := \frac{2\sigma_1(2\sigma_1+2)}{2-(N-2)\sigma_1}.$$

Applying Strichartz estimate (2.9) via Hölder inequality, we get

$$\begin{aligned}
\|\nabla\widetilde{v}\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)} &\lesssim \|\nabla(f(u))\|_{L_T^{\alpha'}(L^{\beta'})} \\
&\lesssim \|\nabla u\|_{L_T^\alpha(L^\beta)}\Big(\||x|^b|u|^{p-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} + \||x|^b|u|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\Big) \\
&+ \|u\|_{L_T^{\alpha_1}(L^{\beta_1})}\Big(\||x|^{b-1}|u|^{p-1}\|_{L_T^{\frac{\theta_1}{2\sigma}}(L^{\frac{\beta_1}{2\sigma_1}})} + \||x|^{b-1}|u|^{q-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L^{\frac{\beta_1}{2\sigma_1}})}\Big).
\end{aligned}$$

Arguing as previously, we obtain

$$\||x|^b|u|^{p-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{p-1}, \quad \||x|^b|u|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1};$$

$$\||x|^{b-1}|u|^{p-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L^{\frac{\beta}{2\sigma_1}})} \lesssim T^{\frac{2\sigma_1}{\theta_1}}(1+\|u_0\|_{H^1_{rd}})^{p-1}, \quad \||x|^{b-1}|u|^{q-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L^{\frac{\beta_1}{2\sigma_1}})} \lesssim T^{\frac{2\sigma_1}{\theta_1}}(1+\|u_0\|_{H^1_{rd}})^{q-1}.$$

So,

$$\begin{aligned}
\|\nabla(f(u))\|_{L_T^{\alpha'}(L^{\beta'})} &\lesssim T^{\frac{2\sigma}{\theta}}\Big((1+\|u_0\|_{H^1_{rd}})^{p-1} + (1+\|u_0\|_{H^1_{rd}})^{q-1}\Big)\|\nabla u\|_{L_T^\alpha(L^\beta)} \\
&+ T^{\frac{2\sigma_1}{\theta_1}+\frac{1}{\alpha_1}}\Big((1+\|u_0\|_{H^1_{rd}})^{p-1} + (1+\|u_0\|_{H^1_{rd}})^{q-1}\Big)\|u\|_{L_T^\infty(L^{\beta_1})}.
\end{aligned}$$

Then, using Sobolev injection (2.4), via the fact that $\sigma_1 \in (0, \frac{2}{N-2})$, it follows that

$$\|\nabla\widetilde{v}\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)} \lesssim (T^{\frac{2\sigma}{\theta}} + T^{\frac{1}{\alpha_1'}})(1+\|u_0\|_{H^1_{rd}})^{q-1}\|u\|_T.$$

Finally,

$$\begin{aligned}
\|\widetilde{v}\|_T &\lesssim (T^{\frac{2\sigma}{\theta}} + T^{\frac{1}{\alpha_1'}})(1+\|u_0\|_{H^1_{rd}})^{q-1}\|u\|_T \\
&\lesssim (T^{\frac{2\sigma}{\theta}} + T^{\frac{1}{\alpha_1'}})(1+\|u_0\|_{H^1_{rd}})^{q}.
\end{aligned}$$

This implies that for $T > 0$ sufficiently small ψ maps $\mathcal{E}_T(1)$ into itself.
Now we prove that ψ is a contraction. Let $v_1, v_2 \in \mathcal{B}_T(1)$. Taking account of Strichartz estimate, we get

$$\|\psi(v_1) - \psi(v_2)\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)} \lesssim \|f(u_1) - f(u_2)\|_{L_T^{\alpha'}(L^{\beta'})}.$$

Compute

$$\begin{aligned}
|f(u_1) - f(u_2)| &\leq |x|^b\Big(|u_1|u_1|^{p-1} - u_2|u_2|^{p-1}| + |u_1|u_1|^{q-1} - u_2|u_2|^{q-1}|\Big) \\
&\lesssim |x|^b\Big(|u_1 - u_2|[|u_1|^{p-1} + |u_2|^{p-1}] + |u_1 - u_2|[|u_1|^{q-1} + |u_2|^{q-1}]\Big) \\
&\leq |u_1 - u_2|\Big(|x|^b|u_1|^{p-1} + |x|^b|u_1|^{q-1} + |x|^b|u_2|^{p-1} + |x|^b|u_2|^{q-1}\Big).
\end{aligned}$$

With Hölder inequality, yields

$$\begin{aligned}
\|f(u_1) - f(u_2)\|_{L_T^{\alpha'}(L^{\beta'})} &\leq \|v_1 - v_2\|_{L_T^\alpha(L^\beta)}\||x|^b|u_1|^{p-1} + |x|^b|u_1|^{q-1} + |x|^b|u_2|^{p-1} + |x|^b|u_2|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} \\
&\leq \|v_1 - v_2\|_{L_T^\alpha(L^\beta)}\sum_{i=1}^2\Big(\||x|^b|u_i|^{p-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})} + \||x|^b|u_i|^{q-1}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\Big).
\end{aligned}$$

This implies via (3.2), that

$$\|f(u_1)-f(u_2)\|_{L_T^{\alpha'}(L^{\beta'})}\lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|v_1-v_2\|_{L_T^{\alpha}(L^{\beta})}.$$

Then,

$$\|\psi(v_1)-\psi(v_2)\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)}\lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|v_1-v_2\|_{L_T^{\alpha}(L^{\beta})}.$$

