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L^p Quantitative Uncertainty Principles for the Generalized Fourier Transform Associated with the Spherical Mean Operator

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Abstract

The aim of this paper is to prove new quantitative uncertainty principles for the Fourier transform connected with the spherical mean operator. The first of these results is an extension of the Donoho and Stark's uncertainty principle. The second result extends the Heisenberg-Pauli-Weyl uncertainty principle. From these two results we deduce a continuous-time principle for the L^p theory, when $1 < p \leq 2$.

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1 Introduction

Uncertainty principles are mathematical results that give limitations on the simultaneous concentration of a function and its Fourier transform. There are many ways to get the statement about concentration precise. The most famous of them is the so called Heisenberg uncertainty Principle [16] where concentration is measured by dispersion and the Hardy uncertainty Principle [14] where concentration is measured in terms of fast decay. A considerable attention has been devoted recently to discovering new formulations and new contexts for the uncertainty principle. Indeed, Morgan [23], Cowling and Price [8], Beurling [3], Miyachi [22] for example interpreted the smallness as sharp pointwise estimates

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or integrable decay of functions and gave qualitative uncertainty principles for the Fourier transforms. Landau and Pollak [20], Slepian and Pollak [30], Benedicks [2] and Donoho and Stark [10] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms. (*see* the surveys [5, 12] and the book [15] for other forms of the uncertainty principle).

The spherical mean operator play an important role and have many applications, for example; in the image processing of so-called synthetic aperture radar (SAR) data [17, 18], or in the linearized inverse scattering problem in acoustics [11]. These operators have been studied by many authors from many points of view [1, 11, 25, 28].

Many uncertainty principles have already been proved for the generalized Fourier transform associated with the spherical mean operator, for examples (cf. [6, 7, 21, 24, 27]).

Our aim here is to prove new uncertainty principles for the generalized Fourier transform associated with the spherical mean operator. The uncertainty principles proved in this paper and in [21], (we recall some of these results in the Appendix), have many applications, for example for the generalized wavelet transform associated with the spherical mean operator, and for the generalized heat and Schrödinger equations. In a forthcoming paper we study these applications.

The remaining part of the paper is organized as follows. In §2, we recall the main results about the spherical mean operator. §3 is devoted to study the generalized versions of Donoho-Stark's uncertainty principle. In the last section we study many variants of Heisenberg's inequalities for $\mathscr{F}$.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence.

2 Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see ([25]).
We denote by

- $C_*(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$, even with respect to the first variable.

- $C_{*,c}(\mathbb{R}^{d+1})$ the subspace of $C_*(\mathbb{R}^{d+1})$ formed by functions with compact support.

- $\mathcal{E}_*(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$, even with respect to the first variable.

- $\mathcal{S}_*(\mathbb{R}^{d+1})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.

- S^d the unit sphere in $\mathbb{R}^{d+1}$,

$$S^d = \Big\{(\eta, \xi) \in \mathbb{R}^{d+1} : \eta^2 + ||\xi||^2 = 1\Big\},$$

  where for $\xi = (\xi_1, ..., \xi_d)$, we have $||\xi||^2 = \xi_1^2 + ... + \xi_d^2$.

- $d\sigma_d$ the normalized surface measure on S^d.
- $\mathbb{R}_+^{d+1} = \Big\{(r,x) \in \mathbb{R}^{d+1} : \ r > 0\Big\}$.

Definition 2.1. The spherical mean operator is defined on $C_*(\mathbb{R}^{d+1})$ by

$$\forall (r,x) \in \mathbb{R}_+^{d+1}, \ \mathcal{R}f(r,x) = \int_{S^d} f(r\eta, x + r\xi) d\sigma_d(\eta,\xi).$$

The spherical mean kernel is the function $\varphi_{\mu,\lambda}$, $(\mu,\lambda) \in \mathbb{C}^{d+1} = \mathbb{C} \times \mathbb{C}^d$, defined by

$$\forall (r,x) \in \mathbb{R}_+^{d+1}, \ \varphi_{\mu,\lambda}(r,x) = \mathcal{R}(\cos(\mu.)e^{-i<\lambda,.>})(r,x).$$

We have

$$\varphi_{\mu,\lambda}(r,x) = j_{\frac{d-1}{2}}(r\sqrt{\mu^2 + \lambda^2})e^{-i\langle\lambda,x\rangle},$$

where

- $\lambda^2 = \lambda_1^2 + ... + \lambda_d^2$, if $\lambda = (\lambda_1,...,\lambda_d) \in \mathbb{C}^d$
- $\langle\lambda,x\rangle = \lambda_1 x_1 + ... + \lambda_d x_d$, if $x = (x_1,...,x_d) \in \mathbb{R}^d$ and $\lambda = (\lambda_1,...,\lambda_d) \in \mathbb{C}^d$
- $j_{\frac{d-1}{2}}$ is the normalized Bessel function defined by

$$j_{(d-1)/2}(x) = \Gamma((d+1)/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma((2k+1+d)/2)} (z/2)^{2k}.$$

Remark 2.2. For all $\nu \in \mathbb{N}^{d+1}, (r,x) \in \mathbb{R}^{d+1}$ and $z = (\mu,\lambda) \in \mathbb{C}^{d+1}$,

$$|D_z^\nu \varphi_{\mu,\lambda}(r,x)| \leq \|(r,x)\|^{|\nu|} \exp(2\|(r,x)\| \, \|\mathrm{Im} z\|), \tag{2.1}$$

where

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_{d+1}^{\nu_{d+1}}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_{d+1}.$$

Now let Γ be the set

$$\Gamma = \mathbb{R}^{d+1} \cup \Big\{(it,x); (t,x) \in \mathbb{R}^{d+1}, |t| \leq \|x\|\Big\}.$$

Γ_+ the subset of Γ, given by

$$\Gamma_+ = \mathbb{R}^{d+1} \cup \Big\{(it,x); (t,x) \in \mathbb{R}^{d+1}, 0 \leq t \leq \|x\|\Big\}.$$

We have for all $(\mu,\lambda) \in \Gamma$,

$$\sup_{(r,x)\in\mathbb{R}^{d+1}} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following, we denote by

