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Abstract

We consider a Gause type predator-prey system with functional response given by $\theta(x) = \arctan(ax)$, where $a > 0$, and give a counterexample to the criterion given in Attili and Mallak [*Commun. Math. Anal.* 1:33–40(2006)] for the nonexistence of limit cycles. When this criterion is satisfied, instead this system can have a locally asymptotically stable coexistence equilibrium surrounded by at least two limit cycles.

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1 Introduction

We consider the predator–prey model studied by Attili and Mallak [1]:

$$\frac{dx}{dt} = rx(1-x) - y\theta(x) = \theta(x)(F(x) - y), \quad (1.1a)$$

$$\frac{dy}{dt} = -Dy + sy\theta(x), \quad (1.1b)$$

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where x and y denote the prey and predator population sizes, respectively, at time t . Parameters r, s , and D are assumed to be positive, and response function $\theta(x) = \arctan(ax)$, where $a > 0$.

In (1.1a),

$$F(x) = \frac{rx(1-x)}{\theta(x)}, \quad x > 0 \quad (1.2)$$

denotes the prey nullcline.

In [1], Attili and Mallak state that for a suitable $a > 0$, system (1.1) has a unique equilibrium point in the first quadrant [only] if

$$k\pi < \frac{D}{s} < k\pi + \frac{\pi}{2} \quad (1.3)$$

where k is an integer, with components (x^*, y^*) given by

$$x^* = \frac{1}{a} \tan\left(\frac{D}{s}\right) \quad \text{and} \quad y^* = \frac{rsx^*(1-x^*)}{D}. \quad (1.4)$$

It is also necessary to assume that $a > 0$ is small enough so that $x^* < 1$ in order to ensure $y^* > 0$.

For biological considerations, we think that we should only allow the principal branches of \arctan and \tan , since $\theta(x)$ is a response function, and so it must be defined at zero and satisfy $\theta(0) = \arctan(0) = 0$. Therefore, we should restrict the range of $\arctan(x)$ to $[0, \frac{\pi}{2})$ and since the component x^* , must satisfy $\theta(ax^*) = \arctan(ax^*) = \frac{D}{s}$, we think that it makes more sense to assume that a coexistence equilibrium point exists only if $0 < \frac{D}{s} < \frac{\pi}{2}$, i.e. taking $k = 0$ in (1.3). Another way to see that this restriction makes more sense biologically is to realize that if the predator cannot survive for a certain value of the parameter D (its per capita death rate in the absence of prey), it should not be able to survive for a larger D . Allowing (1.3) for any positive integer k would violate this.

Since it is well-known that solutions to Gause type predator-prey systems are bounded (see for example [5, 9]) it follows that a necessary condition for the non-existence of limit cycles is that this equilibrium point is stable. In [1], it is then proven (with typos cancelling each other out) that a necessary condition for this equilibrium point to be stable is that

$$\tan\left(\frac{D}{s}\right) \left[\frac{s \tan\left(\frac{D}{s}\right) - 2D \left[1 + \tan^2\left(\frac{D}{s}\right)\right]}{s \tan\left(\frac{D}{s}\right) - D \left[1 + \tan^2\left(\frac{D}{s}\right)\right]} \right] \geq a. \quad (1.5)$$

Their main result, given in Theorem 3.7, states that (1.1) has no limit cycles if and only if (1.5) holds. It is the purpose of this paper to show that this statement is incorrect, and that instead, the system can have at least two limit cycles when (1.5) holds.

2 Main Results

It is well-known (see e.g., [3, 9]), that for system (1.1), the slope of the prey isocline, $F(x)$ at the (x^*, y^*) determines the stability of this equilibrium point. In particular, if the slope is positive the Jacobian of the linearization of system (1.1) at (x^*, y^*) has a pair of complex eigenvalues with positive real parts, and so it is unstable, and if the slope is negative the

eigenvalues are negative or there is a pair of complex eigenvalues with negative real parts, and so it is asymptotically stable. If the slope is zero, i.e. at a local minimum or local maximum of the nullcline, there is a pair of pure imaginary eigenvalues of the Jacobian evaluated at (x^*, y^*) . The predator nullcline that intersects the prey nullcline $y = F(x)$ is the vertical line $x = \frac{1}{a} \tan\left(\frac{D}{s}\right)$. This line moves to the right as D increases. As D increases through a critical value D_c of D where the prey and predator nullclines intersect at (x^*, y^*) and $F'(x^*) = 0$, the real parts of the eigenvalues must therefore change sign and so the transversality condition required for a Hopf bifurcation is also satisfied. Hence, there is a Hopf bifurcation at $D = D_c$. The sign of the following expression, given in [9], and obtained using the algorithm in Marsden and McKracken [6], determines the direction and stability of the family of bifurcating periodic orbits, i.e., the criticality of the Hopf bifurcation. If

$$\Omega = -F''(x^*) \frac{\theta(x^*)\theta''(x^*)}{\theta'(x^*)} + \theta(x^*)F'''(x^*) + 2\theta'(x^*)F''(x^*) \quad (2.1)$$

is positive at the Hopf bifurcation, then the Hopf bifurcation is subcritical, and so there is a family of unstable limit cycles bifurcating from the equilibrium point (x^*, y^*) on the side where the equilibrium is asymptotically stable. If Ω is negative, the Hopf bifurcation is supercritical, giving birth to a family of orbitally asymptotically stable limit cycles on the side where the equilibrium point is unstable.

Example 2.1. Here we provide a counter-example to Theorem 3.7, in [1]. Evaluation of expressions was done using the symbolic computation software, Maple [7].