It remains to estimate the quantity $\|\nabla(\psi(v_1)-\psi(v_2))\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)}$. Applying Strichartz estimate (2.9), we get

$$\|\nabla(\psi(v_1)-\psi(v_2))\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)}\lesssim \||x|^b\nabla(g(u_1)-g(u_2))\|_{L_T^{\alpha'}(L^{\beta'})}+\||x|^{b-1}(g(u_1)-g(u_2))\|_{L_T^{\alpha'_1}(L^{\beta'_1})},$$

where we denote the function $g(u):=\frac{1}{|x|^b}f(u)$, we identify g with a real function on $\mathbb{R}^2$ and Dg denotes the $\mathbb{R}^2$ derivative of the identified function. By Hölder inequality

$$\begin{aligned}
\||x|^b\nabla(g(u_1)-g(u_2))\|_{L_T^{\alpha'}(L^{\beta'})} &\lesssim \sum_{i=1}^{2}\|\nabla u_2(v_1-v_2)(|x|^b|u_i|^{p-2}+|x|^b|u_i|^{q-2})\|_{L_T^{\alpha'}(L^{\beta'})}+\||x|^bDg(u_1)\nabla(v_1-v_2)\|_{L_T^{\alpha'}}\\
&\lesssim \|\nabla u_2\|_{L_T^\alpha(L^\beta)}\sum_{i=1}^{2}\Big(\|(v_1-v_2)|x|^b|u_i|^{p-2}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}+\|(v_1-v_2)|x|^b|u_i|^{q-2}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\\
&+ \|\nabla(v_1-v_2)\|_{L_T^\alpha(L^\beta)}\||x|^bDg(u_1)\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}.
\end{aligned}$$

We have for $i\in\{1,2\}$,

$$\begin{aligned}
\|(v_1-v_2)|x|^b|u_i|^{p-2}\|^{\frac{\beta}{2\sigma}}_{L^{\frac{\beta}{2\sigma}}} &= \int(|x|^{\frac{N-1}{2}}|v_1-v_2|)^{\frac{\beta}{2\sigma}}|x|^{\frac{\beta(N-1)}{4\sigma}(\frac{2b}{N-1}-1)}|u_i|^{\frac{\beta}{2\sigma}(p-2)}\\
&= \int(|x|^{\frac{N-1}{2}}|v_1-v_2|)^{\frac{\beta}{2\sigma}}(|x|^{\frac{N-1}{2}}|u_i|)^{\frac{\beta}{2\sigma}(\frac{2b}{N-1}-1)}|u_i|^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}
\end{aligned}$$

Using Strauss inequality (2.6),

$$(|x|^{\frac{N-1}{2}}|v_1-v_2|)^{\frac{\beta}{2\sigma}}\lesssim(\|\nabla v_1-v_2\|^{\frac12}_{L^2}\|v_1-v_2\|^{\frac12}_{L^2})^{\frac{\beta}{2\sigma}}\lesssim\|v_1-v_2\|_T^{\frac{\beta}{2\sigma}}.$$

With the same way

$$(|x|^{\frac{N-1}{2}}|u_i|)^{\frac{\beta}{2\sigma}(\frac{2b}{N-1}-1)}\lesssim(\|\nabla u\|^{\frac12}_{L^2}\|u\|^{\frac12}_{L^2})^{\frac{\beta}{2\sigma}(\frac{2b}{N-1}-1)}\lesssim(1+\|u_0\|_{H^1_{rd}})^{\frac{\beta}{2\sigma}(\frac{2b}{N-1}-1)}.$$

Now, taking account of Gagliardo-Nirenberg inequality (2.5), we get

$$\int|u_i|^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}\lesssim(1+\|u_0\|_{H^1_{rd}})^{\frac{\beta}{2\sigma}(p-1-\frac{2b}{N-1})}.$$

Consequently, for $i\in\{1,2\}$ we have,

$$\|(v_1-v_2)|x|^b|u_i|^{p-2}\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\lesssim T^{\frac{2\sigma}{\theta}}\|v_1-v_2\|_T(1+\|u_0\|_{H^1_{rd}})^{p-2}.$$

Using previous computation,

$$\|Dg(u_i)\|_{L_T^{\frac{\theta}{2\sigma}}(L^{\frac{\beta}{2\sigma}})}\lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1},\quad for\quad i\in\{1,2\}.$$

Since we have $\|\nabla u_2\|_{L_T^\alpha(L^\beta)} \lesssim (1+\|u_0\|_{H^1_{rd}})$, we obtain

$$\||x|^b\nabla(g(u_1)-g(u_2))\|_{L_T^{\alpha'}(L^{\beta'})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|v_1-v_2\|_T.$$

Moreover, arguing as previously,

$$\begin{aligned}
\||x|^{b-1}(g(u_1)-g(u_2))\|_{L_T^{\alpha_1'}(L^{\beta_1'})} &\leq \|v_1-v_2\|_{L_T^{\alpha_1}(L^{\beta_1})}\||x|^{b-1}|u_1|^{p-1}+|x|^{b-1}|u_1|^{q-1}+|x|^b|u_2|^{p-1}+|x|^b|u_2|^{q-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L} \\
&\leq \|v_1-v_2\|_{L_T^{\alpha_1}(L^{\beta_1})}\sum_{i=1}^{2}\Big(\||x|^{b-1}|u_i|^{p-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L^{\frac{\beta_1}{2\sigma_1}})}+\||x|^{b-1}|u_i|^{q-1}\|_{L_T^{\frac{\theta_1}{2\sigma_1}}(L^{\frac{\beta_1}{2\sigma_1}})}\Big) \\
&\leq \|v_1-v_2\|_{L_T^\infty(H^1)}\Big((1+\|u_0\|_{H^1})^{p-1}+(1+\|u_0\|_{H^1})^{q-1}\Big)T^{\frac{1}{\alpha_1}+2\frac{\sigma_1}{\theta_1}}.
\end{aligned}$$

This implies that

$$\|\nabla(\psi(v_1)-\psi(v_2))\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)} \lesssim (T^{\frac{2\sigma}{\theta}}+T^{\frac{1}{\alpha_1}+2\frac{\sigma_1}{\theta_1}})(1+\|u_0\|_{H^1_{rd}})^{q-1}\|v_1-v_2\|_T.$$

Consequently, we obtain

$$\|\psi(v_1)-\psi(v_2)\|_T \lesssim (T^{\frac{2\sigma}{\theta}}+T^{\frac{1}{\alpha_1}+2\frac{\sigma_1}{\theta_1}})(1+\|u_0\|_{H^1_{rd}})^{q-1}\|v_1-v_2\|_T.$$

Finally, for $T>0$ sufficiently small ψ is a contraction. The existence of solution to (1.1) follows with a standard Picard argument.

