Communications in Mathematical Analysis
Volume 18, Number 1, pp. 69–82 (2015) www.math-res-pub.org/cma
ISSN 1938-9787

1D-Solitons for a Generalized Dispersive Equation

Alex M. Montes*
Department of Mathematics
University of Cauca
Popayán, Kra 3 3N-100, Colombia

(Communicated by Marco Squassina)

Abstract

In this paper we show the existence of one-dimensional solitons (travelling waves of finite energy) for a generalized nonlinear dispersive equation modeling the deformations of a hyperelastic compressible plate. From the Hamiltonian structure for such equation we find the natural space for the travelling wave solutions and characterize travelling waves variationally as minimizers of an energy functional under a suitable constraint. Our approach involves the Lions's Concentration-Compactness Lemma.

AMS Subject Classification: 35Q35, 35Q51, 76B25.

Keywords: Dispersive Equation, Solitons, Concentration-Compactness Lemma.

1 Introduction

The focus of the present work is the one-dimensional generalized dispersive equation

$$\partial_t u-\alpha_1\partial_x^2\partial_t u+\alpha_2\partial_x^4\partial_t u+\partial_x\left(\frac{p+2}{p+1}u^{p+1}-\beta\Big[u\partial_x(\partial_x u)^p+\frac{p}{p+1}(\partial_x u)^{p+1}\Big]\right)=0. \tag{1.1}$$

When $\alpha_1,\alpha_2>0, p\in\mathbb{Z}^+,\beta\in\mathbb{R}$, using a variational approach, we show that (1.1) admits in the energy space $H^2(\mathbb{R})$ travelling wave solutions $u(x,t)=v(x-ct)$.

In a recent paper R. M. Chen (see [4]) derived the following two-dimensional nonlinear dispersive equation,

$$\partial_x\Big(\partial_t u-\partial_x^2\partial_t u+\alpha\partial_x^4\partial_t u+3u\partial_x u-\beta\Big(2\partial_x u\partial_x^2 u+u\partial_x^3 u\Big)\Big)-a\partial_y^2 u+b\partial_x^2\partial_y^2 u=0, \tag{1.2}$$

as a model for the deformations of a hyperelastic compressible plate relative to a uniformly pre-stressed state. In this model u represents vertical displacement of the plate relative to a uniformly pre-stressed state, while x and y are rescaled longitudinal and lateral coordinates

*E-mail address: amontes@unicauca.edu.co

in the horizontal plane. To reduce the full three-dimensional field equation to an approximate two-dimensional plate equation, an assumption has been made that the thickness of the plate is small in comparison to the other dimensions. It is also assumed that the small perturbations superimposed on the pre-stressed state only appear in the vertical direction (the z-direction) and in one horizontal direction (the x-direction). Hence the variation of waves in the transverse direction (the y-direction) is small. Equation (1.2) is obtained under the additional assumption that the wavelength in the x-direction is short.

The parameters in equation (1.2) are all material constants. The scalar α describes the stiffness of the plate which is nonnegative. The coefficients a and b are material constants that measure weak transverse effects. The material constant β occurs as a consequence of the balance between the nonlinear and dispersive effects. Note that there is no dissipation in this model.

Equation (1.2) generalizes several well-known equations including the BBM equation [1] when $\alpha=\beta=a=b=0$, the regularized long-wave Kadomtsev-Petviashvili(KP) equation [2] (also referred as KP-BBM equation, see [8]) when $\alpha=\beta=b=0$, and the Camassa-Holm (CH) equation [3] when $\alpha=a=b=0, \beta=1$. In contrast to the derivation in [4] of nonlinear dispersive waves in a hyperelastic plate, these particular equations are usually derived as models of water waves. In equation (1.2), the two spatial dimensions make the analysis very different from the CH equation. The β-terms include a nonlinear term of fourth order, which makes equation (1.2) very different from the KP- BBM equation.

In the work [5], R. M. Chen showed the global well-posedness for the initial value problem associated to the equation (1.2), in the space W equipped with the norm

$$||u||_W^2 = ||u||_{L^2} + ||\partial_x u||_{L^2} + ||\partial_x^2 u||_{L^2} + ||\partial_y u||_{L^2} + ||\partial_x^{-1}\partial_y u||_{L^2},$$

where $\partial_x^{-1}\partial_y u$ is defined via the Fourier transformation as

$$\widehat{\partial_x^{-1}\partial_y u} = \frac{\eta}{\xi}\widehat{u}(\eta,\xi).$$

In addition, for $\alpha>0$, R. M. Chen established in the space W the existence of 2D-solitons, i.e. the existence of solutions of the form $u(x,y,t)=v(x-ct,y)$.