- $d\nu(r,x)$ the measure defined on $\mathbb{R}_+^{d+1}$ by

$$d\nu(r,x) = k_d r^d dr \otimes dx,$$

with

$$k_d = \frac{1}{2^{(d-1)/2}\Gamma((d+1)/2)(2\pi)^{d/2}}.$$

- $L^p(d\nu), 1 \le p \le \infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$, satisfying

$$\begin{aligned} \|f\|_{L^p(d\nu)} &= \left(\int_{\mathbb{R}_+^{d+1}} |f(r,x)|^p d\nu(r,x)\right)^{1/p} < \infty, \ 1 \le p < \infty, \\ \|f\|_{L^\infty(d\nu)} &= \operatorname*{ess\ sup}_{(r,x)\in\mathbb{R}_+^{d+1}} |f(r,x)| < \infty, \ p = \infty. \end{aligned}$$

- $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \{\theta^{-1}(B) : \quad B \in \mathcal{B}_{Bor}(\mathbb{R}_+^{d+1})\},$$

where θ defined on the set Γ_+ by $\theta(\lambda,\mu) = (\sqrt{\mu^2 + \|\lambda\|^2}, \lambda)$.

- $d\gamma$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall A \subset \mathcal{B}_{\Gamma_+}, \quad \gamma(A) = \nu(\theta(A)).$$

- $L^p(d\gamma), 1 \le p \le \infty$, the space of measurable functions on Γ_+, satisfying

$$\begin{aligned} \|f\|_{L^p(d\gamma)} &= \left(\int_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\right)^{1/p} < \infty, \ 1 \le p < \infty, \\ \|f\|_{L^\infty(d\gamma)} &= \operatorname*{ess\ sup}_{(\mu,\lambda)\in\Gamma_+} |f(\mu,\lambda)| < \infty, \ p = \infty. \end{aligned}$$

We have the following properties.

Proposition 2.3. *i) For every nonnegative measurable function g on Γ_+, we have*

$$\int_{\Gamma_+} f(\mu,\lambda)d\gamma(\mu,\lambda) = k_d\Big[\int_{\mathbb{R}_+^{d+1}} f(\mu,\lambda)(\mu^2 + \|\lambda\|^2)^{(d-1)/2}\mu d\mu d\lambda$$

$$+ \int_{\mathbb{R}^d}\int_0^{\|\lambda\|} f(i\mu,\lambda)(\|\lambda\|^2 - \mu^2)^{(d-1)/2}\mu d\mu d\lambda\Big].$$

ii) For every nonnegative measurable function f on $\mathbb{R}_+^{d+1}$ (resp. integrable on $\mathbb{R}_+^{d+1}$ with respect to the measure $d\nu$), $f\circ\theta$ is a measurable nonnegative function on Γ_+, (resp. integrable on Γ_+ with respect to the measure $d\gamma$) and we have

$$\int_{\Gamma_+} f\circ\theta(\mu,\lambda)d\gamma(\mu,\lambda) = \int_{\mathbb{R}_+^{d+1}} f(r,x)d\nu(r,x). \tag{2.2}$$

In the following we recall some results on the dual of the spherical mean operator $\mathcal{R}$.

Definition 2.4. The dual ${}^t\mathcal{R}$ of the spherical mean operator $\mathcal{R}$ is defined by : $\forall (s,y) \in \mathbb{R}^{d+1}$,

$$ {}^t\mathcal{R}(f)(s,y) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} f(\sqrt{s^2 + \|y-z\|^2}, z)dz,\ f \in C_{*,c}(\mathbb{R}^{d+1}). \tag{2.3} $$

Example 2.5. Let $p \in [1,\infty)$. For all $a > 0, \beta > 0$ we have

$$ \forall\ (s,y) \in \mathbb{R}^{d+1}, \quad {}^t\mathcal{R}(E^p_{a,\beta})(s,y) = C(a,\beta,p)E^p_{\frac{a\beta}{1+\beta},1+\beta}(s,y), \tag{2.4} $$

with $E_{a,\beta}$ is the Gauss kernel associated with the spherical mean operator $\mathcal{R}$ defined by

$$ \forall (r,x) \in \mathbb{C}^{d+1},\ E_{a,\beta}(r,x) = k(a,\beta)e^{-a(\beta r^2 + \|(r,x)\|^2)}, \tag{2.5} $$

where

$$ k(a,\beta) = \frac{2\sqrt{\pi}a^{d+\frac{1}{2}}}{\Gamma(\frac{d+1}{2})}(\frac{\beta}{\pi})^{\frac{d+1}{2}}, \quad \text{and} \quad C(a,\beta,p) = \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}}\Big[\frac{(1+\beta)^{p-1}}{a\beta^p p}\Big]^{\frac{d}{2}}. $$

Proposition 2.6. *The function ${}^t\mathcal{R}(f)$ defined almost everywhere on $\mathbb{R}^{d+1}_+$ by*

$$ {}^t\mathcal{R}(f)(s,y) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} f(\sqrt{s^2 + \|y-x\|^2}, x)dx $$

is Lebesgue integrable on $\mathbb{R}^{d+1}_+$. Moreover for all bounded function $g \in C_*(\mathbb{R}^{d+1})$, we have the formula

$$ \int_{\mathbb{R}^{d+1}_+} {}^t\mathcal{R}(f)(s,y)g(s,y)dsdy = \int_{\mathbb{R}^{d+1}_+} \mathcal{R}(g)(r,x)f(r,x)r^d drdx. \tag{2.6} $$

Remark 2.7. Let f be in $L^1(d\nu)$. By taking $g \equiv 1$ in the relation (2.6) we deduce that

$$ \int_{\mathbb{R}^{d+1}_+} {}^t\mathcal{R}(f)(s,y)dsdy = C(d)\int_{\mathbb{R}^{d+1}_+} f(r,x)r^d drdx, \tag{2.7} $$

where

$$ C(d) := \int_{S^d} d\sigma_d(\eta,\xi). $$

We consider the generalized Fourier transform $\mathcal{F}$ associated with the spherical mean operator $\mathcal{R}$ and we recall its main properties.