3.2 Uniqueness in the energy space

In what follows we prove uniqueness of solution to the Cauchy problem (1.1)-(1.2). Let $u_1,u_2\in C([0,T],H^1_{rd})$ two solutions of the Schrödinger equation (1.1), with the same data. Take $w=u_1-u_2$, then

$$i\partial_t w+\triangle w = f(u_1)-f(u_2)=0,\quad w(0,.)=0.$$

With a continuity argument, take $0<T<1$ such that

$$\max_{i=\{1,2\}}\|u_i\|_{L^\infty([0,T],H^1_{rd})}\leq 1+\|u_0\|_{H^1_{rd}}.$$

With Strichartz estimate, we have

$$\|w\|_{L_T^\infty(L^2)\cap L_T^\alpha(L^\beta)} \lesssim \|f(u_1)-f(u_2)\|_{L_T^{\alpha'}(L^{\beta'})}.$$

With previous computation, we have

$$\|f(u_1)-f(u_2)\|_{L_T^{\alpha'}(L^{\beta'})} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|w\|_{L_T^\alpha(L^\beta)}.$$

So

$$\|w\|_{L_T^\alpha(L^\beta)} \lesssim T^{\frac{2\sigma}{\theta}}(1+\|u_0\|_{H^1_{rd}})^{q-1}\|w\|_{L_T^\alpha(L^\beta)}.$$

Then, for $T>0$ sufficiently small we have

$$\|w\|_{L_T^\alpha(L^\beta)}=0.$$

The proof is closed via a standard time translation argument.

3.3 Global existence in the defocusing case

Take $\epsilon = -1$. Let $u \in C([0,T^*), H^1_{rd})$ to be the maximal solution to (1.1)-(1.2). By contradiction, assume that

$$T^* < +\infty \quad \text{and} \quad \lim_{t\to T^*} \|\nabla u(t)\|_{L^2} = +\infty.$$

Write the energy conservation

$$\int |\nabla u(t)|^2\,dx = 2E(u_0) + \frac{2}{p+1}\int |x|^b |u(t)|^{p+1}\,dx + \frac{2}{q+1}\int |x|^b |u(t)|^{q+1}\,dx.$$

By the estimate (2.7), we get

$$\int |x|^b |u(t)|^{p+1}\,dx \lesssim \|\nabla u(t)\|_{L^2}^{\frac{N(p-1)-2b}{2}} \|u(t)\|_{L^2}^{\frac{2(p+1)-(N(p-1)-2b)}{2}};$$
$$\int |x|^b |u(t)|^{q+1}\,dx \lesssim \|\nabla u(t)\|_{L^2}^{\frac{N(q-1)-2b}{2}} \|u(t)\|_{L^2}^{\frac{2(q+1)-(N(q-1)-2b)}{2}}.$$

With the conservation of the mass, yields

$$\|\nabla u(t)\|_{L^2}^2 \le 2E(u_0) + C\Big[\|\nabla u(t)\|_{L^2}^{\frac{N(p-1)-2b}{2}} + \|\nabla u(t)\|_{L^2}^{\frac{N(q-1)-2b}{2}}\Big].$$

Therefore,

$$\|\nabla u(t)\|_{L^2}^2\Big(1 - C(\|\nabla u(t)\|_{L^2}^{\frac{N(p-1)-2b-4}{2}} + \|\nabla u(t)\|_{L^2}^{\frac{N(q-1)-2b-4}{2}})\Big) \le 2E(0).$$

Since $q < p_c$, it yields $N(q-1)-2b-4 < 0$. Taking $t \to T^*$ in the previous inequality, leads to a contradiction and finishes the proof.

4 Proof of Theorem 2.2

In this section we prove the existence of a ground state solution to (2.3).

Remark 4.1. Note that, in this section

1) $(\alpha,\beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

2) The proof of the Theorem 2.3 is based on several lemmas.

3) We write, for easy notation, $K = K_{\alpha,\beta}$, $K^Q = K^Q_{\alpha,\beta}$, $K^N = K^N_{\alpha,\beta}$, $\mathcal{L} = \mathcal{L}_{\alpha,\beta}$ and $H = H_{\alpha,\beta}$.

Lemma 4.2. *Let $0 \neq \phi \in H^1_{rd}$, then*

1) $\min(\mathcal{L}H(\phi), H(\phi)) > 0$;

2) $\lambda \mapsto H(\phi^\lambda)$ is increasing.

Proof. we have,

$$H(\phi) = \frac{\beta}{2\alpha+N\beta}\|\nabla\phi\|_{L^2}^2 + \frac{\alpha(p-1)+b\beta}{(p+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{p+1}\,dx + \frac{\alpha(q-1)+b\beta}{(q+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{q+1}\,dx > 0.$$

Moreover, denoting $\mu := 2\alpha + N\beta$ and $a = \frac{2\alpha+(N-2)\beta}{2}$, we compute

$$\begin{aligned}\mathcal{L}(H(\phi)) &= 2aH(\phi) + \frac{1}{\mu}(\mathcal{L}-2a)(\mu-\mathcal{L})J(\phi)\\ &\geq \frac{1}{\mu}(\mathcal{L}-2a)(\mu-\mathcal{L})J(\phi).\end{aligned}$$