When we searching the existence of 1D-solitons for the equation (1.2). This is, the existence of travelling wave solutions, $u(x,y,t)=v(x-ct)$, which propagate in the $x-$direction with speed wave $c>0$. One see that the travelling wave profile v should satisfy the ordinary differential equation

$$\left[c\left(v'-v'''+\alpha v'''''\right)-3vv'+\beta\left(2v'v''+vv'''\right)\right]'=0,$$

which, upon integration, yields

$$c\left(v'-v'''+\alpha v'''''\right)-3vv'+\beta\left(2v'v''+vv'''\right)+A=0, \tag{1.3}$$

where A is a constant of integration. Among all the travelling wave solutions we shall focus on solutions which have the additional property that the waves are localized and that v and its derivatives decay at infinity, that is,

$$v^{(n)}(z)\to 0 \quad \text{as} \quad |z|\to\infty,\ \ n\in\mathbb{Z}^+. \tag{1.4}$$

Under this decay assumption the constant of integration in (1.3) vanishes and then we have the equation

$$c(v - v'' + \alpha v'''')' - \frac{3}{2}(v^2)' + \beta\Big(vv'' + \frac{1}{2}(v')^2\Big)' = 0.$$

Using the decay condition (1.4) again, we see that the travelling wave equation takes the form

$$c(v - v'' + \alpha v'''') - \frac{3}{2}v^2 + \beta\Big(vv'' + \frac{1}{2}(v')^2\Big) = 0.$$

Next, we note that $u(x,y,t) = v(x-ct)$ is a travelling wave solution for the equation (1.2) if and only if $u(x,y) = v(x-ct)$ is a travelling wave solution for the equation (1.1), for $\alpha_1 = 1, \alpha_2 = \alpha$ and $p = 1$. So, the main objective in this article is to investigate the existence of solitons for the equation (1.1). For this, we follow a variational approximation by characterizing solitons (travelling waves in the energy space) as multiples of the minimizers of an energy functional subject to a suitable constraint. Using Lions's Concentration-Compactness Principle, we prove that any minimizing sequences converges strongly, after an appropriate translation, to a minimizer. A multiple of this minimizer is a travelling wave solution.

This paper is organized as follows. In Section 2, we describe the Hamiltonian structure for the equation (1.1). From this structure, we find the natural space for the travelling wave solutions, and characterize travelling wave solutions for the equation (1.1) as multiples of the minimizers to a variational problem. In Section 3, we prove the existence of such minimizers by using the Concentration-Compactness Theorem . Throughout this work $H^s = H^s(\mathbb{R})$ denotes the usual Sobolev space of order s and C denotes a generic constant whose value may change from instance to instance.

2 Variational approach for travelling waves

In this section we characterize travelling wave solutions for the equation (1.1) as multiples of the minimizers to a variational problem. First, we show that the evolution equation (1.1) has a Hamiltonian structure.

Proposition 2.1. *The nonlinear evolution equation (1.1) can be expressed in Hamiltonian form,*

$$u_t = -\partial_x\left(I - \alpha_1\partial_x^2 + \alpha_2\partial_x^4\right)^{-1} F'(u),$$

where F is defined on $H^2(\mathbb{R})$ as

$$F(u) = \frac{1}{p+1}\int_{\mathbb{R}}\Big(u^{p+2} + \beta u\,(\partial_x u)^{p+1}\Big)dx.$$

Proof. Since the proof follows the ideas given for other equation models we only present a sketch. First, we will prove that F is well defined on $H^2(\mathbb{R})$. In fact, since the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous we have that there is $C > 0$ such that

$$\int_{\mathbb{R}} u^{p+2}dx \le \|u\|_{L^\infty}^p \|u\|_{L^2}^2 \le C\|u\|_{H^2}^{p+2}.$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}} u(\partial_x u)^{p+1}dx &\le \|u\|_{L^\infty}\|\partial_x u\|_{L^\infty}^{p-1}\|\partial_x u\|_{L^2}^2\\ &\le C\|u\|_{H^1}\|\partial_x u\|_{H^1}^{p-1}\|\partial_x u\|_{L^2}^2\\ &\le C\|u\|_{H^2}^{p+2}.\end{aligned}$$

Therefore

$$|F(u)| \le C\|u\|_{H^2}^{p+2}.$$

Now, we note that if $m = u - \alpha_1\partial_x^2 u + \alpha_2\partial_x^4 u$ then the equation (1.1) can be rewritten as

$$m_t = -\partial_x\left(\frac{p+2}{p+1}u^{p+1} - \beta\Big[u\partial_x(\partial_x u)^p + \frac{p}{p+1}(\partial_x u)^{p+1}\Big]\right),$$

and a simple calculation shows that for every $w \in H^2(\mathbb{R})$,

$$\begin{aligned}\langle F'(u), w\rangle &= \frac{1}{p+1}\int_{\mathbb{R}}\Big((p+2)u^{p+1}w + \beta\Big[(\partial_x u)^{p+1}w + (p+1)u(\partial_x u)^p\partial_x w\Big]\Big)dx\\ &= \frac{1}{p+1}\int_{\mathbb{R}}\Big((p+2)u^{p+1} - \beta\Big[p(\partial_x u)^{p+1} + (p+1)u\partial_x(\partial_x u)^p\Big]\Big)w\,dx.\end{aligned}$$

Then we conclude that

$$F'(u) = \frac{p+2}{p+1}u^{p+1} - \beta\Big[u\partial_x(\partial_x u)^p + \frac{p}{p+1}(\partial_x u)^{p+1}\Big].$$