Definition 2.8. The Fourier transform associated with the spherical mean operator is defined on $L^1(d\nu)$ by

$$ \forall\ (\mu,\lambda) \in \Gamma, \mathcal{F}(f)(\mu,\lambda) = \int_{\mathbb{R}^{d+1}_+} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu(r,x). \tag{2.8} $$

Example 2.9. Let $a > 0$, $\beta > 0$. The Fourier transform of Gauss kernel associated with spherical mean operator is given by

$$ \forall\ \ (\mu,\lambda) \in \Gamma,\ \mathcal{F}(E_{a,\beta})(\mu,\lambda) = C(a,\beta,d)E_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\mu,\lambda), $$

where

$$ C(a,\beta,d) = 2^{2d}\Gamma(\frac{d+1}{2})(a\beta)^{d+\frac{1}{2}}(\frac{\pi}{1+\beta})^{\frac{d}{2}}. $$

Proposition 2.10. *For all f in $L^1(d\nu)$, we have the relation*

$$\forall\ (\mu,\lambda) \in \Gamma, \mathcal{F}(f)(\mu,\lambda) = \mathcal{F}_0 \circ {}^t\mathcal{R}(f)(\mu,\lambda), \tag{2.9}$$

where $\mathcal{F}_0$ is the Fourier-cosine transform on $\mathbb{R}^{d+1}$ defined for f in $\mathcal{S}_*(\mathbb{R}^{d+1})$ by

$$\forall\ (\mu,\lambda) \in \mathbb{R}^{d+1},\ \mathcal{F}_0(f)(\mu,\lambda) = \int_{\mathbb{R}_+^{d+1}} f(r,x)e^{-i\langle\lambda,x\rangle}\cos(r\mu)drdx.$$

In the follow we recall some properties on the Fourier transform $\mathcal{F}$.

For all $f \in L^1(d\nu)$,

$$\|\mathcal{F}(f)\|_{L^\infty(d\gamma)} \leq \|f\|_{L^1(d\nu)}. \tag{2.10}$$

For $f \in L^1(d\nu)$ such that $\mathcal{F}f \in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost every $(r,x) \in \mathbb{R}_+^{d+1}$,

$$f(r,x) = \int_{\Gamma_+} \mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda). \tag{2.11}$$

Theorem 2.11. *(Plancherel formula). For every f in $\mathcal{S}_*(\mathbb{R}^{d+1})$, we have*

$$\int_\Gamma |\mathcal{F}(f)(\lambda,\mu)|^2 d\gamma(\lambda,\mu) = \int_{\mathbb{R}_+^{d+1}} |f(r,x)|^2 d\nu(r,x). \tag{2.12}$$

In particular, the Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

Proposition 2.12. *Let f be in $L^p(d\nu)$, $p \in [1,2]$. Then $\mathcal{F}(f)$ belongs to $L^{p'}(d\gamma)$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and we have*

$$\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \leqslant \|f\|_{L^p(d\nu)}.$$

For $(r,x) \in \mathbb{R}^{d+1}, s > 0$, we note $N_s(r,x)$, by

$$N_s(r,x) := e^{-s(r^2+\|x\|^2)}. \tag{2.13}$$

We have

$$\mathcal{F}\left(N_s(r,x)\right)(t,y) = C(s)e^{-\frac{(t^2+2\|y\|^2)}{4s}}.$$

We define the following functions W_l^s, $\widetilde{W}_l^s$, $l \in \mathbb{N}^{d+1}$, $s > 0$ by

$$\forall\,(r,x) \in \mathbb{R}^{d+1}, \quad W_l^s(r,x) = r^{2k}x^m e^{-s(r^2+\|x\|^2)}, \quad l = (k,m), \tag{2.14}$$

and

$$\forall\,(r,x) \in \mathbb{R}^{d+1}, \quad \widetilde{W}_l^s(r,x) = \mathcal{F}^{-1}(\lambda^{2k}\mu^m e^{-s(\lambda^2+\|\mu\|^2)})(r,x), \quad l = (k,m), \tag{2.15}$$

Notation. We denote by $\mathcal{P}_m(\mathbb{R}^{d+1})$ the set of homogeneous polynomials of degree m.

Proposition 2.13. ([7]). Let $l \in \mathbb{N}^{d+1}$. For all $s > 0$, there exists a homogeneous $Q \in \mathcal{P}_l(\mathbb{R}^{d+1})$ such that

$$\forall\,(r,x) \in \mathbb{R}^{d+1}, \quad \mathcal{F}(W_l^s)(r,x) = Q(r,x)e^{-\frac{1}{4s}(r^2+2\|x\|^2)}. \tag{2.16}$$

3 Donoho-Stark's uncertainty principle

We shall investigate the case where f and $\mathcal{F}(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows. If $f \in L^p(d\nu)$, $1 \le p \le 2$, is ε-concentrated on a measurable set $E \subset \mathbb{R}_+^{d+1}$ if there is a measurable function g vanishing outside E such that $\|f - g\|_{L^p(d\nu)} \le \varepsilon \|f\|_{L^p(d\nu)}$. Therefore, if we introduce a projection operator P_E as

$$P_E f(r,x) = \begin{cases} f(r,x) & \text{if } (r,x) \in E \\ 0 & \text{if } (r,x) \notin E, \end{cases}$$

then f is ε-concentrated on E if and only if $\|f - P_E f\|_{L^p(d\nu)} \le \varepsilon \|f\|_{L^p(d\nu)}$.

We define a projection operator Q_W as

$$Q_W f(r,x) = \mathcal{F}^{-1}\Big(P_W(\mathcal{F}(f))\Big)(r,x). \tag{3.1}$$

Similarly, we say that $\mathcal{F}(f)$ is ε_W-concentrated to W in $L^{p'}(d\gamma)$ if and only if

$$\|\mathcal{F}(f) - \mathcal{F}(Q_W f)\|_{L^{p'}(d\gamma)} \le \varepsilon_W \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}. \tag{3.2}$$

If E and W are sets of finite measure, we define $mes_\nu(E)$ and $mes_\gamma(W)$ as follow

$$mes_\nu(E) := \int_E d\nu(r,x), \quad mes_\gamma(W) := \int_W d\gamma(\mu,\lambda).$$

Lemma 3.1. *Let W a measurable set of $\mathbb{R}_+^{d+1}$ such that $mes_\gamma(W) < \infty$. Let $f \in L^p(d\nu)$ with $p \in [1,2]$. We have*

$$Q_W f(x) = \int_W \overline{\varphi_{\mu,\lambda}(r,x)} \mathcal{F}(f)(\mu,\lambda) d\gamma(\mu,\lambda).$$