Since $(\mathcal{L}-2a)\|\nabla\phi\|_{L^2}^2 = 0 = (\mathcal{L}-\mu)\|\phi\|_{L^2}^2$, we have $(\mathcal{L}-2a)(\mathcal{L}-\mu)\|\phi\|_{H^1_{rd}}^2 = 0$. Then, with a direct computation

$$\begin{aligned}\mathcal{L}(H(\phi)) &\geq \frac{1}{\mu}(\mathcal{L}-2a)(\mu-\mathcal{L})J(\phi)\\ &= \frac{1}{\mu}\frac{(\alpha(p-1)+b\beta)(\alpha(p-1)+\beta(b+2))}{p+1}\int |x|^b|u|^{p+1}\,dx\\ &+ \frac{1}{\mu}\frac{(\alpha(q-1)+b\beta)(\alpha(q-1)+\beta(b+2))}{q+1}\int |x|^b|u|^{q+1}\,dx\\ &> 0.\end{aligned}$$

The second point is a consequence of the equality $\partial_\lambda H(\phi^\lambda) = \mathcal{L}H(\phi^\lambda) > 0$. □

Lemma 4.3. *Assume that $2\alpha+(N-2)\beta \neq 0$ and take (ϕ_n) a bounded sequence of $H^1_{rd} - \{0\}$ such that $\lim_{n\to+\infty} K^Q(\phi_n) = 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $K(\phi_n) > 0$ for all $n \geq n_0$.*

Proof. Since $\alpha,\beta \geq 0$ and $2\alpha+(N-2)\beta \neq 0$,

$$K^Q(\phi_n) = \frac{2\alpha+(N-2)\beta}{2}\|\nabla\phi_n\|_{L^2}^2 + \frac{(2\alpha+N\beta)}{2}\|\phi_n\|_{L^2}^2 \geq C\|\nabla\phi_n\|_{L^2}^2.$$

Applying the estimate (2.7), via the facts that

$$\sup_n \|\phi_n\|_{H^1} \lesssim 1, \quad \|\nabla\phi_n\|_{L^2} \to 0,$$

we have

$$K^N(\phi_n) \lesssim \|\nabla\phi_n\|_{L^2}^{\frac{N(p-1)-2b}{2}} + \|\nabla\phi_n\|_{L^2}^{\frac{N(q-1)-2b}{2}} \lesssim \|\nabla\phi_n\|_{L^2}^{\frac{N(p-1)-2b}{2}}.$$

Now, $\frac{N(p-1)-2b}{2} > 2$ because $p > p_0$, thus

$$K(\phi_n) \simeq K^Q(\phi_n) > 0.$$

The proof is finished. □

Lemma 4.4. *We have*

$$m_{\alpha,\beta} = \inf_{0\neq\phi\in H^1}\{H(\phi),\ s.t\quad K(\phi) \leq 0\}. \tag{4.1}$$

Proof. It sufficient to prove that $m \le m_1$, where m_1 is the right hand side of the previous inequality. Take $\phi \in H^1_{rd}$ such that $K(\phi) < 0$.
Assume that $2\alpha + (N-2)\beta \neq 0$, then by the previous lemma, the facts that $\lim_{\lambda \to -\infty} K^Q(\phi^\lambda) = 0$ and $\lambda \mapsto H(\phi^\lambda)$ is increasing, there exists $\lambda < 0$ such that

$$K(\phi^\lambda) = 0 \qquad H(\phi^\lambda) \le H(\phi). \tag{4.2}$$

Then, $m \le H(\phi^\lambda) \le H(\phi)$. This ends the proof.
Assume now that $\alpha = 0$ and $N = 2$. When as λ tends to zero

$$K^N(\lambda\phi) = o(\lambda^2 K^N(\phi)) = o(\lambda^2 K^Q(\phi)).$$

So $K(\lambda\phi) \simeq \lambda^2 K^Q(\phi) > 0$. Then, with a continuity argument, there exists $\lambda \in (0,1)$ such that $\lambda\phi$ satisfies (4.2). The proof is achieved similarly. □

Proof of Theorem 2.3. The proof contains four steps.
Step 1. We prove that a minimizing sequence is bounded in H^1_{rd}.
let (ϕ_n) to be a minimizing sequence of (2.2), namely

$$0 \neq \phi_n \in H^1_{rd}, \quad K(\phi_n) = 0 \quad \textit{and} \quad \lim_n H(\phi_n) = \lim_n J(\phi_n) = m. \tag{4.3}$$

• First case $\beta \neq 0$. Since

$$\frac{\beta}{2\alpha + N\beta}\|\nabla\phi_n\|_2^2 \le H(\phi_n) \to m,$$

we get

$$\sup_n \|\nabla\phi_n\|_{L^2} \lesssim 1.$$

Assume that $\lim_n \|\phi_n\|_{L^2} = \infty$. Using the estimate (2.7) via the equality $K(\phi_n) = 0$, yields

$$\begin{aligned}
\|\phi_n\|_{L^2}^2 &\lesssim \frac{2\alpha + (N-2)\beta}{2}\|\nabla\phi_n\|_{L^2}^2 + \frac{2\alpha + N\beta}{2}\|\phi_n\|_{L^2}^2 \\
&= \frac{\alpha(p+1) + (N+b)\beta}{p+1}\int |x|^b|\phi_n|^{p+1}\,dx + \frac{\alpha(q+1) + (N+b)\beta}{q+1}\int |x|^b|\phi_n|^{q+1}\,dx \\
&\lesssim \|\nabla\phi_n\|_{L^2}^{\frac{N(p+1)-2(N+b)}{2}}\|\phi_n\|_{L^2}^{\frac{2(N+b)+(p+1)(2-N)}{2}} + \|\nabla\phi_n\|_{L^2}^{\frac{N(q+1)-2(N+b)}{2}}\|\phi_n\|_{L^2}^{\frac{2(N+b)+(q+1)(2-N)}{2}} \\
&\lesssim \|\phi_n\|_{L^2}^{\frac{2(N+b)+(q+1)(2-N)}{2}}.
\end{aligned}$$