Hence, the proof is complete. □

Now, notice that if u is a solution of the equation (1.1) of the type $u(x,t) = v(x-ct)$. Then we see that the travelling wave profile v should satisfy, for $\alpha_0 = 1$, the ordinary differential equation

$$c\sum_{k=0}^{2}(-1)^k\alpha_k v^{(2k+1)} - \left(\frac{p+2}{p+1}v^{p+1} - \beta\Big[v\left((v')^p\right)' + \frac{p}{p+1}(v')^{p+1}\Big]\right)' = 0,$$

which, upon integration, yields

$$c\sum_{k=0}^{2}(-1)^k\alpha_k v^{(2k)} - \left(\frac{p+2}{p+1}v^{p+1} - \beta\Big[v\left((v')^p\right)' + \frac{p}{p+1}(v')^{p+1}\Big]\right) + A = 0, \tag{2.1}$$

where A is a constant of integration. Under the decay assumption (1.4) the constant of integration in (2.1) vanishes and then the travelling wave equation takes the form

$$c\sum_{k=0}^{2}(-1)^k\alpha_k v^{(2k)} - \frac{1}{p+1}\Big((p+2)v^{p+1} - \beta\Big[(p+1)v\left((v')^p\right)' + p(v')^{p+1}\Big]\Big) = 0. \tag{2.2}$$

From the Proposition 2.1 we note that the natural space (energy space) to look for travelling waves is the space $H^2(\mathbb{R})$. Thus, if we multiply the travelling wave equation (2.2) with a test function $w \in H^2(\mathbb{R})$, after integration by parts, a travelling wave solution $v \in H^2(\mathbb{R})$ satisfy the integral equation

$$\int_{\mathbb{R}} \Big[c \sum_{k=0}^{2} \alpha_k v^{(k)} w^{(k)} - \frac{1}{p+1} \Big((p+2) v^{p+1} w + \beta \big((v')^{p+1} w + (p+1) v (v')^p w' \big) \Big) \Big] dx = 0. \tag{2.3}$$

Definition 2.2. We say that $v \in H^2(\mathbb{R})$ is a weak solution of (2.2) if for all $w \in H^2(\mathbb{R})$ the integral equation (2.3) holds.

Our strategy to prove the existence of a solution of (2.3) is to consider the following minimization problem

$$\mathcal{I}_c := \inf\{ I_c(v) \ : \ v \in H^2(\mathbb{R}) \ \text{ with } \ G(v) = 1 \}, \tag{2.4}$$

where the energy I_c and the constraint G are functionals defined in $H^2(\mathbb{R})$ given by

$$I_c(v) = \frac{c}{2} \int_{\mathbb{R}} \Big[v^2 + \alpha_1 (v')^2 + \alpha_2 (v'')^2 \Big] dx, \tag{2.5}$$

$$G(v) = \frac{1}{p+1} \int_{\mathbb{R}} \Big[v^{p+2} + \beta v (v')^{p+1} \Big] dx. \tag{2.6}$$

We start by showing some properties of I_c and G, assuming that α_1, $\alpha_2 > 0$, $p \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}$.

Lemma 2.3. *Let $c > 0$. Then*

1. *The functionals I_c and G are well defined in $H^2(\mathbb{R})$ and smooth.*

2. *The functional I_c is nonnegative. Moreover, there are $C_1(\alpha_1, \alpha_2, c) < C_2(\alpha_1, \alpha_2, c)$ such that*
$$C_1 I_c(v) \le \|v\|^2_{H^2(\mathbb{R})} \le C_2 I_c(v). \tag{2.7}$$

3. *$\mathcal{I}_c$ is finite and positive.*

Proof. 1. I_c is clearly well defined for $v \in H^2(\mathbb{R})$. Moreover, note that if $v \in H^2(\mathbb{R}) \subset H^1(\mathbb{R})$ then, using the fact that the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous, we see that there is a constant $C = C(p, \beta) > 0$ such that

$$|G(v)| \le C \|v\|^p_{H^2} \left(\|v\|^2_{L^2} + \|v'\|^2_{L^2} \right) \le C \|v\|^{p+2}_{H^2}. \tag{2.8}$$

So that, G is well defined. *2.* This property is straightforward. *3.* Note that there exists $v \in H^2(\mathbb{R})$ such that $G(v) \neq 0$. Then for some t we have that

$$G(tv) = t^{p+2} G(v) = 1.$$

On the other hand, the inequalities (2.7)-(2.8) imply that there is $C > 0$ such that for any $v \in H^2(\mathbb{R})$ with $G(v) = 1$,

$$C \left(I_c(v) \right)^{\frac{p+2}{2}} \ge C \|v\|^{p+2}_{H^2} \ge G(v) = 1,$$

meaning that the infimum $\mathcal{I}_c$ is finite and positive. ◻

Theorem 2.4. *If v_0 is a minimizer for the problem (2.4), then $v = \theta v_0$ is a nontrivial weak solution of (2.2) with $\theta = \frac{2}{p+2}\mathcal{I}_c$.*

Proof. By the Lagrange Theorem there is a multiplier θ such that for any $w \in H^1(\mathbb{R})$,

$$\langle I_c'(v_0), w\rangle - \theta\langle G'(v_0), w\rangle = 0. \tag{2.9}$$

Now, a direct calculation shows that

$$\langle I_c'(v_0), w\rangle = c\int_{\mathbb{R}}\Big[v_0 w + \alpha_1 v_0' w' + \alpha_2 v_0'' w''\Big]dx,$$
$$\langle G'(v_0), w\rangle = \frac{1}{p+1}\int_{\mathbb{R}}\Big[(p+2)v^{p+1}w + \beta\Big((v_0')^{p+1}w + (p+1)v_0(v_0')^p w'\Big)\Big]dx.$$

In particular, putting $w = v_0$ we have that

$$\langle I_c'(v_0), v_0\rangle = c\int_{\mathbb{R}}\Big[v_0^2 + \alpha_1(v_0')^2 + \alpha_2(v_0'')^2\Big]dx = 2I(v_0),$$
$$\langle G'(v_0), v_0\rangle = \frac{p+2}{p+1}\int_{\mathbb{R}}\Big[v^{p+2} + \beta(v_0')^{p+2})\Big]dx = (p+2)G(v_0).$$