Proof. Let $f \in L^p(d\nu)$ with $p \in [1,2]$. By Hölder's inequality and Proposition 2.12

$$\begin{aligned} \|P_W(\mathcal{F}(f))\|_{L^1(d\gamma)} &= \int_W |\mathcal{F}(f)(\mu,\lambda)| d\gamma(\mu,\lambda) \\ &\le \Big(mes_\gamma(W)\Big)^{\frac{1}{p}} \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \\ &\le \Big(mes_\gamma(W)\Big)^{\frac{1}{p}} \|f\|_{L^p(d\nu)}. \end{aligned}$$

and

$$\begin{aligned} \|P_W(\mathcal{F}(f))\|_{L^2(d\gamma)} &= \int_W |\mathcal{F}(f)(\mu,\lambda)|^2 d\gamma(\mu,\lambda) \\ &\le \Big(mes_\gamma(W)\Big)^{\frac{p'-2}{p'}} \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \\ &\le \Big(mes_\nu(W)\Big)^{\frac{p'-2}{p'}} \|f\|_{L^p(d\nu)}. \end{aligned}$$

Hence $P_W(\mathcal{F}(f)) \in L^1(d\gamma) \bigcap L^2(d\gamma)$. This combined with (3.1) gives the result. □

Let $B_{L^p(d\nu)}(T)$, $1 \le p \le 2$, the subspace of all $g \in L^p(d\nu)$ such that $Q_T g = g$. We say that f is ε-bandlimited to T if there is a $g \in B_{L^p(d\nu)}(T)$ with $\|f - g\|_{L^p(d\nu)} < \varepsilon \|f\|_{L^p(d\nu)}$. Here we denote by $\|P_E\|_p$ the operator norm of P_E on $L^p(d\nu)$ and by $\|P_E\|_{p,T}$ the operator norm of $P_E : B_{L^p(d\nu)}(T) \to L^p(d\nu)$.

Lemma 3.2. *Let E and T be measurable sets of $\mathbb{R}_+^{d+1}$. For $p \in [1,2]$, we have*

$$\|P_E\|_{p,T} \leq \Big(mes_\nu(E) mes_\gamma(T)\Big)^{\frac{1}{p}}.$$

Proof. If at least one of $mes_\nu(E)$ and $mes_\gamma(T)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and T have finite positive measures.

For $f \in B_{L^p(d\nu)}(T)$ we see that

$$f(r,x) = \int_T \overline{\varphi_{\mu,\lambda}(r,x)} \mathcal{F}(f)(\mu,\lambda) d\gamma(\mu,\lambda).$$

By (2.1), Hölder's inequality and Proposition 2.12

$$\begin{aligned} |f(r,x)| &\leq \Big(mes_\gamma(T)\Big)^{\frac{1}{p}} \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \\ &\leq \Big(mes_\gamma(T)\Big)^{\frac{1}{p}} \|f\|_{L^p(d\nu)}. \end{aligned}$$

Therefore

$$\|P_E f\|_{L^p(d\nu)} = \Big(\int_E |f(r,x)|^p d\nu(r,x)\Big)^{\frac{1}{p}} \leq \Big(mes_\nu(E) mes_\gamma(T)\Big)^{\frac{1}{p}} \|f\|_{L^p(d\nu)}.$$

Then, it follows that for $f \in B_{L^p(d\nu)}(W)$,

$$\frac{\|P_E f\|_{L^p(d\nu)}}{\|f\|_{L^p(d\nu)}} \leq \Big(mes_\nu(E) mes_\gamma(T)\Big)^{\frac{1}{p}},$$

which implies the desired inequality. □

Proposition 3.3. *Let $f \in L^p(d\nu)$. If f is ε_E-concentrated to E and ε_T-bandlimited to T, then*

$$\Big(mes_\nu(E) mes_\gamma(T)\Big)^{\frac{1}{p}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{L^p(d\nu)} = 1$. Since f is ε_E-concentrated to E, it follows that $\|P_E f\|_{L^p(d\nu)} \geq \|f\|_{L^p(d\nu)} - \|f - P_E f\|_{L^p(d\nu)} \geq 1 - \varepsilon_E$. Moreover, since f is ε_T-bandlimited, there is a $g \in B_{L^p(d\nu)}(T)$ with $\|g - f\|_{L^p(d\nu)} \leq \varepsilon_T$. Therefore, it follows that

$$\|P_E g\|_{L^p(d\nu)} \geq \|P_E f\|_{L^p(d\nu)} - \|P_E(g-f)\|_{L^p(d\nu)} \geq \|P_E f\|_{L^p(d\nu)} - \varepsilon_T \geq 1 - \varepsilon_E - \varepsilon_T$$

and $\|g\|_{L^p(d\nu)} \leq \|f\|_{L^p(d\nu)} + \varepsilon_T = 1 + \varepsilon_T$. Then, we see that

$$\frac{\|P_E g\|_{L^p(d\nu)}}{\|g\|_{L^p(d\nu)}} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}.$$

Hence $\|P_E\|_{p,T} \geq \frac{1-\varepsilon_E-\varepsilon_T}{1+\varepsilon_T}$ and Lemma 3.2 yields the desired inequality. □

Proposition 3.4. *Let E and T be measurable subsets of $\mathbb{R}_+^{d+1}$, and $f \in L^p(d\nu)$ for $p \in (1,2]$. If f is ε_E-concentrated to E in $L^p(d\nu)$-norm and $\mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, then*

$$(mes_\nu(E) mes_\gamma(T))^{\frac{1}{p'}} \geq \frac{(1-\varepsilon_E)\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} - \varepsilon_T \|f\|_{L^p(d\nu)}}{\|f\|_{L^p(d\nu)}}.$$

Proof. Let $f\in L^p(d\nu)$ for $p\in(1,2]$. As above

$$\begin{aligned}\|\mathcal{F}(f)-\mathcal{F}(Q_TP_Ef)\|_{L^{p'}(d\gamma)} &\le \|\mathcal{F}(f)-\mathcal{F}(Q_Tf)\|_{L^{p'}(d\gamma)}\\ &\quad+\|\mathcal{F}(Q_Tf)-\mathcal{F}(Q_TP_Ef)\|_{L^{p'}(d\gamma)}\\ &\le \varepsilon_T\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}+\|f-P_Ef\|_{L^p(d\nu)}\\ &\le \varepsilon_T\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}+\varepsilon_E\|f\|_{L^p(d\nu)}\end{aligned}$$

and thus,

$$\begin{aligned}\|\mathcal{F}(Q_TP_Ef)\|_{L^{p'}(d\gamma)} &\ge \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}-\|\mathcal{F}(f)-\mathcal{F}(Q_TP_Ef)\|_{L^{p'}(d\gamma)}\\ &\ge (1-\varepsilon_T)\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}-\varepsilon_E\|f\|_{L^p(d\nu)}.\end{aligned}$$