The condition $q > p_0$ implies that $\frac{2(N+b)+(q+1)(2-N)}{2} < 2$ and leads to a contradiction in the last inequality if letting $n \mapsto +\infty$. Then (ϕ_n) is bounded in H^1_{rd}.
• Second case $\beta = 0$.
In this case (ϕ_n) is bounded in H^1_{rd} because

$$\|\phi_n\|_{H^1}^2 = \int |x|^b\Big(|\phi_n|^{p+1} + |\phi_n|^{q+1}\Big)dx \lesssim H(\phi_n) \to m.$$

Step 2. We prove that the weak limit of (ϕ_n) is nonzero.
Using the first step, for a subsequence, still denoted by (ϕ_n), we have

$$\phi_n \rightharpoonup \phi \quad \text{weakly in} \quad H^1_{rd} \quad \text{and} \quad \phi_n \to \phi \quad \text{in} \quad L^p, \quad \text{for any} \quad 2 < p < \frac{2N}{N-2}.$$

We prove that $\phi \neq 0$. Arguing by contradiction, assume that $\phi = 0$. Since $2 < p+1-\frac{2b}{N-1} < q+1-\frac{2b}{N-1} < \frac{2N}{N-2}$ then, due to the compact injection (2.4), we obtain

$$\int |x|^b|\phi_n|^{p+1}\,dx \leq \int (|x|^{\frac{N-1}{2}}|\phi_n|)^{\frac{2b}{N-1}}|\phi_n|^{p+1-\frac{2b}{N-1}}\,dx \leq C_N\|\phi_n\|_{p+1-\frac{2b}{N-1}}^{p+1-\frac{2b}{N-1}} \longrightarrow 0;$$
$$\int |x|^b|\phi_n|^{q+1}\,dx \leq \int (|x|^{\frac{N-1}{2}}|\phi_n|)^{\frac{2b}{N-1}}|\phi_n|^{q+1-\frac{2b}{N-1}}\,dx \leq C_N\|\phi_n\|_{q+1-\frac{2b}{N-1}}^{q+1-\frac{2b}{N-1}} \longrightarrow 0.$$

Thus

$$K^Q(\phi_n) = K^N(\phi_n) \to 0.$$

• First case $2\alpha + (N-2)\beta \neq 0$.
Using lemma 4.3, there exists n_0 such that $K(\phi_n) > 0$, for all $n > n_0$, which contradicts the fact that $K(\phi_n) = 0$. This implies that $\phi \neq 0$.
• Second case $\alpha = 0$ and $N = 2$.
Without loss of generality, we take $\beta = 1$. Now, $0 = K(\phi_n)$ via the estimate (2.7) yield to the absurdity

$$\|\phi_n\|_{L^2}^2 = \int |x|^b\Big(\frac{|\phi_n|^{p+1}}{p+1} + \frac{|\phi_n|^{q+1}}{q+1}\Big)\,dx \lesssim \|\phi_n\|_{L^2}^{b+2}.$$

Thus $\phi \neq 0$.
Step 3. We prove that ϕ is a minimizer and $m > 0$.
With the lower semi-continuity of H^1_{rd} norm, via the convergence, $\int |x|^b|\phi_n - \phi|^{p+1}\,dx \to 0$,we have

$$\begin{aligned}
0 = \liminf_n K(\phi_n) &\geq \frac{2\alpha+(N-2)\beta}{2}\liminf_n \|\nabla\phi_n\|_{L^2}^2 + \frac{2\alpha+N\beta}{2}\liminf_n \|\phi_n\|_{L^2}^2\\
&- \frac{\alpha(p+1)+(N+b)\beta}{p+1}\lim_n \int |x|^b|\phi_n|^{p+1}\,dx - \frac{\alpha(q+1)+(N+b)\beta}{q+1}\lim_n \int |x|^b|\phi_n|^{q+1}\,dx\\
&\geq \frac{2\alpha+(N-2)\beta}{2}\|\phi\|_{L^2}^2 + \frac{2\alpha+(N-2)\beta}{2}\|\nabla\phi\|_{L^2}^2\\
&- \frac{\alpha(p+1)+(N+b)\beta}{p+1}\int |x|^b|\phi|^{p+1}\,dx - \frac{\alpha(q+1)+(N+b)\beta}{q+1}\int |x|^b|\phi|^{q+1}\,dx\\
&= K(\phi).
\end{aligned}$$

Applying Fatou lemma, we obtain

$$\begin{aligned}
m \geq \liminf_n H(\phi_n) &\geq \liminf_n \frac{\beta}{2\alpha+N\beta}\|\nabla\phi_n\|_2^2 + \liminf_n \frac{\alpha(p-1)+b\beta}{(p+1)(2\alpha+N\beta)}\int |x|^b|\phi_n|^{p+1}\,dx\\
&+ \liminf_n \frac{\alpha(q-1)+b\beta}{(q+1)(2\alpha+N\beta)}\int |x|^b|\phi_n|^{q+1}\,dx\\
&\geq \frac{\beta}{2\alpha+N\beta}\|\nabla\phi\|_2^2 + \frac{\alpha(p-1)+b\beta}{(p+1)(2\alpha+N\beta)}\int |\phi|^{p+1}\,dx + \frac{\alpha(q-1)+b\beta}{(q+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{q+1}\,dx\\
&= H(\phi).
\end{aligned}$$

Then $H(\phi) \leq m$ and ϕ satisfies

$$0 \neq \phi \in H^1_{rd}, \quad K(\phi) \leq 0 \quad \text{and} \quad J(\phi) = H(\phi) \leq m.$$

By (4.2), we can assume that ϕ is a minimizer satisfying

$$0 \neq \phi \in H^1_{rd}, \quad K(\phi) = 0 \quad \text{and} \quad J(\phi) = H(\phi) = m.$$

