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Anisotropic Herz Spaces with Variable Exponents

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Abstract

In this paper, we introduce the anisotropic Herz spaces with two variable exponents and establish their block decomposition. Using this decomposition, we obtain some boundedness on the anisotropic Herz spaces with two variable exponents for a class of sublinear operators.

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1 Introduction

In recent years, the theory of function spaces with variable exponents has developed since the paper [8] of Kováčik and Rákosník appeared in 1991. Lebesgue and Sobolev spaces with integrability exponent have been extensively investigated, see [5] and the references therein. In 2012, Almeida and Drihem [1] introduced the Herz spaces with two variable exponents and proved the boundedness of some operators on these spaces. Meanwhile, extending classic function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings is an important topic. In 2003, Bownik [2] introduced the anisotropic Hardy spaces associated with very general discrete groups of dilations. The above spaces include the classical isotropic Hardy space theory of Fefferman and Stein [6] and parabolic Hardy space theory of Calderón and Torchinsky [3, 4]. In 2006, Lan [9] defined the anisotropic Herz spaces and gave some properties and applications.

Inspired by [9, 10], we introduce the anisotropic Herz spaces with two variable exponents which is a generalization of the anisotropic Herz spaces and the Herz spaces with two variable exponents, and establish their block decomposition. Using this decomposition, we obtain the boundedness of some sublinear operators on the anisotropic Herz spaces with two variable exponents.

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To be precise, we first briefly recall some standard notations in the remainder of this section. In Section 2, we will define the anisotropic Herz spaces with two variable exponents $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ and $K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$, and give their block decomposition. In Section 3, we will give the boundedness of some sublinear operators on $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ and $K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$.

Now, we first recall some notations in variable function spaces. Given an open set $\Omega\subset\mathbb{R}^n$, and a measurable function $p(\cdot):\Omega\longrightarrow[1,\infty)$, $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions f on Ω such that for some $\lambda>0$,

$$\int_\Omega\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx<\infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\Omega)}=\inf\left\{\lambda>0:\int_\Omega\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx\le 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces or, more simply, as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x)=p$ is constant, then $L^{p(\cdot)}(\Omega)$ is isometrically isomorphic to $L^p(\Omega)$. The L^p spaces with variable exponent are a special case of Musielak-Orlicz spaces.

For all compact subsets $E\subset\Omega$, the space $L^{p(\cdot)}_{\text{loc}}(\Omega)$ is defined by $L^{p(\cdot)}_{\text{loc}}(\Omega):=\{f: f\in L^{p(\cdot)}(E)\}$. Define $\mathcal{P}(\Omega)$ to be set of $p(\cdot):\Omega\longrightarrow[1,\infty)$ such that

$$p^-=\operatorname{ess\,inf}\{p(x):x\in\Omega\}>1,\quad p^+=\operatorname{ess\,sup}\{p(x):x\in\Omega\}<\infty.$$

Denote $p'(x)=p(x)/(p(x)-1)$. Let $\mathcal{B}(\Omega)$ be the set of $p(\cdot)\in\mathcal{P}(\Omega)$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\Omega)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1.1. *([8]) Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le r_p\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},$$

where

$$r_p=1+1/p^--1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 1.2. *([7]) Suppose $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C>0$ such that for all balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C.$$

Lemma 1.3. *([7]) Let $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. Then for all balls B in $\mathbb{R}^n$ and all measurable subsets $S\subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\lesssim\frac{|B|}{|S|},$$

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \lesssim \left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$ *are constants.*

Throughout this paper δ_2 is the same as in Lemma 1.3, and the notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, then $f \approx g$.

We can obtain the following two definitions in [1].

Definition 1.4. ([1]) Let a function $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$.
(1) $g(\cdot)$ is locally log-Hölder continuous, if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \le \frac{C}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$ and $|x - y| < 1/2$.
(2) $g(\cdot)$ is locally log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C > 0$ such that

$$|g(x) - g(0)| \le \frac{C}{\log(e + 1/|x|)}$$

for all $x \in \mathbb{R}^n$.
(3) $g(\cdot)$ is locally log-Hölder continuous at infinity (or has a log decay at infinity), if there exist some $g_\infty \in \mathbb{R}^n$ and $C > 0$ such that

$$|g(x) - g_\infty| \le \frac{C}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$.