On the other hand, it is easy to obtain

$$\frac{\|\mathcal{F}(Q_TP_Ef)\|_{L^{p'}(d\gamma)}}{\|f\|_{L^p(d\nu)}}\le\Big(mes_\nu(E)mes_\gamma(T)\Big)^{\frac{1}{p'}}.$$

Hence

$$(mes_\nu(E)mes_\gamma(T))^{\frac{1}{p'}}\|f\|_{L^p(d\nu)}\ge(1-\varepsilon_E)\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}-\varepsilon_T\|f\|_{L^p(d\nu)},$$

which gives the desired result. □

Proposition 3.5. *Let $f\in L^1(d\nu)\cap L^p(d\nu)$, $p\in(1,2]$. If f is ε_E-concentrated to E in $L^1(d\nu)$-norm and $\mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, then*

$$(mes_\nu(E)mes_\gamma(T))^{\frac{1}{p'}}\ge(1-\varepsilon_E)(1-\varepsilon_T)\frac{\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}}{\|f\|_{L^p(d\nu)}}.$$

Proof. Let $f\in L^1(d\nu)\cap L^p(d\nu)$, $p\in(1,2]$. As $\mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}_\gamma$-norm, it follows that

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} &\le \varepsilon_T\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}+\Big(\int_T|\mathcal{F}(f)(\lambda,\mu)|^{p'}d\gamma(\lambda,\mu)\Big)^{\frac{1}{p'}}\\ &\le \varepsilon_T\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}+(mes_\gamma(T))^{\frac{1}{p'}}\|\mathcal{F}(f)\|_{L^\infty(d\gamma)}.\end{aligned}$$

Thus from (2.10),

$$(1-\varepsilon_T)\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}\le(mes_\gamma(T))^{\frac{1}{p'}}\|f\|_{L^1(d\nu)}. \tag{3.3}$$

Similarly, using f is ε_E-concentrated to E in $L^1(d\nu)$-norm, and Hölder inequality, we obtain

$$(1-\varepsilon_E)\|f\|_{L^1(d\nu)}\le(mes_\gamma(E))^{\frac{1}{p'}}\|f\|_{L^p(d\nu)}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain the result. □

4 Generalizations of Heisenberg inequality

In this subsection we study many versions of the Heisenberg uncertainty principle for the generalized Fourier transform.

We put

$$h_t(\lambda,\mu) := e^{-t\|\theta(\lambda,\mu)\|^2}, \quad \text{for all}\,(\lambda,\mu) \in \mathbb{R}_+^{d+1}.$$

Lemma 4.1. *Let* $1 \leq q < \infty$. *There exists a positive constant such that*

$$\|h_t\|_{L^q(d\gamma)} = Ct^{-\frac{2d+1}{2q}}.$$

Proof. Let $1 \leq q < \infty$. Using the relation (2.2), we obtain the result. □

Lemma 4.2. *Let* $1 < p \leq 2$ *and* $0 < a < \frac{2d+1}{p'}$. *Then for all* $f \in L^p(d\nu)$ *and* $t > 0$,

$$\|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \leq Ct^{-\frac{a}{2}}\|\,\|(r,x)\|^a f\|_{L^p(d\nu)}. \tag{4.1}$$

Proof. Inequality (4.1) holds if $\|\,\|(r,x)\|^a f\|_{L^p(d\nu)} = \infty$.
Assume that $\|\,\|(r,x)\|^a f\|_{L^p(d\nu)} < \infty$. For $s > 0$ let $f_s = f\chi_{B(0,s)}$ and $f^s = f - f_s$.

Using Proposition 2.12, and that $|f^s(r,x)| \leq s^{-a}|\,\|(r,x)\|^a f(r,x)|$, we obtain

$$\begin{aligned}
\|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f\chi_{B^c(0,s)})\|_{L^{p'}(d\gamma)} &\leq \|e^{-t\|\theta(\lambda,\mu)\|^2}\|_{L^\infty(d\gamma)}\|\mathcal{F}(f\chi_{B^c(0,s)})\|_{L^{p'}(d\gamma)} \\
&\leq \|f\chi_{B^c(0,s)}\|_{L^p(d\nu)} \\
&\leq s^{-a}\|\,\|(r,x)\|^a f\|_{L^p(d\nu)}.
\end{aligned}$$

On the other hand, by (2.10) and Hölder's inequality

$$\begin{aligned}
\|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f\chi_{B(0,s)})\|_{L^{p'}(d\gamma)} &\leq \|e^{-t\|\theta(\lambda,\mu)\|^2}\|_{L^{p'}(d\gamma)}\|\mathcal{F}(f\chi_{B(0,s)})\|_{L^\infty(d\gamma)} \\
&\leq \|e^{-t\|\theta(\lambda,\mu)\|^2}\|_{L^{p'}(d\gamma)}\|f\chi_{B(0,s)}\|_{L^1(d\nu)} \\
&\leq \|e^{-t\|\theta(\lambda,\mu)\|^2}\|_{L^{p'}(d\gamma)}\|\,\|(r,x)\|^{-a}\chi_{B(0,s)}\|_{L^{p'}(d\nu)}\|\,\|(r,x)\|^a f\|_{L^p(d\nu)}.
\end{aligned}$$

A simple calculation give that

$$\|\,\|(r,x)\|^{-a}\chi_{B(0,s)}\|_{L^{p'}(d\nu)} = C(d,s)s^{\frac{2d+1}{p'}-a}.$$

So

$$\begin{aligned}
\|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f)\|_{L^{p'}(d\gamma)} &\leq \|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f_s)\|_{L^{p'}(d\gamma)} + \|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f^s)\|_{L^{p'}(d\gamma)} \\
&\leq Cs^{-a}(1 + \|e^{-t\|\theta(\lambda,\mu)\|^2}\|_{L^{p'}(d\gamma)}s^{\frac{2d+1}{p'}})\|(r,x)\|^a f\|_{L^p(d\nu)}.
\end{aligned}$$