Moreover

$$H(\phi) = \frac{\beta}{2\alpha + N\beta}\|\nabla\phi\|_2^2 + \frac{\alpha(p-1)+b\beta}{(p+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{p+1} + \frac{\alpha(q-1)+b\beta}{(q+1)(2\alpha+N\beta)}\int |x|^b|\phi|^{q+1} > 0.$$

Thus

$$m > 0.$$

Step 4. We prove that ϕ is a ground state solution of (2.2).
Since ϕ satisfies (2.3), there is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $J'(\phi) = \eta K'(\phi)$. Recall that $\mathcal{L}(\phi) := (\partial_\lambda \phi^\lambda_{\alpha,\beta})_{|\lambda=0}$ and $\mathcal{L}J(\phi) := (\partial_\lambda J(\phi^\lambda_{\alpha,\beta}))_{|\lambda=0}$. Then

$$0 = K(\phi) = \mathcal{L}J(\phi) =< J'(\phi), \mathcal{L}(\phi) >= \eta < K'(\phi), \mathcal{L}(\phi) >= \eta\mathcal{L}^2 J(\phi).$$

Moreover, with previous computation

$$\begin{aligned}
-\mathcal{L}^2 J(\phi) - 2a\mu J(\phi) &= -(\mathcal{L}-2a)(\mathcal{L}-\mu)J(\phi)\\
&= \frac{(\alpha(p-1)+b\beta)(\alpha(p-1)+\beta(b+2))}{p+1}\int |x|^b|u|^{p+1}\,dx\\
&+ \frac{(\alpha(q-1)+b\beta)(\alpha(q-1)+\beta(b+2))}{q+1}\int |x|^b|u|^{q+1}\,dx\\
&\geq 0.
\end{aligned}$$

Because $J(\phi) > 0$, it follows that $\eta = 0$ and $J'(\phi) = 0$. Finally, ϕ is a ground state and m is independent of α and β. □

5 Proof of Theorem 2.3

In this section, we prove that if ϕ is a ground state solution to (2.3), then the standing wave $e^{it}\phi$ of the Schrödinger equation (1.1) is nonlinearly unstable. Here and hereafter, we denote

$$P := K_{\frac{N}{2},-1}, \quad I := K_{1,0} \quad \text{and} \quad \phi^\lambda := \lambda^{\frac{N}{2}}\phi(\lambda\,.).$$

Definition 5.1. For $\varepsilon > 0$, we define

1. the set
$$U_\varepsilon(\phi) := \{v \in H^1_{rd}, \quad \text{s. t} \quad \inf_{t\in\mathbb{R}}\|v - e^{it}\phi\|_{H^1} < \varepsilon\}.$$
2. If $u_0 \in U_\varepsilon(\phi)$ and u is the solution to (1.1)-(1.2) given by Theorem 2.1,
$$T_\varepsilon(u_0) := \sup\{T > 0, \quad \text{s. t} \quad u(t) \in U_\varepsilon(\phi), \quad \text{for any} \quad t \in [0,T)\}.$$
3. We say that $e^{it}\phi$ is orbitally stable if, for any $\sigma > 0$ there exists $\varepsilon > 0$ such that if $u_0 \in U_\varepsilon(\phi)$ and u is the solution to (1.1)-(1.2) given by Theorem 2.1, then $T_\sigma(u_0) = \infty$. Otherwise, the standing wave $e^{it}\phi$ is said to be nonlinearly unstable.

4. Define also the set

$$\Pi_\varepsilon(\phi) := \{v \in U_\epsilon(\phi), \quad \text{s. t} \quad E(v) < E(\phi), \quad \|v\|_{L^2} = \|\phi\|_{L^2} \quad \text{and} \quad P(v) < 0\}.$$

The proof of Theorem 2.3 is based on several Lemmas.

Lemma 5.2. *Let ϕ a ground state solution to* (2.3). *If $\partial_\lambda^2 E(\phi^\lambda)_{|\lambda=1} < 0$, then there exist $\epsilon_0 > 0$, $\sigma_0 > 0$ and a mapping*

$$\lambda : U_{\epsilon_0}(\phi) \longrightarrow [1-\sigma_0, 1+\sigma_0]$$

such that $I(\xi^\lambda) = 0$, for all $\xi \in U_{\epsilon_0}(\phi)$.

Proof. If we assume that $\langle I'(\phi), (\partial_\lambda \phi^\lambda)_{|\lambda=1}\rangle = 0$, then $(\partial_\lambda \phi^\lambda)_{|\lambda=1}$ would be the tangent at ϕ to the set

$$\mathcal{N} := \{0 \neq \phi \in H^1_{rd}, \quad \text{s. t} \quad I(\phi) = 0\}.$$

Therefore, $\langle J''(\phi)(\partial\lambda\phi^\lambda)_{|\lambda=1}, (\partial_\lambda\phi^\lambda)_{|\lambda=1}\rangle \geq 0$ because ϕ is a minimizer. This implies the contradiction

$$0 > \partial_\lambda^2 E(\phi^\lambda)_{|\lambda=1} = \partial_\lambda^2 J(\phi^\lambda)_{|\lambda=1} = \langle J''(\phi)(\partial_\lambda\phi^\lambda)_{|\lambda=1}, (\partial_\lambda\phi^\lambda)_{|\lambda=1}\rangle \geq 0.$$

Therefore

$$\partial_\lambda I(\xi^\lambda)_{|\lambda=1,\xi=\phi} = \langle I'(\phi), (\partial_\lambda\phi^\lambda)_{|\lambda=1}\rangle \neq 0 \quad \text{and} \quad I(\xi^\lambda)_{|\lambda=1,\xi=\phi} = I(\phi) = 0.$$