By $\mathcal{P}_0(\mathbb{R}^n)$ and $\mathcal{P}_\infty(\mathbb{R}^n)$ we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous at the origin and at infinity, respectively.

Next we will introduce some basic definitions and properties of non-isotropic spaces associated with general expansive dilations. A $n \times n$ real matrix A is called an expansive matrix, sometimes called a dilation, if all eigenvalues λ of A satisfy $|\lambda| > 1$. We suppose $\lambda_1, ..., \lambda_n$ are eigenvalues of A (taken according to the multiplicity) so that $1 < |\lambda_1| \le ... \le |\lambda_n|$. A set $\Delta \subset \mathbb{R}^n$ is said to be an ellipsoid if $\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$, for some nondegenerate $n \times n$ matrix P, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a dilation A, there exists an ellipsoid Δ and $r > 1$ such that $\Delta \subset r\Delta \subset A\Delta$, where $|\Delta|$, the Lebesgue measure of Δ, equals 1. Let $B_k = A^k\Delta$ for $k \in \mathbb{Z}$, then we have $B_k \subset rB_k \subset B_{k+1}$, and $|B_k| = b^k$, where $b = |\mathrm{det}A| > 1$. Let w be the smallest integer so that $2B_0 \subset A^w B_0 = B_w$. A quasi-norm associated with an expansive matrix A is a measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\rho_A(x) > 0 \qquad \text{for } x \ne 0,$$

$$\rho_A(Ax) = |\mathrm{det}A|\rho(x) \qquad \text{for } x \in \mathbb{R}^n,$$

$$\rho_A(x + y) \le C(\rho_A(x) + \rho_A(y)) \quad \text{for } x, y \in \mathbb{R}^n,$$

where $C \geq 1$ is a constant. One can show that all quasi-norms associated to a fixed dilation A are equivalent, see [2, Lemma 2.4]. Define the step homogeneous quasi-norm ρ on $\mathbb{R}^n$ induced by dilation A as

$$\rho(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \setminus B_j, \\ 0, & \text{if } x = 0. \end{cases}$$

For any $x, y \in \mathbb{R}^n$, we have

$$\rho(x+y) \leq b^w(\rho(x) + \rho(y)). \tag{1.1}$$

2 The decomposition for the anisotropic Herz spaces with two variable exponents

In this section, we first introduce the definition of anisotropic Herz spaces with two variable exponents. Let $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and non-negative integers, $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$, where χ_{C_k} is the characteristic function of C_k.

Definition 2.1. Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < p \leq \infty$ and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous anisotropic Herz space $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} = \left\{ \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

The non-homogeneous anisotropic Herz space $K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$ associated with the dilation A is defined by

$$K^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n) = \{f \in L^{q(\cdot)}_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K^{\alpha(\cdot),p}_{q(\cdot)}} < \infty\},$$

where

$$\|f\|_{K^{\alpha(\cdot),p}_{q(\cdot)}} = \left\{ \sum_{k=0}^{\infty} \|b^{k\alpha(\cdot)} f\tilde{\chi}_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right\}^{1/p}.$$

Here the usual modifications are made when $p = \infty$.

Next we will consider the decomposition of $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$. We begin with the notation of central block.

Definition 2.2. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$ and $0 < \alpha_l < \infty$. Denote $\alpha_l = \alpha(0), l < 0; \alpha_l = \alpha_\infty, l \geq 0$.

(i) A measurable function $a(x)$ is said to be a central $(\alpha(\cdot), q(\cdot))$-block if

(1) $\operatorname{supp} a \subset B_l$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq b^{-l\alpha_l}$.

(ii) A measurable function $a(x)$ is said to be a central $(\alpha(\cdot), q(\cdot))$-block of restricted type if

(1) $\operatorname{supp} a \subset B_l$ for some $l \geq 0$.

(2) $\|a\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq b^{-l\alpha_\infty}$.

The following decomposition theorem shows that the central blocks are the "building block" of the anisotropic Herz spaces with two variable exponents.

Theorem 2.3. *Let* $0 < p < \infty, q(\cdot) \in \mathcal{P}(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n) \cap \mathcal{P}_\infty(\mathbb{R}^n)$ *and* $0 < \alpha_l < \infty$. *The following two statements are equivalent:*

(i) $f \in \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$.