Thus, from the equation (2.9),

$$2I_c(v_0) - (p+2)\theta G(v_0) = 0. \tag{2.10}$$

But, by using $G(v_0) = 1$ we see that $\theta = \frac{2}{p+2}\mathcal{I}_c$. Moreover, from (2.9) we see that θv_0 is a nontrivial solution of the integral equation (2.3). □

3 Existence of Minimizers

The existence of solitons for the equation (1.1) as multiples of the minimizers of the variational problem (2.4) is based on the existence of a compact embedding (local) result and also on an important result by P. L. Lions, known as the Concentration-Compactness Principle (see [6], [7]).

Theorem 3.1. *(P. L. Lions, [6], [7]) Let $\mathcal{I}$ be a real number and let $\{\nu_m\}$ be a sequence of nonnegative measures on $\mathbb{R}$ such that*

$$\lim_{m\to\infty}\int_{\mathbb{R}} d\nu_m = \mathcal{I}.$$

Then there is a subsequence of $\{\nu_m\}$ (which we denote by the same symbol) that satisfies only one of the following properties.
Vanishing. For any $R > 0$,

$$\lim_{m\to\infty}\left(\sup_{x\in\mathbb{R}}\int_{B_R(x)} d\nu_m\right) = 0, \tag{3.1}$$

where $B_R(x)$ is the ball in $\mathbb{R}$ of radius R centered at x.

Dichotomy. There exist $\theta \in (0, \mathcal{I})$ such that for any $\tau > 0$, there are $R > 0$ and a sequence $\{x_m\}$ in $\mathbb{R}$ with the following property: Given $R' > R$ there are nonnegative measures ν_m^1, ν_m^2 such that

1. $0 \le \nu_m^1 + \nu_m^2 \le \nu_m$,

2. $supp(\nu_m^1) \subset B_R(x_m), \quad supp(\nu_m^2) \subset \mathbb{R} \setminus B_{R'}(x_m)$,

3. $\limsup_{m\to\infty} \left(|\theta - \int_{\mathbb{R}} d\nu_m^1| + |(\mathcal{I} - \theta) - \int_{\mathbb{R}} d\nu_m^2| \right) \le \tau$.

Compactness. There exists a sequence $\{x_m\}$ *in* $\mathbb{R}$ *such that for any* $\tau > 0$, *there is* $R > 0$ *with the property that*

$$\int_{B_R(x_m)} d\nu_m \ge \mathcal{I} - \tau, \ \text{for all } m. \tag{3.2}$$

In order to apply this result to our case, let us assume that $\{v_m\}$ in $H^2(\mathbb{R})$ is a minimizing sequence for $\mathcal{I}_c$, then we define the positive measures $\{\nu_m\}$ by $d\nu_m = \rho_m dx$, where ρ_m is defined as

$$\rho_m = \frac{c}{2}\left(v_m^2 + \alpha_1 (v_m')^2 + \alpha_2 (v_m'')^2 \right), \tag{3.3}$$

From the Concentration-Compactness Theorem (see Theorem 3.1), there exists a subsequence of $\{\nu_m\}$ (which we denote by the same symbol) that satisfies either *vanishing*, or *dichotomy*, or *compactness*. We will see that *vanishing* and *dichotomy* can be ruled out, and so using *compactness* we will establish that the minimizing sequence $\{v_m\}$ is compact in $H^2(\mathbb{R})$, up to translation, as a consequence of a local compact embedding result.

We will establish some technical result. The first one is related with the characterization of "vanishing sequences" in $H^2(\mathbb{R})$.

Lemma 3.2. *(Vanishing sequences) Let* $R > 0$ *be given and let* $\{v_m\}$ *be a bounded sequence in* $H^2(\mathbb{R})$ *such that*

$$\lim_{m\to\infty} \left(\sup_{x\in\mathbb{R}} \int_{B_R(x)} d\nu_m \right) = 0.$$

Then we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}} v_m^{p+2}\, dx = \lim_{m\to\infty} \int_{\mathbb{R}} v_m (v_m')^{p+1}\, dx = 0.$$

In particular, if $\{v_m\}$ *is a minimizing sequence for* $\mathcal{I}_c$, *then vanishing is ruled out.*

Proof. Let $x \in \mathbb{R}$, $R > 0$ and $B_R = B_R(x)$. First, we notice that there is $C > 0$ such that

$$\|v_m\|^2_{H^2(B_R)} \le C \int_{B_R} d\nu_m.$$

Thus, since the embedding $H^1(B_R) \hookrightarrow L^\infty(B_R)$ is continuous, we obtain that

$$\begin{aligned}
\int_{B_R} |v_m|^{p+2} dx &\le \|v_m\|^p_{L^\infty(B_R)} \int_{B_R} |v_m|^2 dx \\
&\le C \|v_m\|^p_{H^2(B_R)} \left(\int_{B_R} (v_m)^2 dx + \int_{B_R} (v_m')^2 dx \right) \\
&\le C \|v_m\|^p_{H^2(B_R)} \int_{B_R} d\nu_m.
\end{aligned}$$