Choosing $s = t^{\frac{1}{2}}$, we obtain (4.1). □

Theorem 4.3. *Let* $1 < p \leq 2$ *and* $0 < a < \frac{2d+1}{p'}$ *and* $b > 0$. *Then for all* $f \in L^p(d\nu)$

$$\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \leq C\|\,\|(r,x)\|^a f\|_{L^p(d\nu)}^{\frac{b}{a+b}}\|\,\|\theta(\mu,\lambda)\|^b\mathcal{F}(f)\|_{L^{p'}(d\gamma)}^{\frac{a}{a+b}}. \tag{4.2}$$

Proof. Let $1 < p \le 2$ and $0 < a < \frac{2d+1}{p'}$. Assume that $b \le 2$. From the previous lemma, for all $t > 0$

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} &\le \|e^{-t\|\theta(\lambda,\mu)\|^2}\mathcal{F}(f)\|_{L^{p'}(d\gamma)} + \|(1-e^{-t\|\theta(\lambda,\mu)\|^2})\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \\ &\le Ct^{-\frac{a}{2}}\|\|(r,x)\|^a f\|_{L^p(d\nu)} + \|(1-e^{-t\|\theta(\lambda,\mu)\|^2})\mathcal{F}(f)\|_{L^{p'}(d\gamma)}.\end{aligned}$$

On the other hand,

$$\|(1-e^{-t\|\theta(\lambda,\mu)\|^2})\mathcal{F}(f)\|_{L^{p'}(d\gamma)} = t^{\frac{b}{2}}\|(t\|\theta(\lambda,\mu)\|^2)^{-\frac{b}{2}}(1-e^{-t\|\theta(\lambda,\mu)\|^2})\|\theta(\mu,\lambda)\|^b\mathcal{F}(f)\|_{L^{p'}(d\gamma_{m,d})}.$$

Since $(1-e^{-t})t^{-\frac{b}{2}}$ is bounded for $t \ge 0$ if $b \le 2$. Then, we obtain

$$\|\mathcal{F}(f)\|_{L^{p'}(d\gamma_{m,d})} \le C\Big(t^{\frac{a}{2}}\|\|(r,x)\|^a f\|_{L^p(d\nu_{m,d})} + t^{\frac{b}{2}}\|\|\theta(\lambda,\mu)\|^b\mathcal{F}(f)\|_{L^{p'}(d\gamma_{m,d})}\Big),$$

from which, optimizing in t, we obtain (4.2) for $0 < a < \frac{2d+1}{p'}$ and $b \le 2$.

If $b > 2$, let $b' \le 2$. For $u \ge 0$ and $b' < b$, we have $u^{b'} \le 1 + u^b$, which for $u = \frac{\|\theta(\lambda,\mu)\|}{\varepsilon}$ gives the inequality $(\frac{\|\theta(\lambda,\mu)\|}{\varepsilon})^{b'} < 1 + (\frac{\|\theta(\lambda,\mu)\|}{\varepsilon})^b$ for all $\varepsilon > 0$.

It follows that

$$\|\|\theta(\lambda,\mu)\|^{b'}\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \le \varepsilon^{b'}\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} + \varepsilon^{b'-b}\|\|\theta(\lambda,\mu)\|^b\mathcal{F}(f)\|_{L^{p'}(d\gamma)}.$$

Optimizing in ε, we get the result for $b > 2$.

$$\|\|\theta(\lambda,\mu)\|^{b'}\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \le \|\mathcal{F}(f)\|_{L^{p'}(d\gamma)}^{\frac{b-b'}{b}}\|\|\theta(\lambda,\mu)\|^b\mathcal{F}(f)\|_{L^{p'}(d\gamma)}^{\frac{b'}{b}}.$$

Together with (4.2) for $b > 2$. □

Corollary 4.4. *Let $a, b > 0$. For all $f \in L^2(d\nu)$, we have*

$$\|f\|_{L^2(d\nu)} \le C\|\|(r,x)\|^a f\|_{L^2(d\nu)}^{\frac{b}{a+b}}\|\|\theta(\mu,\lambda)\|^b\mathcal{F}(f)\|_{L^2(d\gamma)}^{\frac{a}{a+b}}. \tag{4.3}$$

Proof. Using the previous theorem for $p = 2$, and applying Plancherel formula, we obtain the result when $0 < a < \frac{2d+1}{2}$. If $a \ge \frac{2d+1}{2}$, let $a' < \frac{2d+1}{2}$. For $u \ge 0$, $u^{a'} \le 1 + u^a$ which for $u = \frac{\|(r,x)\|}{\varepsilon}$ gives the inequality

$$(\frac{\|(r,x)\|}{\varepsilon})^{a'} \le 1 + (\frac{\|(r,x)\|}{\varepsilon})^a, \quad \text{for all } \varepsilon > 0.$$

It follows that

$$\|\|(r,x)\|^{a'} f\|_{L^2(d\nu)} \le \varepsilon^{a'}\|f\|_{L^2(d\nu)} + \varepsilon^{a'-a}\|\|(r,x)\|^a f\|_{L^2(d\nu)}.$$

Optimizing in ε, we obtain

$$\|\|(r,x)\|^{a'} f\|_{L^2(d\nu)} \le C\|f\|_{L^2(d\nu)}^{\frac{a-a'}{a}}\|\|(r,x)\|^a f\|_{L^2(d\nu)}^{\frac{a'}{a}}. \tag{4.4}$$

Then, by (4.3) for (a' and b), and (4.4), we deduce that

$$\begin{aligned}\|f\|_{L^2(d\nu)} &\leq C\| \|(r,x)\|^{a'} f\|_{L^2(d\nu)}^{\frac{b}{a'+b}} \| |\lambda|^b \mathcal{F}(f)\|_{L^2_\nu(\mathbb{R})}^{\frac{a'}{a'+b}} \\ &\leq C\|f\|_{L^2(d\nu)}^{\frac{b(a-a')}{a(a'+b)}} \| \|(r,x)\|^{a} f\|_{L^2(d\nu)}^{\frac{a'b}{a(a'+b)}} \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^2(d\gamma)}^{\frac{a'}{a'+b}}.\end{aligned}$$