(ii) f *can be represented by*

$$f(x) = \sum_{k=-\infty}^{\infty} \lambda_k b_k(x), \tag{2.1}$$

where each b_k *is a central* $(\alpha(\cdot), q(\cdot))$*-block with support contained in* B_k *and* $\sum\limits_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$.

Proof. We first prove (i) implies (ii). For every $f \in \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$, write

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} f(x)\chi_k(x) \\ &= \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \frac{f(x)\chi_k(x)}{\|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \\ &= \sum_{k=-\infty}^{\infty} \lambda_k b_k(x), \end{aligned}$$

where $\lambda_k = \|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}$ and $b_k(x) = \frac{f(x)\chi_k(x)}{\|b^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}}$.

It is obvious that $\operatorname{supp} b_k \subset B_k$ and $\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} = |B_k|^{-\alpha_l/n}$. Thus, each b_k is a central $(\alpha(\cdot), q(\cdot))$-block with the support B_k and

$$\sum_{k=-\infty}^{\infty} |\lambda_k|^p = \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} = \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} < \infty.$$

Now we prove (ii) implies (i). Let $f(x) = \sum\limits_{k=-\infty}^{\infty} \lambda_k b_k(x)$ be a decomposition of f which satisfies the hypothesis (ii) of Theorem 2.3. For each $j \in \mathbb{Z}$, by the Minkowski inequality, we have

$$\|f\chi_j\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le \sum_{k=j}^{\infty} |\lambda_k| \|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{2.2}$$

Now we consider two cases for the index p.

If $0 < p \leq 1$. From (2.2) it follows that

$$\begin{aligned}
\|f\|^p_{\dot{K}^{\alpha,p}_{q(\cdot)}} &= \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{k=0}^{\infty} b^{k\alpha_\infty p} \|f\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \left(\sum_{j=k}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right) + \sum_{k=0}^{\infty} b^{k\alpha_\infty p} \left(\sum_{j=k}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \\
&= I + II.
\end{aligned}$$

For I, by $0 < \alpha(0), \alpha_\infty < \infty$, we have

$$\begin{aligned}
I &= \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \left(\sum_{j=k}^{-1} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{j=0}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \\
&\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \sum_{j=k}^{-1} |\lambda_j|^p b^{-j\alpha(0)p} + \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \sum_{j=0}^{\infty} |\lambda_j|^p b^{-j\alpha_\infty p} \\
&\lesssim \sum_{k=-\infty}^{-1} \sum_{j=k}^{-1} |\lambda_j|^p b^{(k-j)\alpha(0)p} + \sum_{j=0}^{\infty} |\lambda_j|^p b^{-j\alpha_\infty p} \\
&\lesssim \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{j} |\lambda_j|^p b^{(k-j)\alpha(0)p} + \sum_{j=0}^{\infty} |\lambda_j|^p \\
&\lesssim \sum_{j=-\infty}^{-1} |\lambda_j|^p + \sum_{j=0}^{\infty} |\lambda_j|^p \\
&\lesssim \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}$$

For II, by $0 < \alpha_\infty < \infty$, we have

$$\begin{aligned}
II &= \sum_{k=0}^{\infty} b^{k\alpha_\infty p} \left(\sum_{j=k}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \right) \\
&\lesssim \sum_{k=0}^{\infty} b^{k\alpha_\infty p} \left(\sum_{j=k}^{\infty} |\lambda_j|^p b^{-j\alpha_\infty p} \right) \\
&\lesssim \sum_{j=0}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j} b^{(k-j)\alpha_\infty p} \\
&\lesssim \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}$$

If $1 < p < \infty$, we have

$$\begin{aligned}\|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} &= \sum_{j=-\infty}^{-1} b^{j\alpha(0)p}\|f\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{j=0}^{\infty} b^{j\alpha_\infty p}\|f\chi_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\\ &= III + IV.\end{aligned}$$