Now, covering $\mathbb{R}$ by balls of radius R in such a way that each point of $\mathbb{R}$ is contained in at most two balls, we find that

$$\int_{\mathbb{R}} |v_m|^{p+2} dx \leq 2C \|v_m\|^p_{H^2(\mathbb{R})} \left(\sup_{x \in \mathbb{R}} \int_{B_R(x)} d\nu_m \right).$$

Thus, under the assumptions of the lemma,

$$\lim_{n \to \infty} \int_{\mathbb{R}} v_m^{p+2} = 0.$$

In a similar fashion we obtain the other result, $\lim_{m\to\infty} \int_{\mathbb{R}} v_m (v_m')^{p+1} = 0$. As a consequence of this we see that

$$\lim_{m\to\infty} G(v_m) = 0.$$

If we assume that $\{v_m\}_m$ is a minimizing sequence for $\mathcal{I}_c$, then we have that $G(v_m) = 1$, but from the previous fact we reach a contradiction. □

In order to rule out dichotomy, we will establish a splitting result for a sequence $\{v_m\}$ in $H^2(\mathbb{R})$. Fix a function $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R}^+)$ such that $supp\,\phi \subset B_2(0)$ and $\phi \equiv 1$ in $B_1(0)$. If $R > 0$ and $x_0 \in \mathbb{R}$, we define a split for $v \in H^2(\mathbb{R})$ given by

$$v = v_R^1 + v_R^2,$$

where

$$v_R^1 = v\phi_R,\ \ v_R^2 = v(1-\phi_R),\ \ \phi_R(x) = \phi\Big(\frac{x - x_0}{R}\Big).$$

In addition, we define $A_R(x_0)$ by

$$A_R(x_0) = B_{2R}(x_0) \setminus B_R(x_0).$$

Lemma 3.3. *(A splitting result) Let $R_m > 1$ and $x_m \in \mathbb{R}$ be sequences. Define $A(m) = A_{R_m}(x_m)$ and $\phi_m(x) = \phi\Big(\frac{x - x_m}{R_m}\Big)$. If*

$$\limsup_{m\to\infty} \left(\int_{A(m)} d\nu_m \right) = 0. \tag{3.4}$$

Then as $m \to \infty$ we have that

(a) $I_c(v_m) = I_c(v_m^1) + I_c(v_m^2) + o(1)$.

(b) $G(v_m) = G(v_m^1) + G(v_m^2) + o(1)$.

Proof. First, we will see that

$$I_c(v_m) = I_c(v_m^1) + I_c(v_m^2) + o(1), \ \text{ as } m \to \infty. \tag{3.5}$$

In fact, note that

$$\delta^{(0)} v_m := \int_{\mathbb{R}} \Big[(v_m)^2 - (v_m^1)^2 - (v_m^2)^2 \Big] dx = 2 \int_{A(m)} \phi_m (1 - \phi_m)(v_m)^2 dx.$$

Then

$$|\delta^{(0)} v_m| \le C \int_{A(m)} (v_m)^2 dx \le C \int_{A(m)} d\nu_m \to 0.$$

Similarly we obtain that

$$\begin{aligned}\delta^{(1)} v_m &:= \int_{\mathbb{R}} \left[(v_m')^2 - \left((v_m^1)' \right)^2 - \left((v_m^2)' \right)^2 \right] dx \\ &= 2 \int_{A(m)} \left[\phi_m (1-\phi_m)(v_m')^2 + (1-2\phi_m)(v_m v_m' \phi_m') - 2 v_m^2 (\phi_m')^2 \right] dx.\end{aligned}$$

Since $\left|\phi_m'\right| \le C/R_m$, consequently we have that

$$|\delta^{(1)} v_m| \le C \int_{A(m)} [(v_m)^2 + (v_m')^2] dx \le C \int_{A(m)} d\nu_m \to 0.$$

In a similar fashion if

$$\delta^{(2)} v_m := \int_{\mathbb{R}} \left[(v_m'')^2 - \left((v_m^1)'' \right)^2 - \left((v_m^2)'' \right)^2 \right] dx,$$

we have that $|\delta^{(2)} v_m| \to 0$. Then we obtain that

$$\lim_{m\to\infty} \left[I_c(v_m) - I_c(v_m^1) - I_c(v_m^2) \right] = 0.$$

Next, we will show the item (b). We notice that

$$\begin{aligned}&\int_{\mathbb{R}} \left[(v_m)^{p+2} - \left(v_m^1\right)^{p+2} - \left(v_m^2\right)^{p+2} \right] dx \\ &\le C \int_{A(m)} (v_m)^{p+2} dx \\ &\le C \left(\int_{A(m)} \left[(v_m)^2 + (v_m')^2 + (v_m'')^2 \right] dx \right)^{\frac{p}{2}} \int_{A(m)} \left[(v_m)^2 + (v_m')^2 \right] dx \\ &\le C \left(\int_{A(m)} d\nu_m \right)^{\frac{p+2}{2}} \to 0.\end{aligned}$$

Now, it is not hard to prove that

$$\int_{\mathbb{R}} \left[v_m (v_m')^{p+1} - v_m^1 \left((v_m^1)' \right)^{p+1} - v_m^2 \left((v_m^2)' \right)^{p+1} \right] dx \to 0.$$

Then, from the definition of G, we conclude as $m \to \infty$ that

$$G(v_m) = G(v_m^1) + G(v_m^2) + o(1).$$

□

Using the previous result we have the following lemma.