The implicit function Theorem closes the proof. □

The next auxiliary result reads

Lemma 5.3. *Let ϕ a ground state solution to* (2.3). *If $\partial_\lambda^2 E(\phi^\lambda)_{|\lambda=1} < 0$, then there exist two real numbers $\epsilon_1 > 0$ and $\sigma_1 > 0$ such that for any $\xi \in U_{\epsilon_1}(\phi)$ satisfying $\|\xi\|_{L^2} = \|\phi\|_{L^2}$, holds*

$$E(\phi) < E(\xi) + (\lambda-1)P(\xi) \quad \textit{for some} \quad \lambda \in [1-\sigma_1, 1+\sigma_1].$$

Proof. Since $\partial_\lambda^2 E(\phi^\lambda)|_{\lambda=1} < 0$, with a continuity argument, there exist $\epsilon_1 > 0$ and $\sigma_1 > 0$ such that

$$\partial_\lambda^2 E(\xi^\lambda) < 0, \quad \text{for any} \quad (\lambda, \xi) \in [1-\sigma_1, 1+\sigma_1] \times U_{\epsilon_1}(\phi).$$

Write the Taylor expansion of $E(\xi^\lambda)$ at $\lambda = 1$ and $\xi \in U_{\epsilon_1}(\phi)$,

$$E(\xi^\lambda) = E(\xi^\lambda)_{|\lambda=1} + (\lambda-1)\partial_\lambda E(\xi^\lambda)_{|\lambda=1} + \frac{(\lambda-1)^2}{2}\partial_\lambda^2 E(\xi^\lambda)_{|\lambda\in[1-\sigma_1,1+\sigma_1]}.$$

With a simple calculation, we have $\partial_\lambda E(\xi^\lambda)_{|\lambda=1} = P(\xi)$. Then

$$E(\xi^\lambda) < E(\xi) + P(\xi)(\lambda-1), \quad \text{for any} \quad (\lambda, \xi) \in [1-\sigma_1, 1+\sigma_1] \times U_{\epsilon_1}(\phi). \tag{5.1}$$

By the previous lemma, we can take $0 < \epsilon_1 < \epsilon_0$ and $0 < \sigma_1 < \sigma_0$ such that

$$I(\xi^\lambda) = 0, \quad \text{for any} \quad \lambda \in [1-\sigma_1, 1+\sigma_1] \times U_{\epsilon_1}(\phi).$$

The fact that $\xi^\lambda \in \mathcal{N}$ implies that

$$J(\xi^\lambda) \geq J(\phi), \quad \text{for any} \quad \lambda \in [1-\sigma_1, 1+\sigma_1] \times U_{\epsilon_1}(\phi). \tag{5.2}$$

On the other hand, for any $\xi \in U_{\epsilon_1}(\phi)$ satisfying $\|\xi\|_{L^2} = \|\phi\|_{L^2}$, we have

$$\|\xi^\lambda\|_{L^2} = \|\xi\|_{L^2} = \|\phi\|_{L^2}. \tag{5.3}$$

Therefore, by (5.2)-(5.3), denoting $M(\phi)$ the mass of ϕ, it follows that

$$\begin{aligned} E(\xi^\lambda) &= J(\xi^\lambda) - M(\xi^\lambda) \\ &\geq J(\phi) - M(\xi^\lambda) \\ &= J(\phi) - M(\xi) \\ &= J(\phi) - M(\phi) \\ &= E(\phi). \end{aligned}$$

The proof is completed via (5.1). □

The last intermediary result is the following.

Lemma 5.4. *Let ϕ a ground state solution to* (2.3). *If $\partial_\lambda^2 E(\phi^\lambda)_{|\lambda=1} < 0$, then for any $u_0 \in \Pi_{\varepsilon_1}$, there exists a real number $\sigma_0 = \sigma(u_0) > 0$ such that the solution u to* (1.1)-(1.2) *satisfies*

$$P(u(t)) < -\sigma_0, \quad \textit{for all} \quad 0 \leq t < T_{\varepsilon_1}(u_0).$$

Proof. Let $u_0 \in \Pi_{\varepsilon_1}$, so

$$E(u_0) < E(\phi), \quad \|u_0\|_{L^2} = \|\phi\|_{L^2} \quad \text{and} \quad P(u_0) < 0.$$

Put $\sigma_2 := E(\phi) - E(u_0) > 0$. It follows from the previous lemma that there exists $\lambda \in [1-\sigma_1, 1+\sigma_1]$ such that

$$P(u(t))(\lambda - 1) + E(u(t)) > E(\phi), \quad \text{for any} \quad 0 \leq t < T_{\varepsilon_1}(u_0).$$

By the energy conservation, we get

$$P(u(t))(\lambda - 1) > E(\phi) - E(u(t)) = E(\phi) - E(u_0) = \sigma_2 > 0.$$

Thus

$$P(u(t)) \neq 0, \quad \text{for any} \quad (\lambda, t) \in (1-\sigma_1, 1+\sigma_1) \times [0, T_{\varepsilon_1}(u_0)).$$

Now, with a continuity argument via the fact that $P(u_0) < 0$, yields

$$P(u(t)) < 0, \quad \text{for any} \quad 0 \leq t < T_{\varepsilon_1}(u_0).$$

So, it follows that $-\sigma_1 < \lambda - 1 < 0$. Then

$$P(u(t)) < -\frac{\sigma_2}{\sigma_1} := -\sigma_0 < 0, \quad \text{for any} \quad 0 \leq t < T_{\varepsilon_1}(u_0).$$

The proof is completed. □

Now, we are ready to prove nonlinear instability.