For III, by (2.2), $0 < \alpha(0), \alpha_\infty < \infty$ and the Hölder inequality, we have

$$\begin{aligned}III &\lesssim \sum_{j=-\infty}^{-1} b^{j\alpha(0)p}\left(\sum_{k=j}^{-1}|\lambda_k|\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{k=0}^{\infty}|\lambda_k|\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\
&\lesssim \sum_{j=-\infty}^{-1}\left(\sum_{k=j}^{-1}|\lambda_k|b^{(j-k)\alpha(0)}\right)^p + \sum_{j=-\infty}^{-1}\left(\sum_{k=0}^{\infty}|\lambda_k|b^{-k\alpha_\infty+j\alpha(0)}\right)^p\\
&\lesssim \sum_{j=-\infty}^{-1}\left(\sum_{k=j}^{-1}|\lambda_k|^p b^{(j-k)\alpha(0)p/2}\right)\left(\sum_{k=j}^{-1} b^{(j-k)\alpha(0)p'/2}\right)^{p/p'}\\
&\quad + \sum_{j=-\infty}^{-1} b^{j\alpha(0)p}\left(\sum_{k=0}^{\infty}|\lambda_k|^p b^{-k\alpha_\infty p/2}\right)\left(\sum_{k=0}^{\infty} b^{-k\alpha_\infty p'/2}\right)^{p/p'}\\
&\lesssim \sum_{j=-\infty}^{-1}\left(\sum_{k=j}^{-1}|\lambda_k|^p b^{(j-k)\alpha(0)p/2}\right) + \sum_{k=0}^{\infty}|\lambda_k|^p b^{-k\alpha_\infty p/2}\\
&\lesssim \sum_{k=-\infty}^{-1}\sum_{j=-\infty}^{k}|\lambda_k|^p b^{(j-k)\alpha(0)p/2} + \sum_{k=0}^{\infty}|\lambda_k|^p\\
&\lesssim \sum_{k=-\infty}^{-1}|\lambda_k|^p + \sum_{k=0}^{\infty}|\lambda_k|^p\\
&\lesssim \sum_{k=-\infty}^{\infty}|\lambda_k|^p.\end{aligned}$$

For IV, by (2.2), $0 < \alpha_\infty < \infty$ and the Hölder inequality, we have

$$\begin{aligned}IV &\lesssim \sum_{j=0}^{\infty} b^{j\alpha_\infty p}\left(\sum_{k=j}^{\infty}|\lambda_k|\|b_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p\\
&\lesssim \sum_{j=0}^{\infty} b^{j\alpha_\infty p}\left(\sum_{k=j}^{\infty}|\lambda_k|b^{-k\alpha_\infty}\right)^p\\
&\lesssim \sum_{j=0}^{\infty} b^{j\alpha_\infty p}\left(\sum_{k=j}^{\infty}|\lambda_k|^p b^{-k\alpha_\infty p/2}\right)\left(\sum_{k=j}^{\infty} b^{-k\alpha_\infty p'/2}\right)^{p/p'}\\
&\lesssim \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}|\lambda_k|^p b^{(j-k)\alpha_\infty p/2}\right)\left(\sum_{k=j}^{\infty} b^{(j-k)\alpha_\infty p'/2}\right)^{p/p'}\end{aligned}$$

$$\begin{aligned}&\lesssim \sum_{k=0}^{\infty}|\lambda_k|^p\sum_{j=0}^{k}b^{(j-k)\alpha_\infty p/2}\\&\lesssim \sum_{k=0}^{\infty}|\lambda_k|^p.\end{aligned}$$

This leads to that $f\in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(A;\mathbb{R}^n)$ and then completes the proof of Theorem 2.3. □

Remark 2.4. From the proof of Theorem 2.3, it is easy to see that if $f\in \dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(A;\mathbb{R}^n)$ and $f(x)=\sum_{k=-\infty}^{\infty}\lambda_k b_k(x)$ be a central $(\alpha(\cdot),q(\cdot))$-block decomposition, then

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}}\approx\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}.$$

By an argument similar to the proof of Theorem 2.3, we can obtain the decomposition characterizations of the non-homogeneous anisotropic Herz spaces with two variable exponents as follows.

Theorem 2.5. *Let $0<p<\infty, q(\cdot)\in\mathcal{P}(\mathbb{R}^n), \alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0(\mathbb{R}^n)\cap\mathcal{P}_\infty(\mathbb{R}^n)$ and $0<\alpha_\infty<\infty$. The following two statements are equivalent:*

(i) $f\in K_{q(\cdot)}^{\alpha(\cdot),p}(A;\mathbb{R}^n)$.

(ii) f can be represented by

$$f(x)=\sum_{k=0}^{\infty}\lambda_k b_k(x),\tag{2.3}$$

where each b_k is a central $(\alpha(\cdot),q(\cdot))$-block of restricted type with support contained in B_k and $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$.