Thus

$$\|f\|_{L^2(d\nu)}^{\frac{a'(a+b)}{a(a'+b)}} \leq C\| \|(r,x)\|^{a} f\|_{L^2(d\nu)}^{\frac{a'b}{a(a'+b)}} \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^2(d\gamma)}^{\frac{a'}{a'+b}},$$

which gives the result for $a \geq \frac{2d+1}{2}$. □

Let T be a measurable subset of $\mathbb{R}^{d+1}_+$. Let $b > 0$ and let $f \in L^p(d\nu)$, $p \in [1,2]$. We say that $\|\theta(\mu,\lambda)\|^b \mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, if there is a function h vanishing outside T such that

$$\| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f) - h\|_{L^{p'}(d\gamma)} \leq \varepsilon_T \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^{p'}(d\gamma)}.$$

From (3.2), it follows that $\|\theta(\mu,\lambda)\|^b \mathcal{F}_\Lambda(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, if and only if

$$\| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f) - \|\theta(\mu,\lambda)\|^b \mathcal{F}(Q_T f)\|_{L^{p'}(d\gamma)} \leq \varepsilon_T \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^{p'}(d\gamma)}. \tag{4.5}$$

Corollary 4.5. *Let T be a measurable subset of $\mathbb{R}^{d+1}_+$, and let $1 < p \leq 2$, $f \in L^p(d\nu)$ and $b > 0$. If $\|\theta(\mu,\lambda)\|^b \mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, then for $0 < a < \frac{2d+1}{p'}$*

$$\|\mathcal{F}(f)\|_{L^{p'}(d\gamma)} \leq \frac{C}{(1-\varepsilon_T)^{\frac{a}{a+b}}} \| \|(r,x)\|^a f\|_{L^p(d\nu)}^{\frac{b}{a+b}} \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(Q_T f)\|_{L^{p'}(d\gamma)}^{\frac{a}{a+b}}. \tag{4.6}$$

Proof. Let $f \in L^p(d\nu)$, $1 < p \leq 2$. Since $\|\theta(\mu,\lambda)\|^b \mathcal{F}(f)$ is ε_T-concentrated to T in $L^{p'}(d\gamma)$-norm, then we have

$$\| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^{p'}(d\gamma)} \leq \varepsilon_T \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^{p'}(d\gamma)} + \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(Q_T f)\|_{L^{p'}(d\gamma)}.$$

Thus

$$\| \|\theta(\mu,\lambda)\|^b \mathcal{F}(f)\|_{L^{p'}(d\gamma)}^{\frac{a}{a+b}} \leq \frac{1}{(1-\varepsilon_T)^{\frac{a}{a+b}}} \| \|\theta(\mu,\lambda)\|^b \mathcal{F}_\Lambda(Q_T f)\|_{L^{p'}(d\gamma)}^{\frac{a}{a+b}}.$$

Multiply this inequality by $C\| \|(r,x)\|^a f\|_{L^p(d\nu)}^{\frac{b}{a+b}}$ and applying theorem 4.3 we deduce the desired result. □

Corollary 4.6. *Let T be a measurable subset of $\mathbb{R}^{d+1}_+$, and let $f \in L^2(d\nu)$ and $a, b > 0$.*
If $\|\theta(\mu,\lambda)\|^b \mathcal{F}(f)$ is ε_T-concentrated to T in $L^2(d\gamma)$-norm, then

$$\|f\|_{L^2(d\nu)} \leq \frac{C}{(1-\varepsilon_T)^{\frac{a}{a+b}}} \| \|(r,x)\|^a f\|_{L^2(d\nu)}^{\frac{b}{a+b}} \| \|\theta(\mu,\lambda)\|^b \mathcal{F}(Q_T f)\|_{L^2(d\gamma)}^{\frac{a}{a+b}}. \tag{4.7}$$

Proof. We proceed as the previous corollary and using Corollary 4.4 we obtain the result. □

Appendix

In the following we recall the main results proved in [21].

Proposition 4.7. *Let $s > 0$. Then there exists a constant $C_1(d,s)$ such that for all $f \in L^1(d\nu)\bigcap L^2(d\nu)$*

$$\|f\|_{L^2(d\nu)} \le C_1(d,s)\|f\|_{L^1(d\nu)}^{\frac{2s}{2d+1+2s}}\,\|\,\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|_{L^2(d\gamma)}^{\frac{2d+1}{2d+1+2s}}. \tag{4.8}$$

Proposition 4.8. *Let $s > 0$. Then there exists a constant $C_2(d,s)$ such that for all $f \in L^1(d\nu)\bigcap L^2(d\nu)$*

$$\|f\|_{L^1(d\nu)} \le C_2(d,s)\|f\|_{L^2(d\nu)}^{\frac{2s}{2d+1+2s}}\,\|\,\|(r,x)\|^s f\|_{L^1(d\nu)}^{\frac{2d+1}{2d+1+2s}}. \tag{4.9}$$

From the previous results we deduce the following variation on Heisenberg's uncertainty inequality for the generalized Fourier transform.

Theorem 4.9. *Let $s > 0$. Then for all $f \in L^1(d\nu)\bigcap L^2(d\nu)$*

$$\|f\|_{L^2(d\nu)}\|f\|_{L^1(d\nu)} \le C_1(d,s)C_2(d,s)\|\,\|(r,x)\|^s f\|_{L^1(d\nu)}\|\,\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|_{L^2(d\gamma)}. \tag{4.10}$$

Proposition 4.10. *Let $s > 0$ and let W a measurable subset of Γ with $0 < mes_\gamma(W) < \infty$. Then for all $f \in L^1(d\nu)\bigcap L^2(d\nu)$*

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)} \le C_2(d,s)\sqrt{mes_\gamma(W)}\|f\|_{L^2(d\nu)}^{\frac{2s}{2s+2d+1}}\,\|\,\|(r,x)\|^s f\|_{L^1(d\nu)}^{\frac{2d+1}{2s+2d+1}}. \tag{4.11}$$

We adapt the method of Ghorbal-Jaming [13], we have proved the local uncertainty principle of $\mathcal{F}$.

Theorem 4.11. *Let E, W be a pair of measurable subsets such that*

$$0 < mes_\nu(E), mes_\gamma(W) < \infty.$$

Then the following uncertainty principles hold.