Consider system (1.1) where $r = 1, s = 1, a = 4.341$. Since the condition given in (1.5) is a necessary condition for (x^*, y^*) to be stable, taking into consideration the discussion above, a Hopf bifurcation occurs, if and only if equality holds in (1.5).

A Hopf bifurcation occurs at two values, $D_{HB_1} \approx 0.42639$ and $D_{HB_2} \approx 0.9854847$, that correspond to the Hopf bifurcation at the local minimum and local maximum of $y = F(x)$, respectively. See Table 1 for the corresponding components of the coexistence equilibrium point. Evaluating Ω given in (2.1) at (x^*, y^*) corresponding to D_{HB_1} using Maple [7], we obtain $\Omega = 4.7761 > 0$. Therefore, the Hopf bifurcation at the local minimum of $y = F(x)$ is subcritical, and hence involves a family of unstable limit cycles. The only way this can occur is that either at least two limit cycles exist for all $0 < D < D_{HB_1}$ or a saddle-node of limit cycles bifurcation occurs for some $D < D_{HB_1}$ giving rise to an outer orbitally asymptotically stable limit cycle and an inner unstable limit cycle. In both cases, as D increases through D_{HB_1} , the inner unstable limit cycle disappears in the subcritical Hopf bifurcation. In this example, there must be a saddle-node of limit cycles bifurcation, for some $0 < D < D_{HB_1}$, since Harrison [4] proved, using a Lyapunov function, that if $(\theta(x) - \theta(x^*))(F(x) - F(x^*)) < 0$ for all $x \in [0, x^*) \cup (x^*, 1]$, then (x^*, y^*) is globally asymptotically stable. This holds, for $D > 0$ sufficiently close to zero, so that x^* is also sufficiently close to zero, since $\lim_{x \rightarrow 0} F(x) = 0.23036 > F(x)$ for all $x \in (0, 1]$, noting that the value of $F(x)$ at its local maximum in $(0, 1]$ is approximately 0.2301037 (see Table 1 for the value of y^* when $D = D_{HB_2}$). Therefore, there are no limit cycles for $D > 0$ sufficiently close to zero.

To be specific, if we fix $D = 0.4 < D_{HB_1}$, condition (1.5) holds, since the left-hand side is equal to $4.515467172 > a = 4.341$, but the system has at least two limit cycles as shown in the bifurcation diagram given in Figure 1(a) (generated using XPPAUT [2] and plotted

using Matlab [8]), and the phase portrait given in Figure 1(b) (generated and plotted using Matlab [8]). The unstable limit cycle was found by integrating backward in time. In fact, for any D between the critical value for the saddle-node of limit-cycle bifurcation and the subcritical Hopf bifurcation of (x^*, y^*) (see Table 1), (x^*, y^*) is locally asymptotically stable and so condition (1.5) holds, and at least two limit cycles exist, contradicting Theorem 3.7, in [1], that states that there are no limit cycles in this case.

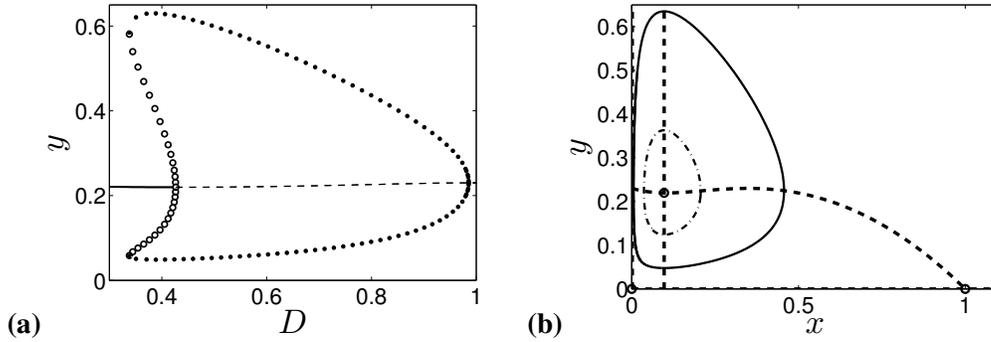


Figure 1. Parameters $r = 1, s = 1$, and $a = 4.341$. **(a)** Bifurcation diagram of system (1.1) showing that as D increases from 0, the system (1.1) undergoes a series of bifurcations at the critical values given in Table 1. First a saddle-node bifurcation of limit cycles (SN) gives rise to two periodic orbits. The amplitude of the inner one (depicted by open circles) is unstable and the outer one (depicted by smaller closed circles) is orbitally asymptotically stable. Next a subcritical Hopf bifurcation occurs where the unstable limit cycle disappears. Finally a supercritical Hopf bifurcation occurs where the orbitally asymptotically stable limit cycle disappears. **(b)** Phase portrait of system (1.1) showing that for $D = 0.4$, system (1.1) has at least two limit cycles surrounding a locally asymptotically stable equilibrium point to the left of the local minimum of the prey nullcline. The inner (dot-dashed) found by integrating backward in time is unstable and the outer (solid) found by integrating forward in time is orbitally asymptotically stable.

Bifurcation type	D	x^*	y^*	Ω
SN	0.33779			
example	0.4	0.097395	0.219774	
HB ₁	0.42639	0.104644	0.219737	4.7761
HB ₂	0.98548	0.347565	0.230104	-8.3121

Table 1. For parameter values $r = 1, s = 1, a = 4.341$, the critical values of D at which there are bifurcations and the corresponding components of the coexistence equilibrium point (see Figure 1(a)), as well as the value of D in the example used for the phase portrait given in Figure 1(b). SN denotes the saddle-node bifurcation of limit cycles, HB₁ the subcritical Hopf bifurcation, and HB₂ the supercritical Hopf bifurcation. The sign of Ω determines the criticality of the Hopf bifurcation.

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