Moreover, the norms $\|f\|_{K_{q(\cdot)}^{\alpha(\cdot),p}}$ and $\inf\left(\sum_{k=0}^{\infty}|\lambda_k|^p\right)^{1/p}$ are equivalent, where the infimum is taken all over all decompositions of f as in (2.3).

3 Boundedness of some sublinear operators

As applications of the decomposition theorems, let us come to investigate the boundedness on the anisotropic Herz spaces with two variable exponents for some sublinear operators.

Theorem 3.1. *Let $0<p<\infty, q(\cdot)\in\mathcal{B}(\mathbb{R}^n), \alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{P}_0(\mathbb{R}^n)\cap\mathcal{P}_\infty(\mathbb{R}^n)$ and $0<\alpha(0),\alpha_\infty<\delta_2$. If a sublinear operator T satisfies*

$$|Tf(x)|\lesssim\int_{\mathbb{R}^n}\frac{|f(y)|}{\rho(x-y)}dy,\quad x\notin \operatorname{supp} f,\tag{3.1}$$

for any $f\in L^{q(\cdot)}(\mathbb{R}^n)$ with a compact support and T is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, then T is bounded on $\dot{K}_{q(\cdot)}^{\alpha(\cdot),p}(A;\mathbb{R}^n)$ and $K_{q(\cdot)}^{\alpha(\cdot),p}(A;\mathbb{R}^n)$, respectively.

Proof. It suffices to prove that T is bounded on $\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$. The non-homogeneous case can be proved in the similar way. Suppose $f \in \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(A;\mathbb{R}^n)$. By Theorem 2.3, $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j b_j(x)$, where each b_j is a central $(\alpha(\cdot), q(\cdot))$-block with support contained in B_j and

$$\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} \approx \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p}.$$

Therefore, we get

$$\begin{aligned}
\|Tf\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} &= \sum_{k=-\infty}^{\infty} \|b^{k\alpha(\cdot)}(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} + \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\|(Tf)\chi_k\|^p_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\
&\quad + \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=k-w}^{\infty} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\
&\quad + \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\
&\quad + \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us first estimate I_1. If $j \le k-w-1$, $x \in C_k$ and $y \in B_j$, by (1.1) we have

$$\rho(x-y) \ge b^{-w}\rho(x) - \rho(y) \ge b^{-w}\rho(x) - b^{-w-1}\rho(x) = b^{-w}(1-1/b)\rho(x).$$

Therefore by (3.1) and the generalized Hölder inequality, we get

$$\begin{aligned}
|Tb_j(x)| &\le C\rho(x)^{-1}\int_{B_j} |b_j(y)|dy \\
&\le Cb^{-k}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

So by Lemma 1.2 and Lemma 1.3, we have

$$\begin{aligned}
\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\lesssim b^{-k}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim b^{-k}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}(|B_k|\|\chi_{B_k}\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)})\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{R}^n)}} \\
&\lesssim b^{\delta_2(j-k)}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \qquad (3.2)
\end{aligned}$$

Therefore, when $0 < p \le 1$, by $0 < \alpha(0) < \delta_2$, we get

$$\begin{aligned} I_1 &= \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\ &\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|^p b^{[\delta_2(j-k)-j\alpha(0)]p}\Big) \\ &\lesssim \sum_{j=-\infty}^{-w-2} |\lambda_j|^p \sum_{k=j+w+1}^{-1} b^{(j-k)[\delta_2-\alpha(0)]p} \\ &\lesssim \sum_{j=-\infty}^{-w-2} |\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \end{aligned} \tag{3.3}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $0 < \alpha(0) < \delta_2$, by (3.2) and the Hölder inequality, we have

$$\begin{aligned} I_1 &\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j| b^{\delta_2(j-k)-j\alpha(0)}\Big)^p \\ &\lesssim \sum_{k=-\infty}^{-1} \Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|^p b^{(j-k)[\delta_2-\alpha(0)]p/2}\Big)\Big(\sum_{j=-\infty}^{k-w-1} b^{(j-k)[\delta_2-\alpha(0)]p'/2}\Big)^{p/p'} \\ &\lesssim \sum_{k=-\infty}^{-1} \Big(\sum_{j=-\infty}^{k-w-1} |\lambda_j|^p b^{(j-k)[\delta_2-\alpha(0)]p/2}\Big) \\ &\lesssim \sum_{j=-\infty}^{-w-2} |\lambda_j|^p \sum_{k=j+w+1}^{-1} b^{(j-k)[\delta_2-\alpha(0)]p/2} \\ &\lesssim \sum_{j=-\infty}^{-w-2} |\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \end{aligned} \tag{3.4}$$