Lemma 3.4. *Let $\{v_m\}$ be a minimizing sequence for $\mathcal{I}_c$. Then dichotomy is not possible.*

Proof. Assume that dichotomy occurs, then we can choose a sequence $R_m \to \infty$ such that

$$supp(v_m^1) \subset B_{R_m}(x_m), \quad supp(v_m^2) \subset \mathbb{R} \setminus B_{2R_m}(x_m) \tag{3.6}$$

and

$$\limsup_{m\to\infty}\left(\left|\theta - \int_{\mathbb{R}} dv_m^1\right| + \left|(\mathcal{I}_c - \theta) - \int_{\mathbb{R}} dv_m^2\right|\right) = 0. \tag{3.7}$$

Using these facts, we have that

$$\limsup_{m\to\infty}\left(\int_{A(m)} dv_m\right) = 0. \tag{3.8}$$

In fact,

$$\begin{aligned}
\int_{A(m)} dv_m &= \left(\int_{\mathbb{R}} - \int_{B_{R_m}(x_m)} - \int_{\mathbb{R}\setminus B_{2R_m}(x_m)}\right) dv_m \\
&\leq \int_{\mathbb{R}} dv_m - \int_{\mathbb{R}} dv_m^1 - \int_{\mathbb{R}} dv_m^2 \\
&\leq \left|\int_{\mathbb{R}} dv_m - \mathcal{I}_c\right| + \left|\theta - \int_{\mathbb{R}} dv_m^1\right| + \left|(\mathcal{I}_c - \theta) - \int_{\mathbb{R}} dv_m^2\right|.
\end{aligned}$$

Using (3.8) and Lemma 3.3 we conclude that

$$\begin{aligned}
&\lim_{m\to\infty}\left[I_c(v_m) - I_c(v_m^1) - I_c(v_m^2)\right] = 0, \\
&\lim_{m\to\infty}\left[G(v_m) - G(v_m^1) - G(v_m^2)\right] = 0.
\end{aligned}$$

Now, let $\lambda_{m,i} = G(v_m^i)$, for $i = 1, 2$. Passing to a subsequence if necessary we have that $\lambda_i := \lim_{m\to\infty} \lambda_{m,i}$ exists. Now, let us prove that $\lambda_i \neq 0$. Assume that $\lim_{m\to\infty} \lambda_{m,1} = 0$, then $\lim_{m\to\infty} \lambda_{m,2} = 1$ (we proceed in a similar way in the other case). Therefore $\lambda_{m,2} > 0$, for m large enough. Then we consider

$$w_m = \lambda_{m,2}^{-\frac{1}{p+2}} v_m^2.$$

So that

$$w_m \in H^2(\mathbb{R}), \quad G(w_m) = 1.$$

We have a contradiction since

$$\begin{aligned}
\mathcal{I}_c &= \lim_{m\to\infty}\left(I_c(v_m^1) + I_c(v_m^2)\right) \\
&\geq \lim_{m\to\infty}\left(\int_{\mathbb{R}} dv_m^1 + \lambda_{m,2}^{\frac{2}{p+2}} \mathcal{I}_c\right) \\
&= \theta + \mathcal{I}_c.
\end{aligned}$$

In other words, $|\lambda_{m,i}| > 0$ for m large enough. Then we are allowed to define

$$w_{m,i} = \lambda_{m,i}^{-\frac{1}{p+2}} v_m^i, \quad i = 1, 2.$$

Note that $w_{m,i} \in H^2(\mathbb{R})$ and $G(w_{m,i}) = 1$. Hence,

$$\begin{aligned}\mathcal{I}_c &= \lim_{m\to\infty}\left(I_c(v_m^1) + I_c(v_m^2)\right)\\ &= \lim_{m\to\infty}\left(|\lambda_{m,1}|^{\frac{2}{p+2}} I_c(w_{m,1}) + |\lambda_{m,2}|^{\frac{2}{p+2}} I_c(w_{m,2})\right)\\ &\geq \left(|\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}}\right)\mathcal{I}_c.\end{aligned}$$

Then

$$1 \geq |\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} \geq (|\lambda_1| + |\lambda_2|)^{\frac{2}{p+2}} \geq |\lambda_1 + \lambda_2|^{\frac{2}{p+2}} = 1.$$

Hence, $|\lambda_1| + |\lambda_2| = 1$. Using that $\lambda_1 + \lambda_2 = 1$ and $\lambda_i \neq 0$, we have that $\lambda_i > 0$ and

$$\lambda_1^{\frac{2}{p+2}} + \lambda_2^{\frac{2}{p+2}} = (\lambda_1 + \lambda_2)^{\frac{2}{p+2}}. \tag{3.9}$$

But (3.9) gives us a contradiction, because for $t \in \mathbb{R}^+$ the function $f(t) = t^{\frac{2}{p+2}}$ is strictly concave, meaning that

$$f(t_1 + t_2) < f(t_1) + f(t_2), \text{ for } t_1, t_2 > 0.$$

Therefore, we have ruled out dichotomy. □

Now we are in position to obtain the main result in this section: the existence of a minimizer for $\mathcal{I}_c$. Since we ruled out vanishing and dichotomy above for a minimizing sequence of $\mathcal{I}_c$, then by P. L. Lion's Concentration-Compactness Theorem, there exists a subsequence of $\{v_m\}$ (which we denote by the same symbol) satisfying *compactness*. We will see as a consequence of local compact embedding that a minimizing sequence $\{v_m\}$ is compact in $H^1(\mathbb{R})$, up to translation.