1) For $0 < s < \frac{2d+1}{2}$, there exists a constant $C_3(d,s)$ such that for all $f \in L^2(d\nu)$

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)} \le C_3(d,s)(mes_\gamma(W))^{\frac{s}{2d+1}}\,\|\,\|(r,x)\|^s f\|_{L^2(d\nu)}. \tag{4.12}$$

2) For $s > \frac{2d+1}{2}$, there exists a constant $C_4(d,s)$ such that for all $f \in L^2(d\nu)$

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)} \le C_4(d,s)\sqrt{mes_\gamma(W)}\|\,\|(r,x)\|^s f\|_{L^2(d\nu)}^{\frac{2d+1}{2s}}\|f\|_{L^2(d\nu)}^{1-\frac{2d+1}{2s}}. \tag{4.13}$$

Theorem 4.12. *(Cowling-Price's theorem for the generalized Fourier transform)*

Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that

$$\int_{\mathbb{R}_+^{d+1}} \frac{e^{ap\|(r,x)\|^2}|f(r,x)|^p}{(1+\|(r,x)\|)^n}d\nu(r,x) < \infty \tag{4.14}$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{e^{4bq\|\theta(\mu,\xi)\|^2}|\mathcal{F}(f)(\mu,\xi)|^q}{(1+\|(\mu,\xi)\|)^s}d\mu d\xi < \infty, \tag{4.15}$$

for some constants $a>0$, $b>0$, $1\leq p,q<\infty$, and for any $n\in(2d+1,2d+1+p]$ and $s\in(d+1,d+1+q]$. Then

i*) If $ab>\frac{1}{4}$, we have $f=0$ almost everywhere.*

ii*) If $ab=\frac{1}{4}$, we have $f=CN_b$.*

iii*) If $ab<\frac{1}{4}$, for all $\delta\in]b,\frac{1}{4a}[$, the functions of the form $f(r,x)=N_\delta(r,x)$, where $P\in\mathcal{P}$, satisfy (4.14) and (4.15).*

The following is an immediate consequence of Theorem 4.12.

Corollary 4.13. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$|f(r,x)|\leq Me^{-a\|(r,x)\|^2}(1+\|(r,x)\|)^m \text{ a.e.} \tag{4.16}$$

and for all $(\mu,\xi)\in\mathbb{R}_+^{d+1}$,

$$|\mathcal{F}(f)(\mu,\xi)|\leq Me^{-4b\|\theta(\mu,\xi)\|^2} \tag{4.17}$$

for some constants $a,b>0$, $r\geq0$ and $M>0$.

i) If $ab>\frac{1}{4}$, then $f=0$ almost everywhere.

ii) If $ab=\frac{1}{4}$, then f is of the form $f(r,x)=CN_b(r,x)$.

iii) If $ab<\frac{1}{4}$, then there are infinity many nonzero f satisfying (4.16) and (4.17).

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the generalized Fourier transform as follows.

Theorem 4.14. *(Beurling's theorem for the generalized Fourier transform)*
Let $N\in\mathbb{N}$, $\delta>0$ and $f\in L^2(d\nu)$ satisfy

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}}\frac{|f(r,x)||\mathcal{F}(f)(t,y)||R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^N}e^{\|(r,x)\|\|(t,y)\|}d\nu(r,x)dtdy<\infty, \tag{4.18}$$

where R is a polynomial of degree m. If $N\geq m\delta+d+3$, then

$$f(r,x)=\sum_{|l|<\frac{N-m\delta-d-1}{2}}a_l^s\widetilde{W}_l^s(r,x) \text{ a.e.}, \tag{4.19}$$

where $s>0$, $a_l^s\in\mathbb{C}$ and $\widetilde{W}_l^s$ is given by (2.15). Otherwise, $f(r,x)=0$ almost everywhere.

As an application of Theorem 4.14, we deduce the following Gelfand-Shilov type theorem for the generalized Fourier transform.

Corollary 4.15. *Let $N,m\in\mathbb{N}$, $\delta>0$, $a,b>0$ with $ab\geq\frac{1}{4}$, and $1<p,q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $f\in L^2(d\nu)$ satisfy*

$$\int_{\mathbb{R}_+^{d+1}}\frac{|f(r,x)|e^{\frac{(2a)^p}{p}\|(r,x)\|^p}}{(1+\|(r,x)\|)^N}d\nu(r,x)<\infty \tag{4.20}$$

and

$$\int_{\mathbb{R}_+^{d+1}}\frac{|\mathcal{F}(f)(t,y)|e^{\frac{(2b)^q}{q}\|(t,y)\|^q}|R(t,y)|^\delta}{(1+\|(t,y)\|)^N}dtdy<\infty \tag{4.21}$$

for some $R \in \mathcal{P}_m$.

i) If $ab > \frac{1}{4}$ *or* $(p,q) \neq (2,2)$*, then* $f(r,x) = 0$ *almost everywhere.*

ii) If $ab = \frac{1}{4}$ *and* $(p,q) = (2,2)$*, then* f *is of the form* (4.19) *whenever* $N \geq \frac{m\delta+d+3}{2}$ *and* $r = 2b^2$*. Otherwise,* $f(x) = 0$ *almost everywhere.*

Theorem 4.16. *(Miyachi's theorem for the generalized Fourier transform)*

Let f *be a measurable function on* $\mathbb{R}_+^{d+1}$ *even with respect to the first variable such that*

$$E_{a,\beta}^{-1} f \in L^p(d\nu) + L^q(d\nu) \tag{4.22}$$

and

$$\int_{\mathbb{R}^{d+1}} \log^+ \frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\alpha,\xi)|\mathcal{F}(f)(\alpha,\xi)|}{\lambda} d\alpha d\xi < \infty, \tag{4.23}$$

for some constants $a > 0$*,* $b > 0$ $\lambda > 0$*,* $1 \leq p,q \leq \infty$*. Then*

If $ab > \frac{1}{4}$*, we have* $f = 0$ *almost everywhere.*

If $ab = \frac{1}{4}$*, we have* $f = CE_{b,\beta}$ *with* $|C| \leq \lambda$*.*

If $ab < \frac{1}{4}$*, for all* $\delta \in (b, \frac{1}{4a})$*, the functions of the form* $f(x) = CE_{\delta,\beta}$*, satisfy (4.22) and (4.23).*

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