Let us now estimate I_2. Similarly, we consider two cases for p. When $0 < p \le 1$, by $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T, we have

$$\begin{aligned} I_2 &= \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p}\Big(\sum_{j=k-w}^{\infty} |\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p \\ &\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=k-w}^{\infty} |\lambda_j|^p \|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big) \\ &\lesssim \sum_{k=-\infty}^{-1} b^{k\alpha(0)p}\Big(\sum_{j=k-w}^{-1} |\lambda_j|^p b^{-j\alpha(0)p} + \sum_{j=0}^{\infty} |\lambda_j|^p b^{-j\alpha_\infty p}\Big) \\ &\lesssim \sum_{k=-\infty}^{-1}\sum_{j=k-w}^{-1} |\lambda_j|^p b^{(k-j)\alpha(0)p} + \sum_{k=-\infty}^{-1} b^{k\alpha(0)p} \sum_{j=0}^{\infty} |\lambda_j|^p b^{-j\alpha_\infty p} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j=-\infty}^{-1}|\lambda_j|^p\sum_{k=-\infty}^{j+w}b^{(k-j)\alpha(0)p}+\sum_{j=0}^{\infty}|\lambda_j|^p\\
&\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^p\lesssim\|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \qquad (3.5)
\end{aligned}$$

When $1<p<\infty$, take $1/p+1/p'=1$. By $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$\begin{aligned}
I_2 &\lesssim \sum_{k=-\infty}^{-1}b^{k\alpha(0)p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=-\infty}^{-1}\Big(\sum_{j=k-w}^{-1}|\lambda_j|b^{(k-j)\alpha(0)}\Big)^p+\sum_{k=-\infty}^{-1}b^{k\alpha(0)p}\Big(\sum_{j=0}^{\infty}|\lambda_j|b^{-j\alpha_\infty}\Big)^p\\
&\lesssim \sum_{k=-\infty}^{-1}\Big(\sum_{j=k-w}^{-1}|\lambda_j|^pb^{(k-j)\alpha(0)p/2}\Big)\Big(\sum_{j=k-w}^{-1}b^{(k-j)\alpha(0)p'/2}\Big)^{p/p'}\\
&\quad+\Big(\sum_{j=0}^{\infty}|\lambda_j|^pb^{-j\alpha_\infty p/2}\Big)\Big(\sum_{j=0}^{\infty}b^{-j\alpha_\infty p'/2}\Big)^{p/p'}\\
&\lesssim \sum_{j=-\infty}^{-1}|\lambda_j|^p\sum_{k=-\infty}^{j+w}b^{(k-j)\alpha(0)p/2}+\sum_{j=0}^{\infty}|\lambda_j|^p\\
&\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^p\lesssim\|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \qquad (3.6)
\end{aligned}$$

For I_3, when $0<p\le 1$, by $0<\alpha(0),\alpha_\infty<\delta_2$, we get

$$\begin{aligned}
I_3 &= \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=-\infty}^{k-w-1}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|^pb^{[\delta_2(j-k)-j\alpha(0)]p}\Big)\\
&\quad+\sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=0}^{k-w-1}|\lambda_j|^pb^{[\delta_2(j-k)-j\alpha_\infty]p}\Big)\\
&\lesssim \sum_{k=0}^{\infty}b^{k(\alpha_\infty-\delta_2)p}\Big(\sum_{j=-\infty}^{-1}|\lambda_j|^pb^{j[\delta_2-\alpha(0)]p}\Big)\\
&\quad+\sum_{j=0}^{\infty}|\lambda_j|^p\Big(\sum_{k=j+w+1}^{\infty}b^{(j-k)(\delta_2-\alpha_\infty)p}\Big)\\
&\lesssim \sum_{j=-\infty}^{-1}|\lambda_j|^p+\sum_{j=0}^{\infty}|\lambda_j|^p
\end{aligned}$$

$$\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \tag{3.7}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. Since $0 < \alpha(0), \alpha_\infty < \delta_2$, by (3.2) and the Hölder inequality, we have