Theorem 3.5. *If $\{v_m\}$ is a minimizing sequence for (2.4), then there is a subsequence (which we denote by the same symbol), a sequence of points $x_m \in \mathbb{R}$, and a minimizer $v_0 \in H^2(\mathbb{R})$ of (2.4), such that the translated functions*

$$\tilde{v}_m = v_m(\cdot + x_m)$$

converge to v_0 strongly in $H^1(\mathbb{R})$.

Proof. Let $\{v_m\}$ be a minimizing sequence for (2.4). In other words,

$$\lim_{m\to\infty} I_c(v_m) = \mathcal{I}_c \quad \text{and} \quad G(v_m) = 1.$$

By *compactness*, there exists a sequence x_m in $\mathbb{R}$ such that for a given $\tau > 0$, there exists $R > 0$ with the following property,

$$\int_{B_R(x_m)} d\nu_m \geq \mathcal{I}_c - \tau, \quad \text{for all } m \in \mathbb{N}. \tag{3.10}$$

Using this we may localize the minimizing sequence $\{v_m\}$ around the origin by defining

$$\tilde{\rho}_m(x) = \rho_m(x + x_m), \quad \tilde{v}_m(x) = v_m(x + x_m).$$

Thus, we have the following localized inequality

$$\int_{B_R(0)} \tilde{\rho}_m dx = \int_{B_R(x_m)} d\nu_m \geq \mathcal{I}_c - \tau, \quad \text{for all } m \in \mathbb{N}, \tag{3.11}$$

and also that

$$G(\tilde{v}_m) = G(v_m) = 1, \quad \lim_{m\to\infty} I_c(\tilde{v}_m) = \lim_{m\to\infty} I_c(v_m) = \mathcal{I}_c. \tag{3.12}$$

Then by (2.7) we note that $\{\tilde{v}_m\}$ is a bounded sequence in $H^2(\mathbb{R})$. On the other hand, since $\tilde{v}_m \in H^2(U)$ for any bounded open set $U \subset \mathbb{R}$ and the embedding $H^2(U) \hookrightarrow L^q(U)$ is compact for $q \in [2,\infty]$, then there exist a subsequence of $\{\tilde{v}_m\}$ (which we denote by the same symbol) and $v_0 \in H^2$ such that

$$\tilde{v}_m \rightharpoonup v_0 \quad \text{in} \quad H^2(\mathbb{R}) \quad \text{and} \quad \tilde{v}_m \rightharpoonup v_0,\ \tilde{v}_m' \rightharpoonup v_0',\ \tilde{v}_m'' \rightharpoonup v_0'' \quad \text{in} \quad L^2(\mathbb{R}),$$

and we also have that

$$\tilde{v}_m \to v_0, \quad \tilde{v}_m' \to v_0', \quad \tilde{v}_m'' \to v_0'' \quad \text{in} \quad L^2_{loc}(\mathbb{R}).$$

Moreover,

$$\tilde{v}_m \to v_0, \quad \tilde{v}_m' \to v_0', \quad \tilde{v}_m'' \to v_0'' \quad \text{a.e in} \quad \mathbb{R}.$$

Using these facts we will show that some subsequence of $\{\tilde{v}_m\}$ (which we denote by the same symbol) converges strongly in $H^2(\mathbb{R})$ to a nontrivial minimizer v_0 of (2.4). This is,

$$\tilde{v}_m \to v_0, \quad \tilde{v}_m' \to v_0', \quad \tilde{v}_m'' \to v_0'' \quad \text{in} \quad L^2(\mathbb{R}). \tag{3.13}$$

In fact, using (3.11), (3.12) and the definition of I_c we have that for $\tau > 0$, there exists $R > 0$ such that for m large enough,

$$\int_{B_R(0)} |\tilde{v}_m|^2 dx \geq \int_{\mathbb{R}} |\tilde{v}_m|^2 dx - 2\tau.$$

Then we have that

$$\begin{aligned}
\int_{\mathbb{R}} |v_0|^2 dx &\leq \liminf_{m\to\infty} \int_{\mathbb{R}} |\tilde{v}_m|^2 dx \\
&\leq \liminf_{m\to\infty} \int_{B_R(0)} |\tilde{v}_m|^2 dx + 2\tau \\
&= \int_{B_R(0)} |v_0|^2 dx + 2\tau \\
&\leq \int_{\mathbb{R}} |v_0|^2 dx + 2\tau.
\end{aligned}$$

Therefore

$$\liminf_{m\to\infty} \int_{\mathbb{R}} |\tilde{v}_m|^2 dx = \int_{\mathbb{R}} |v_0|^2 dx.$$

Thus, there exists a subsequence of $\{\tilde{v}_m\}$ such that $\tilde{v}_m \to v_0$ in $L^2(\mathbb{R})$. Using a similar argument we prove the other part of (3.13). Now, using (3.13) and the fact that the inclusion $H^2(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous we have that

$$G(v_0) = \lim_{m\to\infty} G(\tilde{v}_m) = 1. \tag{3.14}$$

In fact,

$$\int_{\mathbb{R}}\left[\tilde{v}_m(\tilde{v}_m')^{p+1}-v_0\left(v_0'\right)^{p+1}\right]dx=\int_{\mathbb{R}}\left[(\tilde{v}_m-v_0)(\tilde{v}_m')^{p+1}+v_0\left((\tilde{v}_m')^{p+1}-\left(v_0'\right)^{p+1}\right)\right]dx.$$