Proof of Theorem 2.3. Since ϕ is a ground state solution to (2.3) and $P = K_{\frac{N}{2},-1}$, it follows from Proposition 2.12, that

$$P(\phi) = \int \Big[|\nabla\phi|^2 - \frac{N(p-1)-2b}{2(p+1)}|x|^b|\phi|^{p+1} - \frac{N(q-1)-2b}{2(q+1)}|x|^b|\phi|^{q+1}\Big]\,dx = 0.$$

With a simple computation, we obtain that

$$E(\phi^\lambda) = \frac{\lambda^2}{2}\int|\nabla\phi|^2\,dx - \frac{\lambda^{\frac{N(p-1)-2b}{2}}}{p+1}\int|x|^b|\phi|^{p+1}\,dx - \frac{\lambda^{\frac{N(q-1)-2b}{2}}}{q+1}\int|x|^b|\phi|^{q+1}\,dx.$$

Take the derivative

$$\partial_\lambda E(\phi^\lambda) = \lambda\|\nabla\phi\|^2_{L^2} - (\frac{N(p-1)-2b}{2(p+1)})\lambda^{\frac{N(p-1)-2(b+1)}{2}}\int|x|^b|\phi|^{p+1}\,dx - (\frac{N(q-1)-2b}{2(q+1)})\lambda^{\frac{N(q-1)-2(b+1)}{2}}\int|x|^b|\phi|^{q+1}\,dx.$$

We conclude that

$$\partial_\lambda E(\phi^\lambda)_{|\lambda=1} = \|\nabla\phi\|^2_{L^2} - (\frac{N(p-1)-2b}{2(p+1)})\int|x|^b|u|^{p+1}\,dx - (\frac{N(q-1)-2b}{2(q+1)})\int|x|^b|u|^{q+1}\,dx.$$

Take the second derivative

$$\begin{aligned}\partial^2_\lambda E(\phi^\lambda) &= \|\nabla\phi\|^2_{L^2} - \frac{(N(p-1)-2b)(N(p-1)-2(b+1))}{4(p+1)}\lambda^{\frac{N(p-1)-2(b+2)}{2}}\int|x|^b|\phi|^{p+1}\,dx\\ &- \frac{(N(q-1)-2b)(N(q-1)-2(b+1))}{4(q+1)}\lambda^{\frac{N(q-1)-2(b+2)}{2}}\int|x|^b|\phi|^{q+1}\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\partial^2_\lambda E(\phi^\lambda)_{|\lambda=1} &= \|\nabla\phi\|^2_{L^2} - \frac{(N(p-1)-2b)(N(p-1)-2(b+1))}{4(p+1)}\int|x|^b|\phi|^{p+1}\,dx\\ &- \frac{(N(q-1)-2b)(N(q-1)-2(b+1))}{4(q+1)}\int|x|^b|\phi|^{q+1}\,dx.\end{aligned}$$

Since $P(\phi) = 0$ and $p_0 < p$,

$$\begin{aligned}\|\nabla\phi\|^2_{L^2} &= \int_{\mathbb{R}^N}\Big[\frac{N(p-1)-2b}{2(p+1)}|x|^b|\phi|^{p+1} - \frac{N(q-1)-2b}{2(q+1)}|x|^b|\phi|^{q+1}\Big]\,dx\\ &< \frac{(N(p-1)-2b)(N(p-1)-2(b+1))}{4(p+1)}\int|x|^b|\phi|^{p+1}\,dx\\ &+ \frac{(N(q-1)-2b)(N(q-1)-2(b+1))}{4(q+1)}\int|x|^b|\phi|^{q+1}\,dx.\end{aligned}$$

So, we get $\partial^2_\lambda E(\phi^\lambda)_{|\lambda=1} < 0$ and with a continuity argument $\partial^2_\lambda E(\phi^\lambda) < 0$ for λ near to one. So $\partial_\lambda E(\phi^\lambda)$ is a decreasing function and

$$\partial_\lambda E(\phi^\lambda) < \partial_\lambda E(\phi^\lambda)_{|\lambda=1} = P(\phi) = 0, \quad \text{for every} \quad \lambda > 1 \quad \text{near to one.}$$

Arguing as previously,

$$E(\phi^\lambda) < E(\phi), \quad \text{for every} \quad \lambda < 1 \quad \text{near to one.}$$

Moreover, with a direct computation, for every $\lambda > 1$ near to one

$$\begin{aligned}
\lambda^{-1}P(\phi^\lambda) &= \lambda\int|\nabla\phi|^2 - \frac{N(p-1)-2b}{2(p+1)}\lambda^{\frac{N(p-1)-2(b+1)}{2}}\int|x|^b|\phi|^{p+1}\\
&- \frac{N(q-1)-2b}{2(q+1)}\lambda^{\frac{N(q-1)-2(b+1)}{2}}\int|x|^b|\phi|^{q+1}\\
&= \partial_\lambda E(\phi^\lambda)\\
&< 0.
\end{aligned}$$

Finally, for $\lambda > 1$ near to one

$$E(\phi^\lambda) < E(\phi), \quad P(\phi^\lambda) < 0 \quad \text{and} \quad \|\phi^\lambda\|_{L^2} = \|\phi\|_{L^2}.$$

Now we take the initial data $u_0 = \phi^\lambda$, for some $\lambda \longrightarrow 1^+$. Then, there exists $\lambda > 1$ near to one such that

$$u_0 \in \Pi_{\varepsilon_1}.$$

By lemma 5.4, there exists $\sigma_0 = \sigma(u_0) > 0$ such that the solution u to the equation (1.1) with the initial data u_0 satisfies

$$P(u(t)) < -\sigma_0, \quad \text{for any} \quad 0 \le t < T_{\varepsilon_1}(u_0).$$

With the Virial identity (2.8), it follows that

$$\frac{1}{8}(\|xu(t)\|_{L^2}^2)'' = P(u(t)) < -\sigma_0 < 0.$$

Now if $e^{it}\phi$ is orbitally stable, $T_{\varepsilon_1}(u_0) = +\infty$ and $P(u) < -\sigma_0$ on $\mathbb{R}_+$. This implies that $\|xu(t)\|_{L^2}$ becomes negative for long time. This absurdity finishes the proof. □

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