$$\begin{aligned}
I_3 &\lesssim \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=-\infty}^{k-w-1}|\lambda_j| b^{\delta_2(j-k)}\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=-\infty}^{-1}|\lambda_j| b^{\delta_2(j-k)-j\alpha(0)}\Big)^p\\
&\quad+\sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=0}^{k-w-1}|\lambda_j| b^{\delta_2(j-k)-j\alpha_\infty}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty} b^{k(\alpha_\infty-\delta_2)p}\Big(\sum_{j=-\infty}^{-1}|\lambda_j| b^{j[\delta_2-\alpha(0)]}\Big)^p\\
&\quad+\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k-w-1}|\lambda_j| b^{(j-k)(\delta_2-\alpha_\infty)}\Big)^p\\
&\lesssim \Big(\sum_{j=-\infty}^{-1}|\lambda_j|^p b^{j[\delta_2-\alpha(0)]p/2}\Big)\Big(\sum_{j=-\infty}^{-1} b^{j[\delta_2-\alpha(0)]p'/2}\Big)^{p/p'}\\
&\quad+\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{k-w-1}|\lambda_j|^p b^{(j-k)(\delta_2-\alpha_\infty)p/2}\Big)\Big(\sum_{j=0}^{k-w-1} b^{(j-k)(\delta_2-\alpha_\infty)p'/2}\Big)^{p/p'}\\
&\lesssim \sum_{j=-\infty}^{-1}|\lambda_j|^p b^{j[\delta_2-\alpha(0)]p/2}+\sum_{k=0}^{\infty}\sum_{j=0}^{k-w-1}|\lambda_j|^p b^{(j-k)(\delta_2-\alpha_\infty)p/2}\\
&\lesssim \sum_{j=-\infty}^{-1}|\lambda_j|^p+\sum_{j=0}^{\infty}|\lambda_j|^p\sum_{k=j+w+1}^{\infty} b^{(j-k)(\delta_2-\alpha_\infty)p/2}\\
&\lesssim \sum_{j=-\infty}^{\infty}|\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}.
\end{aligned}\tag{3.8}$$

Let us now estimate I_4. Similarly, we consider two cases for p. When $0 < p \le 1$, by $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T, we have

$$\begin{aligned}
I_4 &= \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|^p\|b_j\|^p_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)\\
&\lesssim \sum_{k=0}^{\infty} b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|^p b^{-j\alpha_\infty p}\Big)
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j=-w}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+w}b^{(k-j)\alpha_\infty p}\\
&\lesssim \sum_{j=-w}^{\infty}|\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \qquad (3.9)
\end{aligned}$$

When $1 < p < \infty$, take $1/p + 1/p' = 1$. By $L^{q(\cdot)}(\mathbb{R}^n)$ boundedness of T and the Hölder inequality, we have

$$\begin{aligned}
I_4 &= \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|\|(Tb_j)\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|\|b_j\|_{L^{q(\cdot)}(\mathbb{R}^n)}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|b^{-j\alpha_\infty}\Big)^p\\
&\lesssim \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|^p b^{-j\alpha_\infty p/2}\Big)\Big(\sum_{j=k-w}^{\infty}b^{-j\alpha_\infty p'/2}\Big)^{p/p'}\\
&\lesssim \sum_{k=0}^{\infty}b^{k\alpha_\infty p}\Big(\sum_{j=k-w}^{\infty}|\lambda_j|^p b^{-j\alpha_\infty p/2}\Big)\\
&\lesssim \sum_{j=-w}^{\infty}|\lambda_j|^p\sum_{k=0}^{j+w}b^{(k-j)\alpha_\infty p/2}\\
&\lesssim \sum_{j=-w}^{\infty}|\lambda_j|^p \lesssim \|f\|^p_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}. \qquad (3.10)
\end{aligned}$$

Combining (3.3)-(3.10), we have

$$\|Tf\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}} \lesssim \|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot)}}.$$

Thus, the proof of Theorem 3.1 is completed. □

Remark 3.2. From the proof of Theorem 3.1, it is easy to see that the size condition (3.1) can be replaced by

$$|Tf(x)| \le C\frac{\|f\|_{L^1}}{\rho(x)}, \quad \text{if } \inf_{y\in \operatorname{supp} f}\rho(x-y) \ge b^{-w}(1-1/b)\rho(x), \qquad (3.11)$$

for all $f \in L^{q(\cdot)}(\mathbb{R}^n)$ with compact support.

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