But we see that

$$\begin{aligned}\int_{\mathbb{R}}(\tilde{v}_m-v_0)(v_m')^{p+1}\,dx &\le \|\tilde{v}_m-v_0\|_{L^\infty}\|\tilde{v}_m'\|_{L^\infty}^{p-1}\|v_m'\|_{L^2}^2\\ &\le C\,\|\tilde{v}_m-v_0\|_{H^2}\|\tilde{v}_m\|_{H^2}^{p+1}\\ &\le C\,[I_c(\tilde{v}_m)]^{\frac{p+1}{2}}\,\|\tilde{v}_m-v_0\|_{H^2}\\ &\le C\,\|\tilde{v}_m-v_0\|_{H^2}\to 0,\end{aligned}$$

and also, using that the embedding $H^1(U)\hookrightarrow L^q(U)$ is continuous for $q\in[2,\infty]$,

$$\begin{aligned}\int_{\mathbb{R}}v_0\left((\tilde{v}_m')^{p+1}-\left(v_0'\right)^{p+1}\right)dx &\le C\|v_0\|_{L^\infty}\|\tilde{v}_m'-v_0'\|_{L^2}\sum_{j=0}^{p}\|(\tilde{v}_m')^{p-j}(v_0')^j\|_{L^2}\\ &\le C\|v_0\|_{H^2}\|\tilde{v}_m-v_0\|_{H^2}\sum_{j=0}^{p}\|\tilde{v}_m'\|_{L^4}^{p-j}\|v_0'\|_{L^4}^{j}\\ &\le C\|v_0\|_{H^2}\|\tilde{v}_m-v_0\|_{H^2}\sum_{j=0}^{p}\left(\|\tilde{v}_m\|_{H^2}^{2(p-j)}+\|v_0\|_{H^2}^{2j}\right).\end{aligned}$$

Next, there are $r_1,r_2>0$ such that

$$\|\tilde{v}_m\|_{H^2}^{2(p-j)}\le\|\tilde{v}_m\|_{H^2}^{2r_1},\quad \|v_0\|_{H^2}^{2j}\le\|v_0\|_{H^2}^{2r_2},\quad j=1,...,p.$$

So that

$$\begin{aligned}\int_{\mathbb{R}}v_0\left((\tilde{v}_m')^{p+1}-\left(v_0'\right)^{p+1}\right)dx &\le C\|v_0\|_{H^2}\|\tilde{v}_m-v_0\|_{H^1}\left(\|\tilde{v}_m\|_{H^2}^{2r_1}+\|v_0\|_{H^2}^{2r_2}\right)\\ &\le C[I(v_0)]^{\frac{1}{2}}\|\tilde{v}_m-v_0\|_{H^1}\left([I_c(\tilde{v}_m)]^{r_1}+[I_c(v_0)]^{r_2}\right)\to 0\end{aligned}$$

In a similar fashion we have that

$$\int_{\mathbb{R}}\left[(\tilde{v}_m)^{p+2}-v_0^{p+2}\right]dx\to 0.$$

Then from the definition of G we conclude that (3.14) holds, which implies that $v_0\ne 0$. On the other hand, from (3.13), we see that

$$\lim_{m\to\infty}I_c(\tilde{v}_m)=I_c(v_0)=\mathcal{I}_c,\quad \lim_{m\to\infty}I_c(\tilde{v}_m-v_0)=0.$$

Moreover, the sequence $\{\tilde{v}_m\}$ converges to v_0 in $H^2(\mathbb{R})$, since

$$\|\tilde{v}_m-v_0\|_{H^2(\mathbb{R})}\le C_1I_c(\tilde{v}_m-v_0).$$

Then we concluded that $\{\tilde{v}_m\}$ converges to v_0 in $H^2(\mathbb{R})$ and v_0 is a minimizer for $\mathcal{I}_c$. □

Combining the Theorem 2.4 and Theorem 3.5 we obtain the main result of this work. This is, the result of existence of travelling wave solutions for the equation (1.1).

Corollary 3.6. *Let $\alpha_1,\alpha_2>0, p\in\mathbb{Z}^+,\beta\in\mathbb{R}$ and $c>0$. Then the equation (1.1) admits in the space $H^2(\mathbb{R})$ travelling wave solutions, $u(x,t)=v(x-ct)$, with wave speed $c>0$.*

Acknowledgments

A. M. Montes was supported by University of Cauca under the projet I.d. 3982. The author thanks the referees for their careful reading of the manuscript and insightful comments.

References

[1] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, *Philos. Trans. R. Soc. Lond. A.* **227** (1972), pp 47-78.

[2] J. L. Bona, Y. Liu, and M. M. Tom, The Cauchy problem and stability of solitary-wave solutions for the RLW-KP equations, *J. Diff. Eq.* **185** (2002), pp 437-482.

[3] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), pp 1661-1664.

[4] R. M. Chen, Some nonlinear dispersive waves arising in compressible hyperelastic plates, *International Journal of Engineering Science.* **44** (2006), pp 1188-1204.

[5] R. M. Chen, The Cauchy problem and the stability of solitary waves of a hyperelactic dispersive equation, *Indiana U. J. Math.* **57** (2008), pp 2949-2979.

[6] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **1** (1984), pp 109-145.

[7] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **1** (1984), pp 223-283.

[8] J. C. Saut and N. Tzvetkov, Global well-posedness for the KP-BBM equations, *AMRX.* **1**, (2004), pp 1-16.